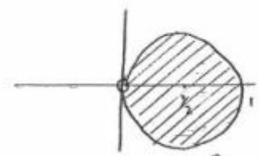
پاسخ کوییز دوم

$$3)\ Re\left(\frac{1}{z}\right) \geqslant 1 \longrightarrow Re\left(\frac{1}{x+yi}\right) \geqslant 1 \xrightarrow{z \neq 0}$$



$$\longrightarrow Re\left(\frac{1}{x+yi} \times \frac{x-yi}{x-yi}\right) \geqslant 1$$

$$Re\left(\frac{x-yi}{x^2+y^2}\right) \geqslant 1 \longrightarrow \frac{x}{x^2+y^2} \geqslant 1 \longrightarrow x^2+y^2-x \leqslant 0 \longrightarrow \left(x-\frac{1}{2}\right)^2+y^2 \leqslant \frac{1}{4}$$

۴ سؤال) $|\cos z|$ را محاسبه کنید.

$$\cos z = \cos(x+yi) = \cos x \cos(yi) - \sin x \sin(yi)$$

$$= \cos x \cosh y - i \sin x \sinh y \longrightarrow |\cos z| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

$$= \sqrt{\cos^2 x \cosh^2 y + (1-\cos^2 x)\sinh^2 y} = \sqrt{\cos^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y} = \sqrt{\cos^2 x + \sinh^2 y}$$

مثال) نشان دهید $U = 2^x \cos(y\ln 2)$ همساز است پس مزدوج همساز و تابع تحلیلی $f(z)$ را بیابید.

$$\begin{cases} U_x = 2^x \ln 2 \cos(y\ln 2) \longrightarrow U_{xx} = 2^x \ln^2 2 \cos(y\ln 2) \\ U_y = (-\ln 2)\, 2^x \sin(y\ln 2) \longrightarrow U_{yy} = -2^x \ln^2 2 \cos(y\ln 2) \end{cases} \longrightarrow U_{xx} + U_{yy} = 0$$

U همساز است

$$C\text{-}R\ (1) \longrightarrow U_x = V_y \longrightarrow 2^x \ln 2 \cos(y\ln 2) = V_y \xrightarrow{\int dy} 2^x \sin(y\ln 2) + g(x) = V$$

$$C\text{-}R\ (2) \longrightarrow -U_y = V_x \longrightarrow \ln 2\, 2^x \sin(y\ln 2) = 2^x \ln 2 \sin(y\ln 2) + g'(x)$$

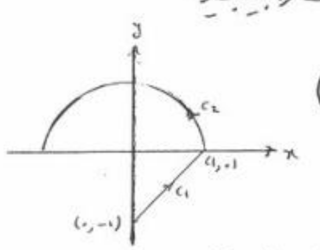
$$\longrightarrow g'(x) = 0 \xrightarrow{\int dx} g(x) = C$$

پس $V = 2^x \sin(y\ln 2) + C$ مزدوج همساز $\xrightarrow{\text{برای راحتی } C=0}$ $f(z) = U + Vi$

$$f(z) = 2^x \cos(y\ln 2) + i\left(2^x \sin(y\ln 2)\right) = 2^x\left(\cos(y\ln 2) + i\sin(y\ln 2)\right)$$

$$= 2^x e^{(y\ln 2)i} = 2^x e^{\ln 2^{yi}} = 2^x\, 2^{yi} = 2^{x+yi} = 2^z$$ تابع تحلیلی

مثال) حاصل انتگرال $\int_C \mathrm{Re}(i\bar{z})\,dz$ را که C شکل زیر است را بیابید.



$C = C_1 + C_2$

نکته) معادله خط واصل بین دو نقطه (x_1, y_1) و (x_2, y_2) عبارت است از:

$$\begin{cases} m = \dfrac{y_2 - y_1}{x_2 - x_1} \\ y_2 - y_1 = m(x_2 - x_1) \end{cases}$$

$$c_1 \to m = 1 \to y - 0 = 1(x-1) \to \underset{\text{معادله خط } c_1}{y = x - 1} \to \begin{cases} x = t \\ y = t - 1 \end{cases}$$

$$z = t + (t-1)i \to dz = (1+i)\,dt \qquad 0 \leq t \leq 1$$

$$\int_{c_1} Re(i\bar{z})\,dz = \int_0^1 Re\left[i(t-(t-1)i)\right](1+i)\,dt = (1+i)\int_0^1 (t-1)\,dt$$

$$= (1+i)\left(\frac{t^2}{2} - t\right)\Big|_0^1 = -\frac{1+i}{2}$$

$$c_2 \to \begin{cases} x = \cos t \\ y = \sin t \end{cases} \to z(t) = \cos t + i\sin t \to dz = (-\sin t + i\cos t)\,dt$$

$$0 \leq t \leq \pi$$

$$\int_0^\pi Re\left[i(\cos t - i\sin t)\right]\left[-\sin t + i\cos t\right] dt = \int_0^\pi (\sin t)(-\sin t + i\cos t)\,dt$$

$$= -\int_0^\pi \sin^2 t\,dt + i\int_0^\pi \sin t\cos t\,dt = -\int_0^\pi \frac{1-\cos 2t}{2}\,dt + i\int_0^\pi \sin t\cos t\,dt$$

$$= -\frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^\pi + i\left(\frac{\sin^2 t}{2}\right)\Big|_0^\pi = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$$

$$= \int_c Re(i\bar{z})\,dz = \int_{c_1} Re(i\bar{z})\,dz + \int_{c_2} Re(i\bar{z})\,dz = -\frac{1+i}{2} + \left(-\frac{\pi}{2}\right) = -\left(\frac{1}{2} + \frac{\pi}{2}\right) - \frac{i}{2}$$