Partial Differential Equations (PDEs)

معادلات دیفرانسیل جزئی

Ghasemzadeh

فهرست عناوین و فصول

معادلات نیمه خطی مرتبه اول

- First-order PDE in z(x,y)

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z)$$

solution $u(x,y,z) = c$

$\Rightarrow Pu_x + Qu_y + Ru_z = 0$ or $\vec{v}.\nabla\vec{u} = 0$

$\vec{v} = P\hat{i} + Q\hat{j} + R\hat{k}$

بردار v بر گرادیان u عمود است یعنی مماس بر u است حرکت در جهت v منحنی مشخصه معادله را می دهد که برای آن داریم

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \begin{cases} u_1(x,y,z) = c_1 \\ u_2(x,y,z) = c_2 \end{cases}$$

از تلاقی دو سطح جواب بدست می آید

3 $f(u_1, u_2) = 0$ or $u_2(x,y,z) = f(u_1(x,y,z))$

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مثال

$$xz_x + yz_y = z$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \Rightarrow \begin{cases} \frac{dx}{x} = \frac{dy}{y} \\ \frac{dx}{x} = \frac{dz}{z} \end{cases} \Rightarrow \begin{cases} \ln x = \ln y + \ln c_1 \\ \ln x = \ln z + \ln c_2 \end{cases}$$

$$\Rightarrow \begin{cases} x = c_1 y & c_1 = x/y \\ x = c_2 z & c_2 = x/z \end{cases}$$

از تلاقی دو سطح جواب بدست می آید

$$c_2 = f(c_1) \Rightarrow x/z = f(x/y)$$

$$z = x/f(x/y)$$

رابطه f از شرایط مرزی بدست می آید

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مثال

$$\begin{cases} uu_x + u_y = y \\ u(x,2) = x \end{cases}$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R} \Rightarrow \frac{dx}{u} = \frac{dy}{1} = \frac{du}{y} \Rightarrow \begin{cases} ydy = du \\ dx = udy \end{cases}$$

$$\Rightarrow \begin{cases} u = 1/2y^2 + c_1 \qquad \Rightarrow c_1 = u - 1/2y^2 \\ dx = (1/2y^2 + c_1)dy \Rightarrow x = 1/6y^3 + c_1 y + c_2 \end{cases}$$

از تلاقی دو سطح جواب بدست می آید

$$c_2 = f(c_1) \Rightarrow c_2 = f(u - 1/2y^2)$$

$$x = 1/6y^3 + (u - 1/2y^2)y + f(u - 1/2y^2)$$

جواب

$$u = 1/3y^2 + x/y - 1/y\, f(u - 1/2y^2)$$

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ادامه مثال

جواب

$$u = 1/3y^2 + x/y - 1/y\, f(u - 1/2y^2)$$

رابطه f از شرایط مرزی بدست می آید

$$u(x,2) = x \Rightarrow x = 4/3 + x/2 - 1/2\, f(x-2)$$

$$\Rightarrow f(x-2) = -x + 8/3$$

$$x - 2 \to x \Rightarrow f(x) = -x + 2/3$$

جایگذاری درجواب

$$u = 1/3y^2 + x/y - 1/y(-(u - 1/2y^2) + 2/3)$$

$$u = \frac{x + 1/3y^3 - 1/2y^2 - 2/3}{y - 1}$$

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معادلات خطی مرتبه دوم با ضرایب ثابت

$$az_{xx} + bz_{xy} + cz_{yy} = 0$$

فرض جواب

$$z = f(y+mx)$$

$$z_{xx} = m^2 f''(y+mx)$$

$$z_{xy} = mf''(y+mx)$$

$$z_{yy} = f''(y+mx)$$

جایگذاری جواب در معادله

$$am^2 + bm + c = 0 \qquad \Rightarrow m_1, m_2$$

جواب

$$z = f(y+m_1x) + g(y+m_2x)$$

$$\begin{cases} b^2-4ac<0: & \text{complex} & :\text{elliptic} \\ b^2-4ac=0: & \text{double} & :\text{parabolic} \\ b^2-4ac>0: & \text{real} & :\text{hyperbolic} \end{cases}$$

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معادلات خطی مرتبه دوم با ضرایب ثابت

مثال

$$z_{xx} - 3z_{xy} + 2z_{yy} = 0$$

فرض جواب

$$m^2 - 3m + 2 = 0 \quad \Rightarrow m_1 = 1, m_2 = 2$$

$$z = f(y+x) + g(y+2x)$$

روابط f,g از شرایط مرزی بدست می آیند

مثال

$$z_{xx} - 4z_{xy} + 4z_{yy} = 0$$

$$m^2 - 4m + 4 = 0 \qquad \Rightarrow m = 2$$

$$z = f(y+2x) + xf(y+2x)$$

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خطوط مشخصه

- First-order PDE in (x,t)

$$Au_t + Bu_x = C$$

$$\frac{dt}{P}=\frac{dx}{Q}=\frac{du}{R} \Rightarrow \frac{dt}{A}=\frac{dx}{B}=\frac{du}{C} \Rightarrow \begin{cases} Adx - Bdt = 0 \\ Adu - Cdt = 0 \end{cases} \Rightarrow \begin{cases} Ax - Bt = c_1 = \xi \\ Au - Ct = c_2 \end{cases}$$

solution $c_2 = f(c_1)$

$$\begin{cases} \text{homogeneous solution: } u_h = f(\xi) = f(Ax - Bt) \\ \text{particular solution: } u_p = \dfrac{C}{A}t \quad \text{or} \quad \dfrac{C}{B}x \end{cases}$$

Characteristic lines: $\xi = Ax - Bt =$ const

Characteristic direction: $d\xi = 0$

propagation velocity of the characteristic line $v = dx/dt = B/A$

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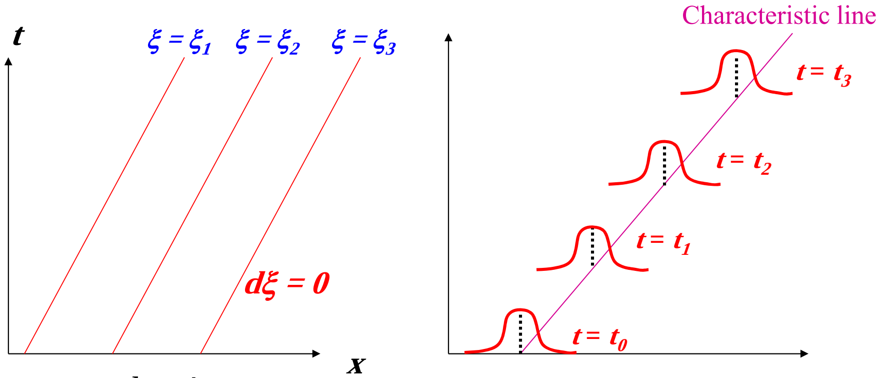
خطوط مشخصه

- $u = f(\xi)$ along the characteristic direction $\xi =$ constant



$$slope = \frac{dt}{dx} = \frac{A}{B}$$

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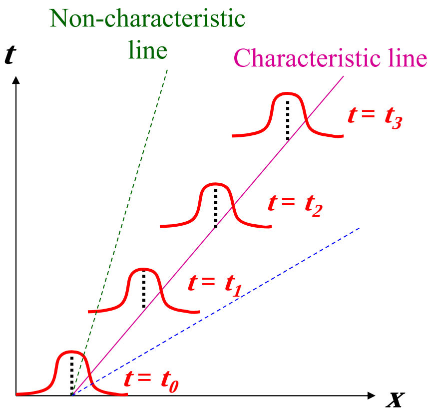
خطوط مشخصه

- Along the characteristic direction $d\xi = 0,\ \xi = \text{constant}$
- $u = f(\xi) = constant$
- The solution remains the same along the characteristic direction
- An observer moving with $\xi =$ constant sees no changes (stationary) in wave form u
- The profile will change if the observer moves faster or slower than the characteristic line
- Hyperbolic PDE - involves only total differentials along the characteristic directions



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خطوط مشخصه

First-Order Hyperbolic PDEs

- Consider a transport quantity ϕ

$$\boxed{\frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x} = 0}$$

v : convective velocity of ϕ

- Along $\xi = x - vt = const$ $(v = dx/dt)$, the property ϕ remains the same (i.e., $d\phi/dt = 0$)
- Consider an airplane (or a train) moving at a velocity v, the passengers inside the airplane (or train) see everything remains stationary while a ground observer sees partial derivatives $(d\phi/dt = 0,\ but\ \partial\phi/\partial t \neq 0)$
- Lagrangian description along the characteristic line

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خطوط مشخصه

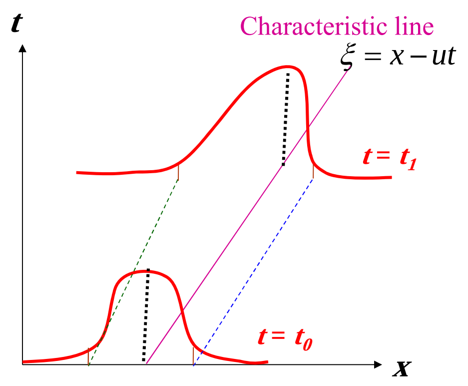
First-Order Hyperbolic PDEs

- First-order PDE in (x,t)

$$u_t + uu_x = 0$$

$$\frac{dt}{P} = \frac{dx}{Q} = \frac{du}{R} \Rightarrow \frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$$

$$\Rightarrow \begin{cases} dx - udt = 0 \\ du = 0 \end{cases} \qquad x - ut = \xi$$



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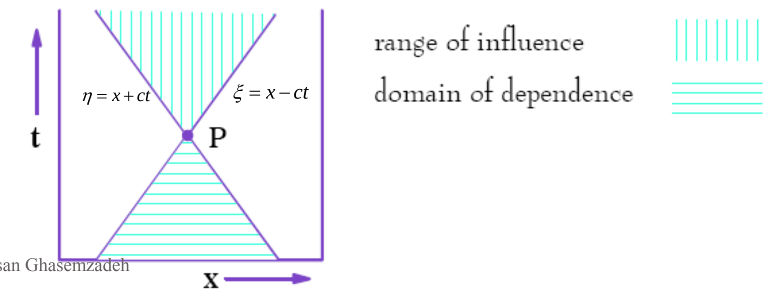
خطوط مشخصه

Second-Order Hyperbolic PDEs

$$u_{tt} - c^2 u_{xx} = 0$$

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0 \qquad \begin{cases} (dx - cdt) = 0 \\ (dx + cdt) = 0 \end{cases} \qquad \begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u = f(x - ct)G(x + ct)$$



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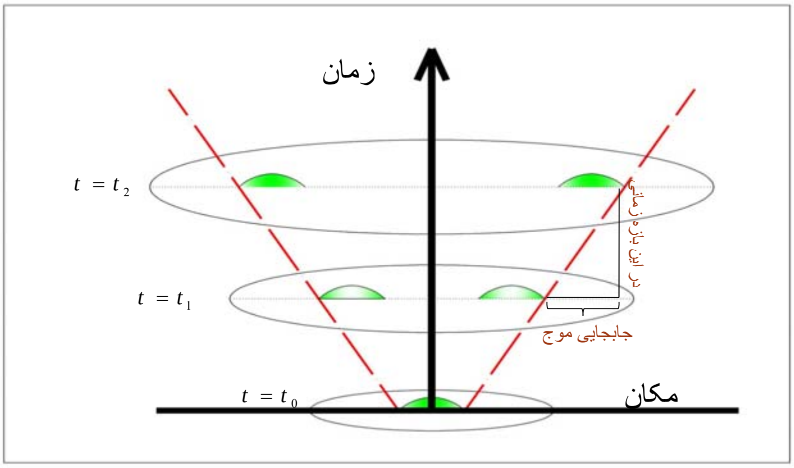
خطوط مشخصه

Second-Order Hyperbolic PDEs



❖ مطالعه رابطه فوق نشان می دهد که تابع در نقطه x,t تنها تحت تاثیر شرایط اولیه در فاصله x+ct, x-ct قرار می گیرد.

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Hyperbolic PDEs

- Two real roots, two characteristic directions
- **Two propagation (marching) directions**
- **Domain of dependence**
- **Domain of influence**
- (u_x, u_y, v_x, v_y) are not uniquely defined along the characteristic lines, discontinuity may occur
- Boundary conditions must be specified according to the characteristics

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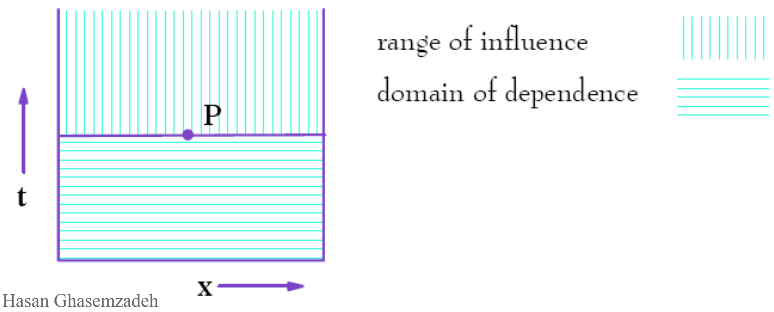
خطوط مشخصه

Second-Order Parabolic PDEs

$$u_{xx} - u_t = 0$$

$$-(\partial_x - 0.\partial_t)(\partial_x - 0.\partial_t)u + \partial_t u = 0 \qquad \frac{dx}{1} = \frac{dt}{0} \qquad dt = 0$$

$$t = c$$



Parabolic PDEs

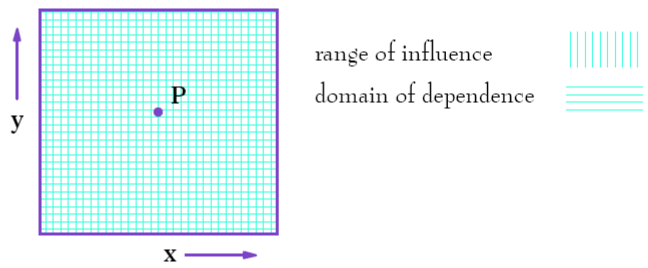
- One real (double) root, one characteristic direction (typically t = const)
- **The solution is marching in time (or spatially) with given initial conditions**
- The solution will be modified by the boundary conditions (time-dependent, in general) during the propagation
- Any change in boundary conditions at t_1 will not affect solution at $t < t_1$, but will change the solution after $t = t_1$
- **Irreversible: You can control your future, but not changing what already happened (history!)**

خطوط مشخصه

Second-Order Elliptic PDEs

$$u_{xx} + u_{yy} = f$$

$$(\partial_x - i\partial_y)(\partial_x + i\partial_y)u = f \qquad \begin{cases} (dx - idy) = 0 \\ (dx + idy) = 0 \end{cases}$$



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Elliptic PDEs

- **det [C] $\neq$ 0 in every direction**
- The derivatives (u_x,u_y,v_x,v_y) can always be uniquely determined at every point in the solution domain
- **No marching or propagation direction !**
- Boundary conditions needed on all boundaries
- The solution will be continuous (smooth) in the entire solution domain
- **Jury problem** - all boundary conditions must be satisfied simultaneously

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System of N Equations

- Classification for first-order equations in n variables
- **Hyperbolic:** n real roots
- **Parabolic:** m real roots, $1 \le m < n$, and no complex roots
- **Elliptic:** no real roots
- **Mixed:** some real and some complex roots, assumed to be elliptic if any complex roots occur

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