MAT2377

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Comments

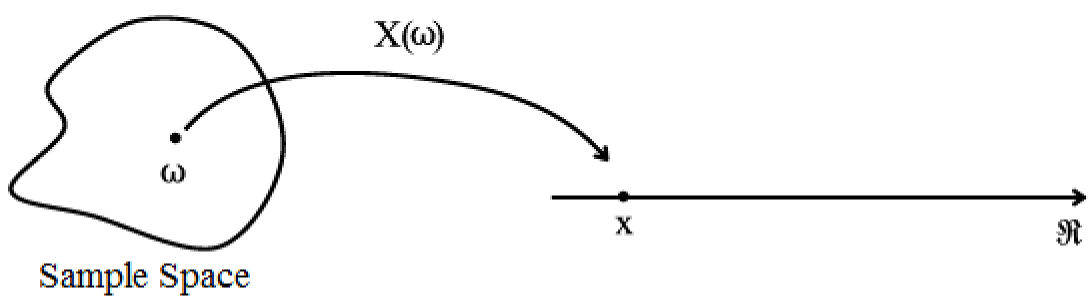
- These slides cover material from Chapter 2.
- In class, I may use a blackboard. I recommend reading these slides before you come to the class.
- I am planning to spend 2 lectures on this chapter.
- I am not re-writing the textbook. The reference book contains many interesting and practical examples.
- There may be some typos. The final version of the slides will be posted *after* the chapter is finished.

Concept of a Random Variable

It is often important to allocate a numerical description to the outcome.

- Remember in the flipping a (fair) coin twice example:
  The **discrete** sample space was $S = \{HH, HT, TH, TT\}$.
  Define, X = number of "Heads".

$$X(\{TT\}) = 0$$
$$X(\{TH\}) = 1, \quad X(\{HT\}) = 1,$$
$$X(\{HH\}) = 2.$$



A random variable is called a discrete random variable if its set of possible outcomes is countable.

When a random variable can take on values on a continuous scale, it is called a continuous random variable.

Examples:

- You flip a (fair) coin repeatedly until you observe one "Heads". Define, $X =$ number of trials until you get one "Heads". Then,
$$X(\{H\}) = 1, \quad X(\{TH\}) = 2, \quad X(\{TTH\}) = 3, \quad \ldots$$

- Interest centers around the proportion of people who vote for a specific candidate.
Let Y be that proportion.
Y is a random variable that takes on all values y for which $0 \leq y \leq 1$.

Notation for Random Variables (RVs)

- Capital letters e.g. X, Y are usually used to denote the RVs.
- Corresponding lower case letters e.g. x, y are usually used to denote *generic values taken* by RV.
- A RV is a way to define events: if X takes values $0, 1, 2, \ldots$ then we can define events $\{X = 0\}$, $\{X = 1\}$, $\{X = 2\}$,. . . etc.
- The probability (mass) function is

$$f(x) = P\left(\{X = x\}\right) = P(X = x), \quad x \in S_X,$$

where S_X, is the support of the random variable X (the set of values that the random variable can take).

Example:

- Flip a fair coin twice.
  The discrete sample space is $S = \{HH, HT, TH, TT\}$.
  Define, X = number of "Heads".
  $S_X = \{0, 1, 2\}$, is the support of the random variable X. Thus,

$$P(X = 0) = P(\{TT\}) = \frac{1}{4}$$

$$P(X = 1) = P(\{TH, HT\}) = \frac{1}{2},$$

$$P(X = 2) = P(\{HH\}) = \frac{1}{4}.$$

The probability (mass) function is

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Formal Definition of the Probability (Mass) Function

The function f is said to be a probability (mass) function for the discrete random variable X with the support S_X, if

1. For each $x \in S_X$, $f(x) \geq 0$;

2. $\sum_{x \in S_X} f(x) = 1$.

Example:

- A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives. Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school.
  Then x can only take the numbers 0, 1, and 2. Now,

$$P(X=0)=\frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}}=\frac{68}{95}, \qquad P(X=1)=\frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}}=\frac{51}{190},$$

$$P(X=2)=?$$

x	0	1	2
$P(X=x)$	$\frac{68}{95}$	$\frac{51}{190}$	?

Cumulative Distribution Function for a Discrete RV

The cumulative distribution function $F(x)$ of a discrete random variable X with probability function $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} P(X = t), \quad x \in \Re.$$

Example:

- Consider the following probability function

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Find the cumulative distribution function.

$$F(0) = P(X \leq 0) = P(X = 0) = \frac{1}{16},$$

$$F(\frac{1}{2}) = P(X \leq \frac{1}{2}) = P(X = 0) = \frac{1}{16},$$

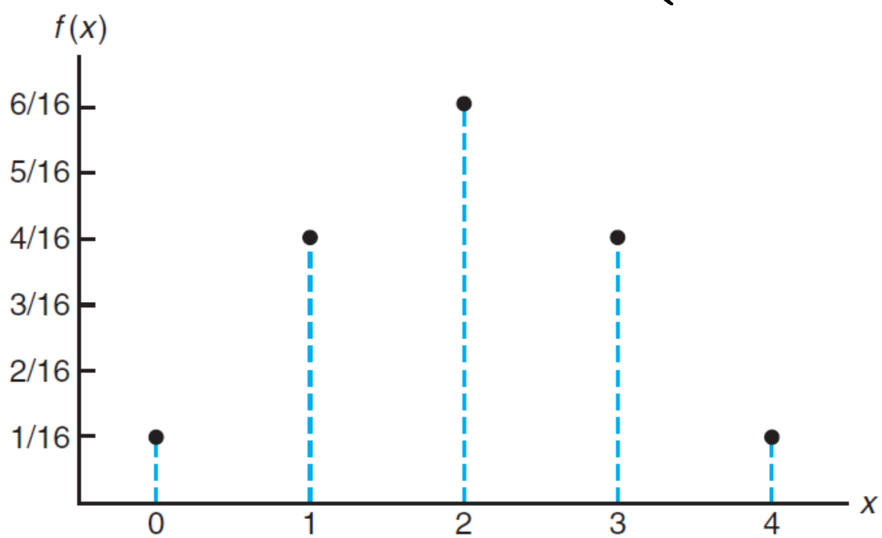
$$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{16} + \frac{1}{4} = \frac{5}{16},$$

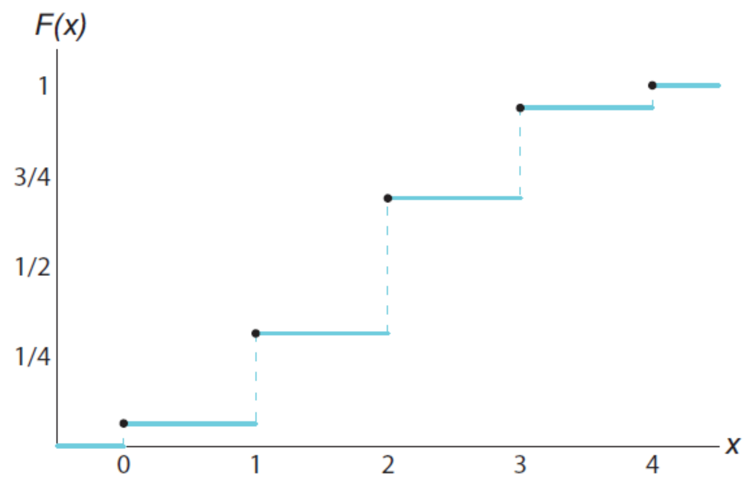
$$\vdots$$

$$F(4) = P(X \leq 4) = P(X = 0) + \ldots + P(X = 4) = 1.$$

Example (Continued): The cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16} & 0 \le x < 1 \\ \frac{1}{4} & 1 \le x < 2 \\ \frac{3}{8} & 2 \le x < 3 \\ \frac{1}{4} & 3 \le x < 4 \\ 1 & x \ge 4 \end{cases}$$





Formal Definition of the Probability Density Function

The function f is said to be a probability density function (or density function) for the continuous random variable X, if

1. For each $x \in \Re$, $f(x) \geq 0$;
2. $\int_{-\infty}^{\infty} f(x) = 1$.

$$\begin{aligned} P(a \leq X \leq b) &= P(a < X \leq b) \\ &= P(a \leq X < b) \\ &= P(a < X < b) = \int_a^b f(t)dt \end{aligned}$$

Example:

- Determine the value of c such that the following function can serve as a probability density function.

$$f(x) = \begin{cases} cx^2 & \text{for } -1 < x < 2 \\ 0 & elsewhere \end{cases}$$

Find $P(0 \leq X < 1)$.

Cumulative Distribution Function for a Continuous RV

The cumulative distribution function $F(x)$ of a continuous random variable X with probability density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, \quad x \in \Re.$$

Two interesting results:

- $P(a < X < b) = F(b) - F(a)$;
- $f(x) = \frac{\partial}{\partial x}F(x)$, if the derivative exists.

Example:

- In the former example, derive the cumulative distribution function.

Expectation

Example-Motivation:

- An electric device is regularly sold for \$1000 but now, it is on an online non-refundable sale for \$600.
  Suppose based on a valid reference, %42 of products of the company that are sold online do not function very well and customers are not satisfed with their purchase.
  Will you buy it?

Formal Definition of Expectation

Let X be a random variable with probability distribution $f(x)$.
The expected value or mean of X, or simply $\mathrm{E}(X)$ is defined as below:

- If X is discrete

$$\mu = \mathrm{E}(X) = \sum_x x f(x),$$

- If X is continuous,

$$\mu = \mathrm{E}(X) = \int_x x f(x) dx.$$

Statisticians refer to $\mathrm{E}(X)$ as the population mean of the random variable X or the mean of the probability distribution of X. The expectation is just a property of a probability distribution, but we can interpret it as a long-run average.

Example:

- Flip a fair coin twice. Define, $X =$ number of "Heads".
  Remember that

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

These probabilities are just the relative frequencies for the given events in the long run.
Therefore,

$$\mu = \mathrm{E}(X) = \sum_{x=0}^{2} x f(x) = (0)\frac{1}{4} + (1)\frac{1}{2} + (2)\frac{1}{4} = 1$$

This means that a person who tosses two fair coins over and over again will, on the average, get 1 head per two tosses.

Example:

- A lot containing 7 components is sampled by a quality inspector. The lot contains 4 good components and 3 defective components.
  A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

  Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \quad x = 0, 1, 2, 3.$$

  Thus,

$$\mu = \mathrm{E}(X) = \sum_{x=0}^{3} x f(x) = \ldots = 1.7.$$

Example:

- The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport has the density function

$$f(y) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}y} & \text{for } y > 0 \\ 0 & elsewhere \end{cases}$$

Compute $\mathrm{E}(Y)$.

Mean of a Function of a RV

Let X be a random variable with probability distribution $f(x)$.
The expected value of a function of X, say $h(X)$, is defined as below:

- If X is discrete

$$\mathrm{E}[h(X)] = \sum_x h(x)P(X = x) = \sum_x h(x)f(x),$$

- If X is continuous,

$$\mathrm{E}[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx.$$

Example:

- Toss a balanced six-sided die. If Z is the number that shows on the top face, find $\mathrm{E}(Z^2)$ and $\mathrm{E}\left[(Z-3.5)^2\right]$.

$$\begin{aligned}
\mathrm{E}\left[Z^2\right] &= \sum_{z=0}^{6} z^2 P(Z=z) = 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + \ldots + 6^2 \times \frac{1}{6} \\
&= \frac{1}{6}(1^2 + 2^2 + \cdots + 6^2) = \frac{91}{6} = 15\frac{1}{6}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[(Z-3.5)^2\right] &= \sum_{z} (z-3.5)^2 P(Z=z) \\
&= (1-3.5)^2 \times \frac{1}{6} + \ldots + (6-3.5)^2 \frac{1}{6} = \ldots = 2\frac{5}{6}.
\end{aligned}$$

Example:

- The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport has the density function

$$f(y) = \begin{cases} \frac{1}{2}e^{-\frac{1}{2}y} & \text{for } y > 0 \\ 0 & elsewhere \end{cases}$$

Compute $\mathrm{E}[Y^2]$ and $\mathrm{E}[(Y-2)^2]$.

Properties of Expectation

By definition, we can verify that

- $\mathrm{E}[cX] = c\,\mathrm{E}(X), \quad c \in \Re,$
- $\mathrm{E}[X \pm d] = \mathrm{E}(X) \pm d, \quad d \in \Re,$
- $\mathrm{E}[cX \pm d] = c\,\mathrm{E}(X) \pm d, \quad c, d \in \Re,$
- $\mathrm{E}[ch(X) \pm d] = c\,\mathrm{E}[h(X)] \pm d, \quad c, d \in \Re,$
- $\mathrm{E}[c_1 h(X) \pm c_2 g(X)] = c_1\,\mathrm{E}[h(X)] \pm c_2\,\mathrm{E}[g(X)], \quad c_1, c_2 \in \Re.$

Variance and Standard Deviation of a RV

Let X be a random variable with probability distribution $f(x)$.

- Variance of X is defined as the expected squared difference from the expectation:

$$\mathrm{Var}(X) \quad = \quad \mathrm{E}\left[(X - \mathrm{E}(X))^2\right]$$

- Standard Deviation (SD) of X is defined as

$$SD(X) = \sqrt{\mathrm{Var}(X)}\,.$$

Variance and SD allow us to compare probability distributions: those with higher variance/SD are *more spread out about the expectation*.

Examples:

- Take a look at Slide No. 21.

- Take a look at Slide No. 22.

- Let X and Y be RVs with the following probability functions:

x	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

y	-4	-1	0	1	4
$P(Y=y)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Calculate expected values.
Compare variances.

Solution: We have $\mathrm{E}(X) = \mathrm{E}(Y) = 0$ and $\mathrm{Var}(X) < \mathrm{Var}(Y)$,

Properties of Variance

By definition, we can verify that

- $\mathrm{Var}[cX] = c^2\,\mathrm{Var}(X), \quad c \in \Re,$
- $\mathrm{Var}[X \pm d] = \mathrm{Var}(X), \quad d \in \Re,$
- $\mathrm{Var}[cX \pm d] = c^2\,\mathrm{Var}(X), \quad c, d \in \Re,$
- $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}^2(X),$
- $\mathsf{SD}[cX] = |c|\mathsf{SD}(X), \quad c \in \Re.$