MAT2377

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Comments

- These slides cover material from Chapter 1.
- In class, I may use a blackboard. I recommend reading these slides before you come to the class.
- I am planning to spend 3 lectures on this chapter.
- I am not re-writing the textbook. The reference book contains many interesting and practical examples.
- There may be some typos. The final version of the slides will be posted *after* the chapter is finished.

Sample spaces and events

- Any process that generates a **set of data (observations)** is referred to as an **experiment**. A simple example of an experiment is the tossing of a coin. The experiment leads to some possible **outcomes**.

- For any experiment, the **sample space** is defined as the set of all possible outcomes. This is often denoted by the symbol S (or Ω).

- An **event** is a collection of outcomes from the sample space S.

- Events will be denoted by A, B, E, G_1, G_2, etc.

Examples:

- Tossing a (fair) coin:
  The (discrete) sample space is $S = \{H, T\}$, where "H" stands for "Head" and "T" stands for "Tail".
  $A = \{H\}$ is an *event* (the event of observing "Heads" in tossing one (fair) coin.

- Flipping a (fair) coin twice:
  The (discrete) sample space is $S = \{HH, HT, TH, TT\}$.
  $B = \{HH, HT, TH\}$ is an *event* (the event of observing at least one "Heads" in twice flipping a (fair) coin.

  If a sample space contains a finite number of possibilities or an unending sequence, it is called a discrete sample space.

Examples:

- Rolling a (balanced) six-sided die:
  The (discrete) sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
  $E = \{2, 4, 6\}$ is an *event* (the event of observing "even" numbers in rolling one die.

- Tossing a (balanced) six-sided die twice:
  The (discrete) sample space is $S = \{(i, j) : i, j = 1, 2, \ldots, 6\}$.
  "The sum of the results of the two tosses is equal to 7" is an *event*.
  We might denote it by $C = \{(i, j) : i, j = 1, 2, \ldots, 6,\ i + j = 7\}$.

Examples:

- Choosing a point from the interval $(0, 1)$:
  The (continuous) sample space is $S = (0, 1)$.
  $D = (1/3, 1/2)$ is an *event*.

- Measuring the weight (in grams) of a chemical sample:
  The (continuous) sample space is $S = (0, \infty)$, the positive half line.
  $E_1 = (0, 453.592)$ and $E_2 = (0, 1000)$ can be two *events*.

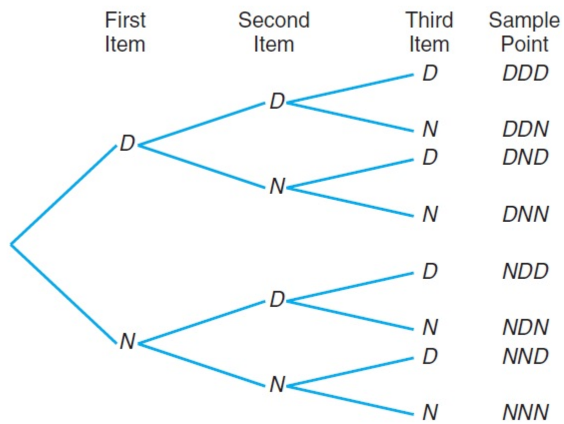
- Measuring the lifetime of a lightbulb:
  The (continuous) sample space is $S = [0, \infty)$, the non-negative half line.
  What can be an *event* here?

If a sample space contains an infinite number of possibilities, it is called a continuous sample space.

Examples:

- **Tree Diagram Usage:**

Three items are selected at random from a manufacturing process. Each item is inspected and classified defective (D) or nondefective (N).



Three diagram of Example 1.5 (Ref. Book).

The *sample space* consists of eight sample points:

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}.$$

Examples: Sample spaces with large or infinite number of sample points:

- If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, the sample space is
$$S = \{x : x \text{ is a city with a population over 1 million}\}.$$

- If S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the origin, the sample space is
$$S = \{(x, y) : x^2 + y^2 \leq 4\}.$$

- Sampling until k defectives: The experiment is to sample items randomly until one defective item is observed. The sample space is
$$S = \{D, ND, NND, NNND, \ldots, \}.$$

Sets of Events

For any events A and B in S:

- $A \cup B$: The union of A and B is the event containing all the elements that belong to A or B or both.
- $A \cap B$: The intersection of A and B is the event containing all elements that are common to A and B.
- $A \setminus B$: The relative complement of B in A (A difference B) is the event containing all elements in A but not in B.
- A' or A^c: The complement of A with respect to S is the subset of all elements of S that are not in A.
- $A \cap B = \emptyset$: If A and B have no outcomes in common, they are mutually exclusive or disjoint. In particular, A and A^c are mutually exclusive.

Examples:

- In the rolling a six-sided die example, let $A = \{2, 3, 5\}$ (a prime number) and $B = \{3, 6\}$ (multiple of 3). Then

$$
\begin{aligned}
A \cup B &= \{2, 3, 5, 6\}, \\
A \cap B &= \{3\}, \\
A^c &= \{1, 4, 6\}.
\end{aligned}
$$

The relationship between events and the corresponding sample space can be illustrated graphically by means of Venn diagrams.

Counting techniques

Permutation is an arrangement of all or part of a set of objects.

Examples:

- Consider the three letters a, b, and c.
  The possible permutations are

$$abc, acb, bac, bca, cab, cba.$$

  Thus, there are 6 distinct (ordered) arrangements.

- The number of permutations of n objects is $n!$.
  - $n! = n(n-1)(n-2)\cdots 1$,
  - $0! = 1$ (by convention).

Examples:

- Consider the four letters a, b, c and d.
  How many arrangements can be set?

- Consider permutations that are possible by taking two letters at a time from the four letters.
  The possible permutations are $ab, ac, ad, bc, bd, cd, ba, ca, da, cb, db, dc$.
  (Consider that we have two positions to fill, with $n_1 = 4$ choices for the first letter and then $n_2 = 3$ choices for the second one).

The number of permutations of n distinct objects taken r at a time is

$$nPr = \frac{n!}{(n-r)!}.$$

Examples: In a permutation, ordering is important.

- Three awards (research, teaching, and service) will be given to a class of 25 graduate students in an engineering department. If each student can receive at most one award, how many possible selections are there?

- A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if
  - (a) there are no restrictions;
  - (b) A will serve only if he is president?

- Number of permutations of n objects arranged in a circle is $(n-1)!$.

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}.$$

Examples:

- How many distinct permutations can be made from the letters of the word SMS? What about the word STATISTICS?

- In how many ways can 5 graduate students be assigned to 1 triple and 1 double hotel rooms during a conference?

- A counting problem without regard to the ordering, is a combination.

- A combination is actually a partition with two cells, the one cell containing the r objects selected and the other cell containing the $(n-r)$ remaining objects.

The number of combinations of n distinct objects taken r at a time is

$$nCr = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Example:

- Flip a coin for n times. What is the sample space? In how many ways you get $x(< n)$ Heads and $n - x$ Tails?
  This number is equal to the number of partitions of n outcomes into two groups with x in one group and $n - x$ in the other group.

Classical Probability

For situations where we have a random experiment which has exactly c possible mutually exclusive, equally likely simple outcomes, we can assign a probability to an event A by counting the number of simple outcomes that correspond to A. If the count is a then

$$P(A) = \frac{a}{c}.$$

Example:

- In the fair coin tossing example, the probability of observing a Head is $P(A) = P(\{H\}) = \frac{1}{2}$.

Examples:

- In the rolling a balanced six-sided die example, the probability of observing even numbers is

$$\text{Prob}(\{\text{number is even}\}) = P(E) = \frac{3}{6} = \frac{1}{2}.$$

- In a group of 1000 people it is known that 545 have high blood pressure.
  1 person is selected randomly.
  What is the probability that this person has high blood pressure?

  Relative frequency of people with high blood pressure is $\frac{545}{1000}$.
  Via the classical definition, this is the probability we are looking for.

Axioms of Probability

1. For any event A, $0 \leq P(A) \leq 1$.

2. For the complete sample space S, $P(S) = 1$.

3. If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Corollaries:

- $P(A^c) = 1 - P(A)$,
- $P(\emptyset) = 0$,
- If $A_1, A_2, \ldots, A_n$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n),$$

- If $A_1, A_2, \ldots, A_n$ is a sequence of mutually exclusive and exhaustive events (a partition) of the sample space S, then

$$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n) = P(S) = 1.$$

Examples:

- In the rolling a balanced six-sided die example.

  Let $A = \{3, 6\}$, the number is a multiple of 3, and $B = \{1, 2\}$, the number is less than 3.
  A and B are mutually exclusive. Thus,

$$P(A \text{ or } B \text{ occurs}) = P(A \cup B) = P(A) + P(B) = \frac{2}{6} + \frac{2}{6} = \frac{2}{3}.$$

- I have an urn containing 4 green balls, 3 red balls and 1 blue ball. Draw one ball and let $G = \{\text{the ball is green}\}$, $R = \{\text{the ball is Red}\}$ and $B = \{\text{the ball is blue}\}$. Then

$$P(G) = 4/8 = 1/2, \quad P(R) = 3/8, \quad P(G \text{ or } R) = P(G \cup R) = 7/8.$$

Examples:

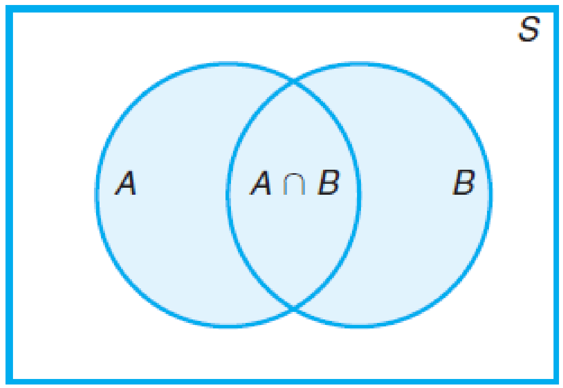
- Pick a real number between 0 and 2 with all real values in $[0, 2]$ equally likely, and let X be the randomly selected point.
  What is probability that the point falls between 0 and 0.5?

- Throw a dart on a dartboard with radius 2. Suppose you always hit the dartboard and all points on the board are hit equally likely.
  What is probability that the thrown dart does not land within the circle of radius 1 centered at origin?
  Define $W = \{$The thrown dart does not land within the circle of radius 1$\}$ and $W' = \{$The thrown dart lands within the circle of radius 1$\}$.Thus,

$$P(W) = 1 - P(W') = 1 - \frac{\pi\,(1^2)}{\pi\,(2^2)} = 1 - \frac{1}{4} = \frac{3}{4},$$

General Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,.$$



Additve rule of Probability (Ref. Book).

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &-P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+P(A \cap B \cap C) \end{aligned}$$

Example:

- An electronic assembly consists of two components, namely A and B. Experience tells us that $P\{A \text{ fails}\} = 0.2$, $P\{B \text{ fails}\} = 0.3$ and $P\{\text{both } A \text{ and } B \text{ fail}\} = 0.15$.

  Find $P\{\text{at least one of } A \text{ and } B \text{ fails}\}$ and $P\{\text{neither } A \text{ nor } B \text{ fails}\}$.
  Write simply A for "A fails" and B for "B fails". Then,

$$\begin{aligned} P\{\text{at least one fails}\} &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) = 0.35\,; \end{aligned}$$

$$P\{\text{neither A nor B fail}\} = 1 - P\{\text{at least one fails}\} = 0.65\,.$$

Independent Events

Any two events A and B are said to be independent if

$$P\left(A \cap B\right) = P(A)P(B).$$

Example:

- In the flipping a fair coin twice example, let $A = \{HH, HT\}$ and $B = \{HH, TH\}$. These events refer "Heads on the first flip", and "Heads on the second flip", respectively.

$$P\left(A\right) = P(\{HH\}) + P(\{HT\}) = \frac{1}{2}, \; P\left(B\right) = P(\{HH\}) + P(\{TH\}) = \frac{1}{2}.$$

$$P(A \cap B) = P(\{HH\}) = \frac{1}{4} \quad \longrightarrow \quad P(A \cap B) = P(A)P(B).$$

Examples:

- A six-sided die is loaded in such a way that the probability of getting each value is *proportional* to that value.
  Find the probability that 3 is shown on the top face.

  For some value k, $P(\{1\}) = k$, $P(\{2\}) = 2k$, …, $P(\{6\}) = 6k$.
  Since these add to 1,

$$1 = k + 2k + 3k + 4k + 5k + 6k = 21k\,.$$

  Hence $k = 1/21$ and $P(\{3\}) = 3k = 3/21 = 1/7$.

- Now roll the loaded die twice *independently*.
  What is probability of getting two 3's?

Example:

- Airplane engines will fail with probability $1 - p$, independently from engine to engine. If an airplane needs more than 50 percent of its engines operative to complete a successful flight, for what values of p is a 3-engine plane preferable to a 2-engine plane?

  For a 3-engine plane:

  $$\begin{aligned} P(\{\text{ either 3 or 2 engines fail}\}) &= P(\{\text{3 engines fail}\}) + P(\{\text{2 engines fail}\}) \\ &= (1-p)^3 + 3p(1-p)^2 \end{aligned}$$

  For a 2-engine plane: $P(\{\text{of the two engines both fail}\}) = (1-p)^2$

  The 3-engine plane is less likely to crash if ...

Conditional Probability

The probability of an event B occurring when it is known that some event A has occurred is called a conditional probability and is denoted

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

Note that this only makes sense when "A can happen" i.e. $P(A) > 0$.

Example:

From a group of 100 people, 1 is selected.

What is the probability that this person has high blood pressure? This is (unconditional) probability.

Now, from this group we select first all people with high cholesterol level, and then from the latter group we select 1 person.
What is the probability that this person has high blood pressure?
This is conditional probability; the probability of selecting a person with high blood pressure, given high cholesterol level.

Example:

Suppose our sample space S is the population of adults in a small town who have completed the requirements for a college degree.

	Employed	**Unemployed**	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

One of these individuals is to be selected at random for a tour.

Let M be the event that the randomly selected adult is Male and suppose E is the event that the one chosen is employed.

$$P(M|E) \quad = \quad \frac{P(M\cap E)}{P(E)} = \frac{460/900}{600/900}, \quad P(E|M) = ?$$

- The events A and B are independent if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities.
Otherwise, A and B are dependent.

- Multiplicative rule (or product rule): If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A),$$

provided $P(A) > 0$.

Example:

- Suppose that an urn contains 8 red balls and 4 green balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls are Red?

  Let $R_1 = \{\text{the first ball is Red}\}$ and $R_2 = \{\text{the second ball is Red}\}$. Then

$$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \frac{8}{12}\frac{7}{11}$$

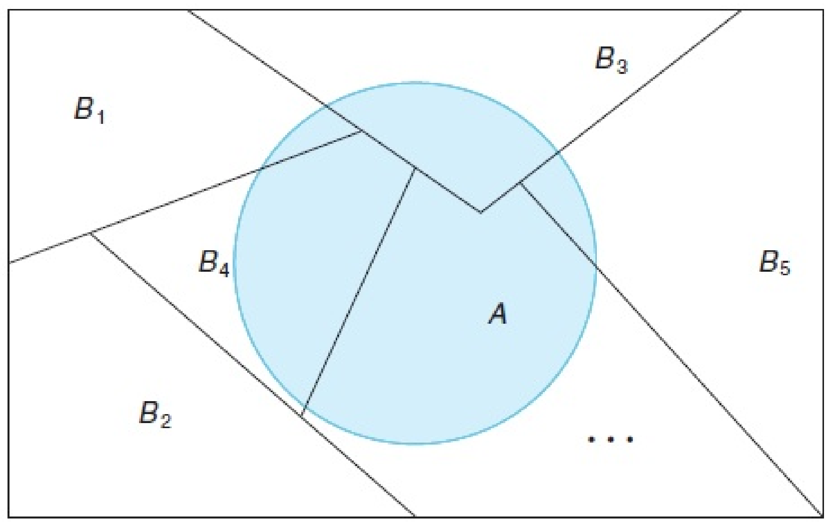
  or we can compute another way

$$P(R_1 \cap R_2) = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{\frac{8!}{2!6!}}{\frac{12!}{2!10!}} = \frac{\frac{8\times7}{2!}}{\frac{12\times11}{2!}} = \frac{8\times7}{12\times11}$$

Total Probability Rule

If $B_1, \ldots, B_n$ are mutually exclusive and exhaustive (i.e. $B_i \cap B_j = \emptyset$ if $i \neq j$ and $B_1 \cup, \ldots, \cup B_n = S$), then for any event A

$$P(A) = P(A \mid B_1)P(B_1) + \ldots + P(A \mid B_n)P(B_n)$$



Bayes Theorem

After an experiment generates an outcome, we are interested in the probability that a condition was present given an outcome.

If $B_1, \ldots, B_n$ are mutually exclusive and exhaustive (i.e. $B_i \cap B_j = \emptyset$ if $i \neq j$ and $B_1 \cup, \ldots, \cup B_n = S$), then for any event A and for each i,

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A)} = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + ... + P(A \mid B_n)P(B_n)}$$

Example: A manufacturing firm employs three analytical plans for the design and development of a particular product. Plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively.
The defect rate is different for the three procedures as follows

$$P(D|P_1) = 0.01, \quad P(D|P_2) = 0.03, \quad P(D|P_3) = 0.02,$$

where $P(D|P_j)$ is the probability of a defective product, given plan j.
If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

From the statement of the problem

$$P(P_1) = 0.30, \quad P(P_2) = 0.20, \quad P(P3) = 0.50.$$

We must find $P(P_j|D)$ for $j = 1, 2, 3$.