پاسخ تمرین سوم

سول یک:

$$f(z) = \sinh z \quad ; \quad z = \pi i \qquad f'(z) = \cosh z \ , \ f''(z) = \sinh z$$

$$f(z) = f(\pi i) + f'(\pi i)(z-\pi i) + \frac{f''(\pi i)}{2!}(z-\pi i)^2 + \cdots$$

$$= \sinh \pi i + \cosh \pi i \,(z-\pi i) + \frac{\sinh(\pi i)}{2!}(z-\pi i)^2 + \cdots$$

$$= i\sin\pi + \cos\pi\,(z-\pi i) + \frac{i\sin\pi}{2!}(z-\pi i)^2 + \cdots$$

$$= -(z-\pi i) - \frac{1}{3!}(z-\pi i)^3 - \frac{1}{5!}(z-\pi i)^5 - \cdots$$

$$f(z) = \sin z^2 \qquad z = 0$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \Rightarrow \sin z^2 = z^2 - \frac{z}{3!} + \frac{z^{10}}{5!} - \cdots$$

سوال دو:

b) $f(z) = z \cot z$

$$g(z) = \cot z = \frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}$$

$$\begin{array}{l|l}
1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots & z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
\hline
-\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!}\right) \cdots & \frac{1}{z} + \left(\frac{1}{3!} - \frac{1}{2!}\right) z + \alpha z^3 + \cdots \\
\hline
\left(\frac{1}{3!} - \frac{1}{2!}\right) z^2 + \left(\frac{1}{4!} - \frac{1}{5!}\right) z^4 + \cdots & \\
-\left[\left(\frac{1}{3!} - \frac{1}{2!}\right) z^2 - \frac{1}{3!}\left(\frac{1}{3!} - \frac{1}{2!}\right) z^4 + \cdots\right] & \\
\hline
\left[\left(\frac{1}{4!} - \frac{1}{5!}\right) + \frac{1}{3!}\left(\frac{1}{3!} - \frac{1}{2!}\right)\right] z^4 + \cdots &
\end{array}$$

$$\cot z = \frac{1}{z} + \left(\frac{1}{3!} - \frac{1}{2!}\right) z + \alpha z^3 + \cdots$$

$$z \cdot \cot z = 1 + \left(\frac{1}{3!} - \frac{1}{2!}\right) z^2 + \alpha z^4 \cdots$$

سوال سه :

$$f(z) = z \sin z^2$$

الان مثال ۱۵ سوری ؛

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + - \cdots \qquad , \quad z = 0 + 1z + 0z^2 + 0z^3 + \cdots$$

طبق قضیه ضرب سری ها

$$f(z) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \cdots$$

$$\frac{e^z}{(z-1)^2} \; ; \; z_0 = 1$$

ابتدا سری تیلور e^z را حول $z_0=1$ می‌نویسیم.

$$e^z = 1 + (z-1) + \frac{(z-1)^2}{2!} + \frac{(z-1)^3}{3!} + \cdots$$

$$f(z) = (z-1)^{-2} e^z = (z-1)^{-2} + (z-1)^{-1} + \frac{1}{2!}(z-1)^0 + \frac{(z-1)}{3!} + \cdots$$

د) $\frac{1}{z^2+1} \quad ; \quad z_0 = i$

$$\frac{1}{(z-i)(z+i)} = \frac{\frac{1}{z+i}}{z-i} = \frac{P(z)}{z-i} \quad , \quad P(z) = \frac{1}{z+i} \quad , \quad P^{(n)}(z) = \frac{(-1)^n n!}{(z+i)^{n+1}}$$

$$\frac{P^{(n)}(i)}{n!} = \frac{(-1)^n}{(2i)^{n+1}} \Rightarrow P(z) = \frac{1}{2i} + \frac{(z-i)}{2^2 \times 1} + \frac{(z-i)^2}{2^3 i^3} + \frac{(z-i)^3}{2^4 \times 1} + \cdots$$

$$\Rightarrow P(z) = -\frac{i}{2} + \frac{(z-i)}{4} + \frac{i}{8}(z-i)^2 + \frac{1}{16}(z-i)^4 + \cdots$$

$$\frac{1}{z^2+1} = \frac{P(z)}{z-i} = -\frac{i}{2}(z-i)^{-1} + \frac{1}{4} + \frac{i}{8}(z-i) + \frac{1}{4}(z-i)^2 + \cdots$$

c) $\dfrac{z+i+1}{(z+i)^2}$; $z_0 = -i$ $\qquad P(z) = z+1+i$ چون P(z) چندجمله‌ای است

پس سری تیلورش با خودش یکی است.

$$P(z) = 1 + (z+i) + 0\,(z+i)^2 + \dots$$

$$(z+i)^{-2} P(z) = (z+i)^{-2} + (z+i)^{-1} + 0 + 0 + 0 + \dots$$

.) $\dfrac{\sin z}{\left(z-\frac{\pi}{4}\right)^3}$; $z_0 = \frac{\pi}{4}$

$$P(z) = \sin z = \left(z-\frac{\pi}{4}\right) - \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} + \frac{\left(z-\frac{\pi}{4}\right)^5}{5!} \dots$$

$$f(z) = \left(z-\frac{\pi}{4}\right)^{-3} P(z) = \left(z-\frac{\pi}{4}\right)^{-2} - \frac{1}{3!} + \frac{1}{5!}\left(z-\frac{\pi}{4}\right)^2 + \dots$$

سوال چهار:

[illegible]

$$\oint_{|z|=1} z\cot z \, dz = 0$$

چون [illegible] سوال حل شده [illegible] $z\cot z$ [illegible] است.

c) $\displaystyle\oint \frac{\sin\pi z}{z^2}\, dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i\, C_{-1}$

$$\sin\pi z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \qquad \frac{\sin\pi z}{z^2} = z^{-3} - \frac{z^{-1}}{3!} + \frac{z}{5!} - \dots \Rightarrow C_{-1} = -\frac{1}{6}$$

$$\Rightarrow \oint_{|z|=1} f(z) = 2\pi i \cdot \frac{-1}{6} = \frac{-\pi i}{3} \;;$$

f) $$\oint_{|z|=1} \frac{4z^2-4z+1}{(z-2)(1+4z^2)}\, dz$$

$$1+4z^2=0 \Rightarrow z^2 = i^2\frac{1}{4} \quad z=\pm\frac{i}{2} \;;$$

$$\oint f(z)\,dz = 2\pi i\left[\operatorname*{Res}_{z=-\frac{i}{2}} f(z) + \operatorname*{Res}_{z=\frac{i}{2}} f(z)\right]$$

$$\operatorname*{Res}_{z=\frac{i}{2}} f(z) = \lim_{z\to\frac{i}{2}} \left(z-\frac{i}{2}\right)\frac{4z^2-4z-1}{4(z-2)(z-\frac{i}{2})(z+\frac{i}{2})} = \frac{-\frac{2}{2}-2i+1}{4i(-2+\frac{i}{2})} = \alpha$$

$$\operatorname*{Res}_{z=-\frac{i}{2}} f(z) = \lim \left(z+\frac{i}{2}\right)\frac{4z^2-4z-1}{4(z-2)(z-\frac{i}{2})(z+\frac{i}{2})} = \frac{1+2i-\frac{2}{2}}{4i(2+\frac{i}{2})} = \beta$$

$$\oint_{|z|=1} f(z)\,dz = 2\pi i(\alpha+\beta)$$

h) $$\oint_{|z|=1} \frac{e^{(z-i)\pi/6}}{\sin z}\,dz = \oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \lim_{z\to 0}\frac{e^{(z-i)\pi/6}}{\cos z}$$
$$= 2\pi i\cdot\frac{e^{-i\pi/6}}{1} = -2\pi i$$

* اگر $f(z)=\frac{p(z)}{q(z)}$ ، $p(z_0)=0$ ، $q(z_0)=0$ آنگاه:

$$\operatorname{Res} f(z) = \lim (z-z_0)\, f(z) = \lim_{z\to z_0}\frac{(z-z_0)p(z)}{q(z)} = \lim_{z\to z_0}\frac{p(z)}{\frac{q(z)-0}{z-z_0}}$$
$$= \lim_{z\to z_0}\frac{p(z)}{\frac{q(z)-q(z_0)}{z-z_0}} = \frac{p(z_0)}{q'(z_0)}$$

$$\oint_{|z|=1} \frac{e^{z^2}\cosh z}{\cos z/2}\, dz = 0$$

---

m) $\displaystyle\int_{|z|=2} \frac{-z^2-22z+8}{z^3-5z^2+4z}$ [illegible] $= z(z-1)(z-4)=0$

$$I = \oint_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right]$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{-z^2-22z+8}{3z^2-10z+4} = \frac{8}{4} = 2$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{-z^2-22z+8}{3z^2-10z+4} = \frac{-1-22+8}{-3} = 5 \qquad I = 14\pi i$$