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Lokenath Debnath

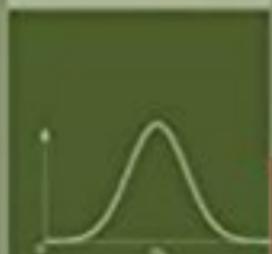


$$\nabla^2 u = 0$$

# Linear Partial Differential Equations

for Scientists  
and Engineers

Fourth Edition



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Tyn Myint-U  
Lokenath Debnath

Linear Partial  
Differential Equations  
for Scientists and Engineers

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