

**E**astern  
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**Third Edition**

**Textbook of**

**Matrix**

**Algebra**



**Suddhendu Biswas**

# Textbook of Matrix Algebra

THIRD EDITION

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**TEXTBOOK OF MATRIX ALGEBRA, Third Edition**

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# Contents

<i>Preface</i>	<i>ix</i>
<i>Notation</i>	<i>xi</i>
<b>Introduction</b>	<b>1–3</b>
<u>History of Linear Algebra and Matrix Theory</u>	<u>1</u>
<b>1. Elements of Matrix Theory</b>	<b>4–22</b>
1.1 <u>Matrix Operation</u>	<u>4</u>
1.1.1 <u>Addition of Two or More Matrices</u>	<u>4</u>
1.1.2 <u>Scalar Multiplication</u>	<u>4</u>
1.1.3 <u>Addition Law of Matrices</u>	<u>5</u>
1.1.4 <u>Matrix Multiplication</u>	<u>5</u>
1.1.5 <u>Multiplication Law Matrices</u>	<u>5</u>
1.2 <u>Row and Column Vector</u>	<u>6</u>
1.3 <u>Transpose and Conjugate Transpose of a Matrix</u>	<u>6</u>
1.4 <u>Orthogonal and Unitary Matrix</u>	<u>7</u>
1.5 <u>Partitioning of Matrices</u>	<u>7</u>
1.6 <u>Trace of a Matrix</u>	<u>10</u>
1.6.1 <u>Properties of Trace</u>	<u>11</u>
<u>More Solved Examples</u>	<u>12</u>
<b>2. Some Special Matrices</b>	<b>23–39</b>
2.1 <u>Triangular Matrix</u>	<u>23</u>
2.2 <u>Idempotent Matrix</u>	<u>26</u>
2.3 <u>Involutory Matrix</u>	<u>27</u>
2.4 <u>Nilpotent Matrix</u>	<u>27</u>
2.5 <u>Non-negative Matrices</u>	<u>27</u>
2.6 <u>Stochastic Matrix</u>	<u>28</u>
2.7 <u>Hermitian and Skew Hermitian Matrices</u>	<u>31</u>
2.8 <u>Some Special Matrices in Statistics</u>	<u>33</u>

2.8.1 Hadamard Matrix 33

2.8.2 Unitary Matrix 34

2.8.3 Leslie Matrix 35

2.8.4 Infinite Matrices 35

2.9 A Special Case for Symmetric Matrix 38

### **3. Scalar Function and Inverse of a Matrix 40–111**

3.1 Determinants 40

3.2 Properties of Determinants 40

3.3 Adjoint of a Matrix 43

3.4 Inverse of a Matrix 44

3.5 Non-singular and Singular Matrix 45

3.6 Cramer's Method of Solving a System of Linear Equations 48

3.7 Certain Useful Methods of Expanding a Determinant 49

3.7.1 Method of Pivotal Condensation 49

3.7.2 Cauchy's Method of Bordered Expansion 52

3.7.3 Laplace Method of Expansion 55

*More Solved Examples 62*

### **4. Certain Basic Algebraic Concepts 112–129**

4.1 Sets 112

4.2 Groups 112

4.2.1 Abelian Group 113

4.2.2 Some Characteristics of Groups 113

4.3 Rings 114

4.3.1 Types of Rings 116

4.4 Field 116

4.5 Vector 117

4.6 Vector Space 118

4.7 Linear Independence 118

4.8 Certain Elementary Propositions 118

4.9 Linear Manifold (Subspace of a given vector space) 119

4.10 Gram-Schmidt's Orthogonalisation Process 120

4.11 Orthogonal and Orthonormal Basis 121

4.12 Orthogonal Projection of a Vector 123

4.13 Orthogonal Transformation 125

4.13.1 Properties of Orthogonal Transformation 125

4.14 Projective Transformation 126

4.15 Statistical Application 127

4.15.1 Application of Projections in Least Square Theory 127

4.15.2 Derivation of Normal Equations 129

<b>5. Rank of a Matrix</b>	<b>130–162</b>
5.1 Rank of a Matrix	130
5.1.1 Definition of Rank	130
5.1.2 Nullity of a Matrix	131
5.1.3 Rank of a Symmetrical Matrix	131
5.2 Elementary Transformation of a Matrix	132
5.3 Equivalent Matrices	132
5.4 Elementary Matrices	132
5.4.1 Illustrations of Elementary Transformations	133
5.4.2 A Result	135
5.5 Echelon Matrix	137
5.6 Hermite Canonical Form	138
5.7 Sylvester's Law	143
5.8 Frobenius Inequality	144
5.9 Certain Results on the Rank of an Idempotent Matrix	145
5.9.1 Some Results on Idempotent Matrix	145
<i>More Solved Examples</i>	147
<b>6. Theory of Linear Equations</b>	<b>163–185</b>
<i>More Solved Examples</i>	170
<b>7. Eigenvalues and Eigenvectors</b>	<b>186–230</b>
7.1 Characteristic Roots and Vectors	186
7.2 Spectral Decomposition of a Symmetric Matrix	198
7.3 Spectral Decomposition of a Symmetric Matrix When Rank of Matrix $<$ Its Order (Special Case) (Rank is Not Full)	202
7.4 An Example of Spectral Decomposition	205
7.5 Cayley-Hamilton Theorem	207
7.6 Geometric and Algebraic Multiplicity of Characteristic Roots	209
7.7 Spectral Decomposition of Asymmetric Matrix	209
7.8 Similarity of Two Matrices: Jordan Canonical Form	211
7.9 A Result	211
7.10 A More Generalised Result	211
7.11 Factorization of a Matrix	211
7.12 Modification of the Process When $A = RQ$	213
7.13 Eigenvalues and Eigenvectors for Solution of Differential Equations	214
<i>More Solved Examples</i>	216
<b>8. Generalised Inverse of a Matrix</b>	<b>231–263</b>
8.0 Introduction	231
8.1 Different Classes of Generalised Inverse	234
8.2 Properties of $g$ -Inverse ( $A^{\#}$ )	244
8.3 Properties of Reflexive $g$ -Inverse ( $A^r$ )	244

- 8.4 [Properties of Left Weak  \$g\$ -Inverse \( \$A^{\#}\$ \)](#) 244
- 8.5 [Properties of Right Weak  \$g\$ -Inverse \( \$A^{\#}\$ \)](#) 245
- 8.6 [Properties of Moore and Penrose \(MP\) \(or Pseudo Inverse\)  \$g\$ -Inverse \( \$A^{\dagger}\$ \)](#) 245
  - 8.6.1 [Weighted Pseudo Inverse](#) 246
- 8.7 [Applications of Generalised Inverses in The Solution of System of Linear Equations](#) 248
- 8.8 [Solution of Linear Equations Least Squares Properties of Moore-Penrose \(MP\) Generalised Inverse](#) 251
  - 8.8.1 [Notation and Definition](#) 251
- 8.9 [Applications of MP Inverse For the Solution of Optimization Problems](#) 252
- 8.10 [Some Further Results on the Applications of  \$g\$ -Inverse on Optimization Techniques](#) 260

## 9. Quadratic Forms and Inequalities 264–309

- 9.1 [Classification of the Quadratic Forms](#) 264
- 9.2 [Positive Semi-Definite Quadratic Form](#) 265
- 9.3 [Gram Matrix](#) 269
- 9.4 [Quadratic Form into Sum of Squares](#) 272
- 9.5 [Lagrange's Method for Transforming a Quadratic Form into Terms of the Form  \$\alpha\_1 y\_1^2 + \alpha\_2 y\_2^2 + \dots + \alpha\_n y\_n^2\$](#)  272
- 9.6 [Cocharan's Theorem](#) 276
  - 9.6.1 [Statistical Translation of Cochran's Theorem](#) 279
- 9.7 [Another Related Result](#) 279
  - 9.7.1 [Statistical Interpretation of the Result in Section 9.7](#) 280
- 9.8 [Inequalities](#) 280
  - 9.8.1 [Cauchy-Schwarz \(CS\) Inequality](#) 280
  - 9.8.2 [CS Inequality in Integral Form](#) 281
  - 9.8.3 [The Lagrange Identity](#) 282
  - 9.8.4 [Hölder's Inequality](#) 282
  - 9.8.5 [Hölder's Inequality in Integral Form](#) 285
  - 9.8.6 [Minkowski's Inequality](#) 285
  - 9.8.7 [Hadamard's Inequality](#) 286
  - 9.8.8 [An Inequality Involving Moments \(Statistical\)](#) 290
  - 9.8.9 [Convex Functions and Jensen's Inequality \(Statistical\)](#) 292
  - 9.8.10 [Problem of Constrained Minimization](#) 292
  - 9.8.11 [Some Results on Inequalities](#) 293

*More Solved Examples* 303

## 10. Some Applications of Algebra of Matrices 310–337 (Theory of Markov Chains and Linear Programming Techniques)

- 10.1 [Introduction](#) 310
- 10.2 [Markov Chain](#) 310
  - 10.2.1 [Fundamental Matrix of a Markov Chain](#) 321

<a href="#">10.3</a>	<a href="#">Linear Programming (LP)</a>	<a href="#">323</a>
<a href="#">10.4</a>	<a href="#">Basic Concepts in LP Problems</a>	<a href="#">325</a>
<a href="#">10.5</a>	<a href="#">Simplex Method for Solving a Linear Programming Problem</a>	<a href="#">327</a>
<a href="#">10.6</a>	<a href="#">Mathematical Justification in the Steps of Simplex Method</a>	<a href="#">330</a>

## **11. Matrices in the Infinite Dimensional Vector Space** **338–359**

<a href="#">11.1</a>	<a href="#">A Matrix Conceived as a Linear Transformation</a>	<a href="#">338</a>
	<a href="#">11.1.1 Linear Transformations</a>	<a href="#">338</a>
<a href="#">11.2</a>	<a href="#">Concept of Banach Space</a>	<a href="#">339</a>
<a href="#">11.3</a>	<a href="#">Bounded Map</a>	<a href="#">339</a>
	<a href="#">11.3.1 Banach Space</a>	<a href="#">340</a>
	<a href="#">11.3.2 Some Examples of Banach Space</a>	<a href="#">340</a>
<a href="#">11.4</a>	<a href="#">Hilbert Space</a>	<a href="#">341</a>
	<a href="#">11.4.1 Example of Hilbert Space</a>	<a href="#">342</a>
	<a href="#">11.4.2 Some Important Properties of Hilbert Space</a>	<a href="#">343</a>
	<a href="#">11.4.3 Another Useful Result of Hilbert Space in Optimization Theory</a>	<a href="#">348</a>
<a href="#">11.5</a>	<a href="#">Finite Dimensional Spectral Analysis</a>	<a href="#">349</a>
	<a href="#">11.5.1 A Few Result</a>	<a href="#">350</a>
	<a href="#">11.5.2 An Analogous Result to the Reality of the Characteristic Root of Hermitian Matrix Over a Complex Field in Hilbert Space</a>	<a href="#">350</a>
	<a href="#">11.5.3 An Analogous Result of Orthogonality of Characteristic Vectors for a Hermitian Matrix in Hilbert Space</a>	<a href="#">351</a>
	<a href="#">11.5.4 An Analogous Result of the Characteristic Root and Vectors for an Orthogonal Matrix in Hilbert Space</a>	<a href="#">351</a>
	<a href="#">11.5.5 An Analogous Result to Orthogonality of Characteristic Vectors for an Orthogonal (Unitary) Matrix Extended in Hilbert Space</a>	<a href="#">352</a>
	<a href="#">11.5.6 Some Results on Projection in the Hilbert Spaces</a>	<a href="#">353</a>
	<a href="#">11.5.7 An Analogous Result of Projection Matrix to be Symmetric (Hermitian)</a>	<a href="#">354</a>
	<a href="#">11.5.8 A Note on the Generalisation of Spectral Resolution on the Infinite Dimensional Hilbert Space</a>	<a href="#">358</a>

## **12. Computational Tracts in Matrices** **360–387**

### **Appendix** **389–424**

### **Suggested Further Reading** **425–426**

### **Question Bank** **427–439**

### **Index** **441–444**