Partial Differential Equations (PDEs)

معادلات دیفرانسیل جزئی

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۲- معادلات دیفرانسیل جزئی
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یادآوری معادلات دیفرانسیل معمولی

- معادلات با ضرایب ثابت
- معادلات هم بعد
- معادلات کامل
- کاهش مرتبه
- تبدیل به معادله شناخته شده
- روش سری توان

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معادلات با ضرایب ثابت

$Ly = 0 \qquad y = e^{rx}$ جواب

$y'' - 5y' + 4y = 0 \quad \Rightarrow e^{rx}(r^2 - 5r + 4) = 0$

$(r-1)(r-4) = 0$

ریشه ها $r = 1, r = 4$

حل عمومی $\Rightarrow y = c_1 e^{x} + c_2 e^{4x}$

ریشه های تکراری $y = e^{rx}, xe^{rx}, x^2 e^{rx}, \ldots$

$y'' - 4y' + 4y = 0 \qquad e^{rx}(r^2 - 4r + 4) = 0$

$(r-2)^2 = 0 \qquad r = 2$

$\Rightarrow y = c_1 e^{2x} + c_2 x e^{2x}$

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معادلات هم بعد(اولر)

$$x^2 y'' - axy' + by = 0$$ با تبدیل x به ax تغییر در معادله ایجاد نمی شود

$x = e^t$ تغییر متغیر $\Rightarrow$ حالت قبل

جواب $y = x^r$

مثال

$$4y'' + y/4x^2 = 0$$

$$4r(r-1)x^{r-2} + \frac{x^r}{4x^2} = 0 \qquad 4r(r-1) + \frac{1}{4} = 0$$

$$r = 1 \pm \sqrt{3/4}$$

$$\Rightarrow y = c_1 x^{1+\sqrt{3/4}} + c_2 x^{1-\sqrt{3/4}}$$

ریشه های تکراری $y = x^{r_1}, x^{r_1}\ln x, x^{r_1}(\ln x)^2, \ldots$

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معادلات کامل

دیفرانسیل کامل

$$L_1 y = \frac{d}{dx}(L_2 y) = 0$$

معادله ساده تر ولی غیر همگن

$$\Rightarrow L_2 y = c$$

مثال

$$y'' + xy' + y = 0$$

$$\frac{d}{dx}(y' + xy) = 0 \qquad y' + xy = c$$

فاکتور انتگرال در معادله ضرب شود معادله کامل بدست می آید

$$\mu(x) = e^{x^2/2}$$

$$e^{x^2/2}y' + xe^{x^2/2}y = ce^{x^2/2} \qquad \frac{d}{dx}\left(e^{x^2/2}y\right) = ce^{x^2/2}$$

$$e^{x^2/2}y = c\int_0^x e^{x^2/2}dx + c_1 \qquad y = \left(c\int_0^x e^{x^2/2}dx + c_1\right)e^{-x^2/2}$$

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Differential Equations

Ordinary Differential Equations	Partial Differential Equations
$F(x,y,y',y'',y''',\ldots)=f(x)$	$F(x,y,z,\ldots,u,u',u'',u''',\ldots)=f(x,y,z,\ldots)$

Three major categories of DE

- Initial-value problems – involve time-dependent equations with given initial conditions:
  $\left.\frac{\partial u}{\partial t}\right|_{t=0} = v_0 \qquad u|_{t=0} = u_0$
- Boundary-value problems – involve differential equations with specified boundary conditions:
  $u|_{t=T} = u_0$
- Eigenvalue problems – involve solutions for selected parameters in the equations

  Buckling and stability of structures, natural frequency in vibration, resonance in acoustics

  In reality, a problem may have more then just one of the categories above

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First-Order PDEs

- **First-order linear wave equation (advection eq.)**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

- Propagation of wave with speed c
- Advection of passive scalar with speed c
- **First-order nonlinear wave equation (inviscid Burgers's equation)**

gas dynamics and traffic flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

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Second-Order PDEs

- Advection-diffusion equation (linear)

$$\frac{\partial T}{\partial t} + c\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}$$

- Burger's equation (nonlinear)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial x^2}$$

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Other Common PDEs

- Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Nonlinear dispersive wave

waves on shallow water surfaces

- Laplace and Poisson's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad \begin{cases} f = 0: \text{Laplace} \\ f \neq 0: \text{Poisson} \end{cases}$$

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Other Common PDEs

- Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

Time-dependent harmonic waves

Propagation of acoustic waves

- Tricomi equation

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{cases} y > 0 \ : \ \text{elliptic} \\ y < 0 \ : \ \text{hyperbolic} \end{cases}$$

Mixed-type

transonic flow

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Other Common PDEs

- Wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- Fourier equation (Heat equation)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

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Navier-Stokes Equations

- Navier-Stokes equation
- Vorticity / stream function formulation

$$\begin{cases} \nabla^2 \psi = -\omega \\ \dfrac{\partial \omega}{\partial t} + u\dfrac{\partial \omega}{\partial x} + v\dfrac{\partial \omega}{\partial y} = \nu\left(\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}\right) \end{cases}$$

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Navier-Stokes Equations

- Navier-Stokes equation
- Primitive variables

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

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RANS Equations: Turbulent Flows

- Reynolds-Averaged Navier-Stokes equation

$$\begin{cases} \dfrac{\partial U}{\partial x}+\dfrac{\partial V}{\partial y}=0 \\ \dfrac{\partial U}{\partial t}+U\dfrac{\partial U}{\partial x}+V\dfrac{\partial U}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial P}{\partial x}+\nu\left(\dfrac{\partial^2 U}{\partial x^2}+\dfrac{\partial^2 U}{\partial y^2}\right)-\dfrac{\partial \overline{uu}}{\partial x}-\dfrac{\partial \overline{uv}}{\partial y} \\ \dfrac{\partial V}{\partial t}+U\dfrac{\partial V}{\partial x}+V\dfrac{\partial V}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial P}{\partial y}+\nu\left(\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}\right)-\dfrac{\partial \overline{uv}}{\partial x}-\dfrac{\partial \overline{vv}}{\partial y} \end{cases}$$

$$\begin{cases} \dfrac{\partial k}{\partial t}+U\dfrac{\partial k}{\partial x}+V\dfrac{\partial k}{\partial y}=(\nu+\nu_t)\left(\dfrac{\partial^2 k}{\partial x^2}+\dfrac{\partial^2 k}{\partial y^2}\right)+G-\varepsilon \\ \dfrac{\partial \varepsilon}{\partial t}+U\dfrac{\partial \varepsilon}{\partial x}+V\dfrac{\partial \varepsilon}{\partial y}=(\nu+\nu_t)\left(\dfrac{\partial^2 \varepsilon}{\partial x^2}+\dfrac{\partial^2 \varepsilon}{\partial y^2}\right)+\dfrac{\varepsilon}{k}(C_{\varepsilon 1}G-C_{\varepsilon 2}\varepsilon) \end{cases}$$

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Classification of PDEs

- Linear second-order PDE in two independent variables (x,y), (x,t), etc.

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu+G=0$$

- A, B, C, …, G are constant coefficients (may be generalized)
- The equation types are coordinate invariant, i.e., coordinate transformation will not change the type of equations
- Physical processes are independent of coordinates

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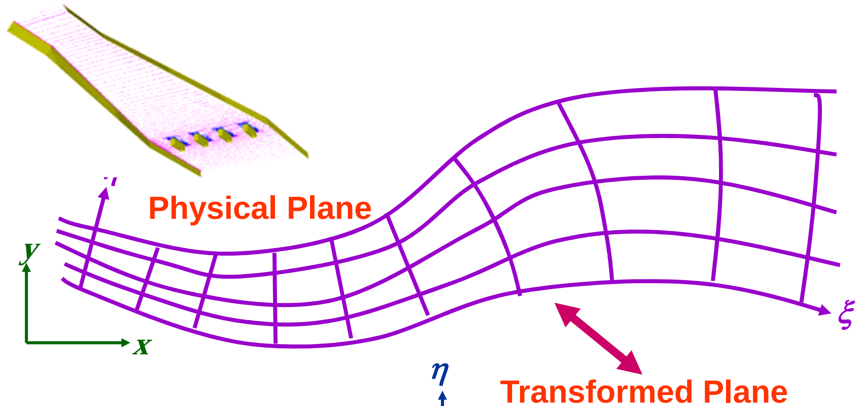
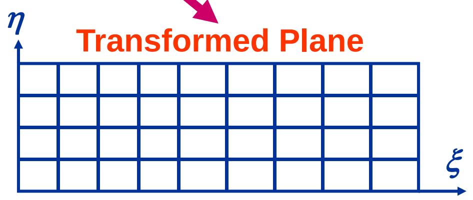
Coordinate Transformation

- Physical plane ⇔ Transformed





$$\begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases} \Leftrightarrow \begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$$

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Classification of PDEs

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Classification $\begin{cases} B^2 - 4AC < 0 : \text{elliptic} \\ B^2 - 4AC = 0 : \text{parabolic} \\ B^2 - 4AC > 0 : \text{hyperbolic} \end{cases}$

- The classification depends only on the highest-order derivatives (independent of D, E, F, G)
- For nonlinear problems [A,B,C = f(x,y,u)], the discriminant can still be used.

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Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(1) Hyperbolic PDEs (Propagation)

Advection equation

Wave equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & \text{(first - order)} \\ \dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0 & \text{(second - order)} \end{cases}$$

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$$B^2 - 4AC = 4c^2 > 0 : \text{hyperbolic}$$

Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(2) Parabolic PDEs (Time- or space-marching)

Burger's equation

Fourier equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} = \nu\dfrac{\partial^2 \phi}{\partial x^2} \\ \dfrac{\partial \phi}{\partial t} = \alpha\dfrac{\partial^2 \phi}{\partial x^2} \end{cases}$$

Diffusion / dispersion

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$$B^2 - 4AC = 0 : \text{pararabolic}$$

Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(3) Elliptic PDEs (Diffusion, equilibrium problems)

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0 & \text{Laplace equation} \\ \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = f(x,y) & \text{Possion's equation} \\ \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + c^2\phi = 0 & \text{Helmholtz equation} \end{cases}$$

$$B^2 - 4AC = -4 < 0 : \text{elliptic}$$

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Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(4) Mixed-type PDEs

Steady, compressible potential flow

$$(1 - M^2)\frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial n^2} = 0 \qquad \begin{cases} M < 1: & \text{subsonic} \\ M > 1: & \text{supersonic} \end{cases}$$

$$-4(1 - M^2) = -4 + 4M^2 = 0 \Rightarrow M = 1 \qquad \begin{cases} M < 1: \text{elliptic} \\ M = 1: \text{parabolic} \\ M > 1: \text{hyperbolic} \end{cases}$$

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Classification of PDEs

- **General form of second-order PDEs (2 variables)**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

(5) System of Coupled PDEs

Navier-Stokes Equations

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

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