Partial Differential Equations (PDEs)

معادلات دیفرانسیل جزئی

Ghasemzadeh

فهرست عناوین و فصول

حل دستگاه معادلات دیفرانسیل جزئی

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یادآوری- جبر خطی

مثال

$$A=\begin{bmatrix}0 & 8\\ 2 & 0\end{bmatrix} \quad \det(A-\lambda I)=0 \quad \begin{vmatrix}-\lambda & 8\\ 2 & -\lambda\end{vmatrix}=0 \quad \lambda=\pm 4$$

$$\Lambda=\begin{bmatrix}4 & 0\\ 0 & -4\end{bmatrix}$$

$$Ax-\lambda x=0$$

$$\lambda=4 \quad \begin{bmatrix}-4 & 8\\ 2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0 \Rightarrow P_1=\begin{bmatrix}2\\ 1\end{bmatrix}$$

$$\lambda=-4 \quad \begin{bmatrix}4 & 8\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0 \Rightarrow P_2=\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$P=\begin{bmatrix}2 & -2\\ 1 & 1\end{bmatrix}$$

$P=$ eigen vectors of A

$$P^{-1}A\,P=\begin{bmatrix}2 & -2\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}0 & 8\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & -2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & -4\end{bmatrix}=\Lambda$$

$$P^{-1}A\,P=\Lambda$$

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مثال: مقادیر ویژه و بردارهای ویژه را محاسبه نمایید

$$y'=\bar{A}y=\begin{bmatrix}2 & -4\\ 1 & -3\end{bmatrix}y$$

جواب خصوصی با ارضای شرط ذیل

$$y^T(0)=\begin{bmatrix}3 & 0\end{bmatrix}$$

$$\det(A-\lambda I)=0 \quad \begin{vmatrix}2-\lambda & -4\\ 1 & -3-\lambda\end{vmatrix}=0 \quad \lambda^2+\lambda-2=0$$

$$\lambda=-2$$

$$\lambda=1$$

مقادیر ویژه اعداد صحیح و مختلف العلامه

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$\lambda = -2 \quad (A-\lambda I)X_1 = 0 \quad \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 1 \quad (A-\lambda I)X_2 = 0 \quad \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$y(x) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-2x} + c_2\begin{bmatrix} 4 \\ 1 \end{bmatrix}e^{x}$$

$$y_1 = c_1 e^{-2x} + 4c_2 e^{x}$$
$$y_2 = c_1 e^{-2x} + c_2 e^{x}$$

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مثال

$c_1 = 0 \Rightarrow y_1 = 4c_2 e^{x}, y_2 = c_2 e^{x} \qquad y_1 = 4y_2$

$c_2 = 0 \Rightarrow y_1 = c_1 e^{-2x}, y_2 = c_1 e^{-2x} \qquad y_1 = y_2$

شرط مرزی

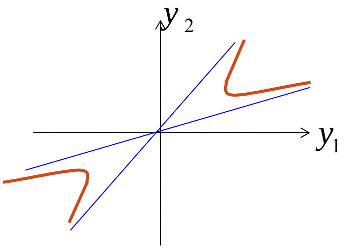
$y_1(0) = c_1 + 4c_2 = 3$

$y_2(0) = c_1 + c_2 = 0$

$c_1 = -1, c_2 = 1$

$y_1 = -e^{-2x} + 4e^{x}$

$y_2 = -e^{-2x} + e^{x}$



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دستگاه معادلات همگن مرتبه اول

$$\begin{cases} x_1' - a_{11}x_1 - a_{12}x_2 - \dots a_{1n}x_n = 0 \\ x_2' - a_{21}x_1 - a_{22}x_2 - \dots a_{2n}x_n = 0 \\ \vdots \\ x_n' - a_{n1}x_1 - a_{n2}x_2 - \dots a_{nn}x_n = 0 \end{cases}$$

حالت استاندارد- معادله همگن

$$\begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix}' = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} \quad \{X\}' = [A]\{X\} \quad \{X\} = \begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix}$$

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فرض جواب معادله بصورت نمایی

$$\begin{Bmatrix} x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{Bmatrix} e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{Bmatrix} \quad \{X\} = \{c_1\}e^{\lambda_1 t} + \dots + \{c_n\}e^{\lambda_n t}$$

$$\{c_k\} = \begin{Bmatrix} c_{1k} \\ \vdots \\ c_{nk} \end{Bmatrix}$$

$$\{X\}' = \{c_1\}\lambda_1 e^{\lambda_1 t} + \dots + \{c_n\}\lambda_n e^{\lambda_n t}$$
$$= [A]\{X\} = [A]\{c_1\}e^{\lambda_1 t} + \dots + [A]\{c_n\}e^{\lambda_n t}$$

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برای برقراری رابطه باید داشته باشیم

$$\{c_1\}\lambda_1 = [A]\{c_1\}$$
$$\vdots$$
$$\{c_n\}\lambda_n = [A]\{c_n\} \qquad \lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$$

$$[A]\{c_k\} = \lambda_k\{c_k\} = \lambda_k I\{c_k\}$$
$$([A] - \lambda_k I)\{c_k\} = 0$$

بردارهای ویژه

$$\Rightarrow ([A] - \lambda_k I) = 0$$

با حل این دستگاه معادلات مقادیر ویژه تعیین می شوند

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مثال

- **Consider two coupled first-order PDEs**

$$\begin{matrix} -2 \\ + \end{matrix}\begin{cases} x' + 2x + y' + 6y = 0 \\ 2x' + 3x + 3y' + 8y = 0 \end{cases} \qquad \begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

$$\begin{cases} -x + y' - 4y = 0 \\ x' + 3x + 10y = 0 \end{cases} \qquad \begin{cases} y' = x + 4y \\ x' = -3x - 10y \end{cases}$$

Standard Form

$$\begin{Bmatrix} x \\ y \end{Bmatrix}' = \begin{bmatrix} -3 & -10 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}$$

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حالت استاندارد- معادله همگن

$$([A]-\lambda_k I)=0$$

برای حالت همگن می توان از روش مقادیر ویژه استفاده نمود

$$\begin{bmatrix}-3 & -10\\ 1 & 4\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=0 \quad\Rightarrow\quad \begin{cases}\lambda_1=2\\ \lambda_2=-1\end{cases}$$

بردارهای ویژه باید محاسبه شوند

بردارهای ویژه

$$\lambda=2\Rightarrow\begin{bmatrix}-3-2 & -10\\ 1 & 4-2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0\Rightarrow P_1=\begin{bmatrix}-2\\ 1\end{bmatrix}$$

$$\lambda=-1\Rightarrow\begin{bmatrix}-3+1 & -10\\ 1 & 4+1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0\Rightarrow P_2=\begin{bmatrix}1\\ -1/5\end{bmatrix}$$

$$x=\begin{bmatrix}1\\ -1/2\end{bmatrix}e^{2t}+\begin{bmatrix}1\\ -1/5\end{bmatrix}e^{-t}=\begin{bmatrix}e^{2t} & e^{-t}\\ -1/2\,e^{2t} & -1/5\,e^{-t}\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\end{Bmatrix}=\underline{X}\,\underline{C}\quad x=\begin{bmatrix}1 & 1\\ -1/2 & -1/5\end{bmatrix}\begin{bmatrix}e^{2t}\\ e^{-t}\end{bmatrix}$$

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معادله همگن- جواب مضاعف

$$\begin{Bmatrix}x\\ y\end{Bmatrix}'=\begin{bmatrix}0 & 1\\ -4 & 4\end{bmatrix}\begin{Bmatrix}x\\ y\end{Bmatrix}$$

روش مقادیر ویژه

$$([A]-\lambda_k I)=0$$

مقادیر ویژه مضاعف

$$\begin{vmatrix}0-\lambda & 1\\ -4 & 4-\lambda\end{vmatrix}=\lambda^2-4\lambda+4=0 \qquad \lambda_1=2$$

بردار ویژه

$$\lambda=2\Rightarrow\begin{bmatrix}0-2 & 1\\ -4 & 4-2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=0\quad c_1=1\Rightarrow P_1=\begin{bmatrix}1\\ 2\end{bmatrix}e^{2t}\qquad X_1=\begin{bmatrix}1\\ 2\end{bmatrix}e^{2t}$$

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معادله همگن - جواب مضاعف

جواب دوم به صورت مقابل است

$$X_2 = \begin{bmatrix}1\\2\end{bmatrix} te^{2t} + \begin{bmatrix}c_3\\c_4\end{bmatrix} e^{2t}$$

جایگذاری در معادله

$$\begin{bmatrix}1\\2\end{bmatrix} e^{2t} + \begin{bmatrix}1\\2\end{bmatrix} 2te^{2t} + \begin{bmatrix}c_3\\c_4\end{bmatrix} 2e^{2t} = \begin{bmatrix}0 & 1\\-4 & 4\end{bmatrix}\left(\begin{bmatrix}1\\2\end{bmatrix} te^{2t} + \begin{bmatrix}c_3\\c_4\end{bmatrix} e^{2t}\right)$$

$$\begin{bmatrix}-2 & 1\\-4 & 2\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \quad \Rightarrow \begin{bmatrix}c_3\\c_4\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$X = c_1 \begin{bmatrix}1\\2\end{bmatrix} e^{2t} + c_2\left(\begin{bmatrix}1\\2\end{bmatrix} te^{2t} + \begin{bmatrix}0\\1\end{bmatrix} e^{2t}\right)$$

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معادله همگن - جواب مختلط

$$\begin{Bmatrix}x\\y\end{Bmatrix}' = \begin{bmatrix}2 & -5\\2 & -4\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}$$

روش مقادیر ویژه

$$([A] - \lambda_k I) = 0$$

مقادیر ویژه مزدوج

$$\begin{vmatrix}2-\lambda & -5\\2 & -4-\lambda\end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0 \quad \lambda = -1 \pm i$$

بردارهای ویژه

$$\lambda = -1+i \Rightarrow \begin{bmatrix}3-i & -5\\2 & -3-i\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = 0 \quad c_1 = 5 \Rightarrow P_1 = \begin{bmatrix}5\\3-i\end{bmatrix}$$

بردار ویژه مزدوج

$$\lambda = -1-i \Rightarrow \begin{bmatrix}3+i & -5\\2 & -3+i\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = 0 \quad c_1 = 5 \Rightarrow P_2 = \begin{bmatrix}5\\3+i\end{bmatrix}$$

جواب

$$X = c_1 \begin{bmatrix}5\\3-i\end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix}5\\3+i\end{bmatrix} e^{(-1-i)t}$$

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معادله ناهمگن

$$\begin{Bmatrix} x \\ y \end{Bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} f(t) \\ g(t) \end{Bmatrix}$$

معاداه همگن

$$\begin{Bmatrix} x \\ y \end{Bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

جواب

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_c = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

جواب خصوصی

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_p = c_1(t) \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2(t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

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معادله ناهمگن

جایگذاری در معادله

$$c_1(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + 2c_2(t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_1'(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2'(t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} =$$

$$\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1(t)e^t + \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_2(t)e^{2t} + \begin{Bmatrix} f(t) \\ g(t) \end{Bmatrix}$$

$$c_1'(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2'(t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} = \begin{Bmatrix} f(t) \\ g(t) \end{Bmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{bmatrix} c_1'(t)e^t \\ c_2'(t)e^{2t} \end{bmatrix} = \begin{Bmatrix} f(t) \\ g(t) \end{Bmatrix}$$

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معادله ناهمگن

جواب از کرامر

$$c_1'(t)e^t = \frac{\begin{vmatrix} f(t) & 1 \\ g(t) & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2f(t) - g(t) \qquad c_1'(t) = \left(2f(t) - g(t)\right)e^{-t}$$

$$c_2'(t)e^{2t} = \frac{\begin{pmatrix} 1 & f(t) \\ 1 & g(t) \end{pmatrix}}{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}} = g(t) - f(t) \qquad c_2'(t) = \left(g(t) - f(t)\right)e^{-2t}$$

$$\left(g(t) = 1, f(t) = e^t\right) \quad \Rightarrow c_1(t) = 2t + e^{-t}, c_2(t) = -0.5e^{-2t} + e^{-t}$$

جواب خصوصی

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_p = \left(2t + e^{-t}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \left(-0.5e^{-2t} + e^{-t}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}_p = \left(2te^t + 1\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(e^t - 0.5\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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حل به روش اپراتور

$$\begin{cases} x' + y' + 2x + 2y = 0 \\ 2x' + y' + 3x - 3y = 0 \end{cases} \qquad D = \frac{d}{dt}$$

$$\begin{bmatrix} D+2 & D+2 \\ 2D+3 & D-3 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} x = c_{11}e^{\lambda_1 t} + c_{12}e^{\lambda_2 t} \\ y = c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t} \end{cases}$$

$$\begin{cases} Dx = \lambda_1 c_{11}e^{\lambda_1 t} + \lambda_2 c_{12}e^{\lambda_2 t} \\ Dy = \lambda_1 c_{21}e^{\lambda_1 t} + \lambda_2 c_{22}e^{\lambda_2 t} \end{cases}$$

$$\begin{cases} (D+2)x + (D+2)y = 0 \\ (2D+3)x + (D-3)y = 0 \end{cases} \quad \Rightarrow$$

$$\begin{cases} (\lambda_1 c_{11} + 2c_{11})e^{\lambda_1 t} + (\lambda_2 c_{12} + 2c_{12})e^{\lambda_2 t} + (\lambda_1 c_{21} + 2c_{21})e^{\lambda_1 t} + (\lambda_2 c_{22} + 2c_{22})e^{\lambda_2 t} = 0 \\ (2\lambda_1 c_{11} + 3c_{11})e^{\lambda_1 t} + (2\lambda_2 c_{12} + 3c_{12})e^{\lambda_2 t} + (\lambda_1 c_{21} - 3c_{21})e^{\lambda_1 t} + (\lambda_2 c_{22} - 3c_{22})e^{\lambda_2 t} = 0 \end{cases}$$

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حل به روش اپراتور

$$\begin{cases} (\lambda_1+2)(c_{11}+c_{21})\,e^{\lambda_1 t}+(\lambda_2+2)(c_{12}+c_{22})\,e^{\lambda_2 t}=0 \\ \left[(2\lambda_1+3)\,c_{11}+(\lambda_1-3)\,c_{21}\right]e^{\lambda_1 t}+\left[(2\lambda_2+3)\,c_{12}+(\lambda_2-3)\,c_{22}\right]e^{\lambda_2 t}=0 \end{cases}$$

$$\begin{cases} (\lambda_1+2)\,c_{11}+(\lambda_1+2)\,c_{21}=0 \\ (2\lambda_1+3)\,c_{11}+(\lambda_1-3)\,c_{21}=0 \end{cases} \quad , \quad \begin{cases} (\lambda_2+2)\,c_{12}+(\lambda_2+2)\,c_{22}=0 \\ (2\lambda_2+3)\,c_{12}+(\lambda_2-3)\,c_{22}=0 \end{cases}$$

$$\begin{vmatrix} \lambda_1+2 & \lambda_1+2 \\ 2\lambda_1+3 & \lambda_1-3 \end{vmatrix}=0$$

مقایسه شود با

$$\begin{bmatrix} D+2 & D+3 \\ 2D+3 & D-3 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix}=0$$

$$\begin{vmatrix} \lambda+2 & \lambda+2 \\ 2\lambda+3 & \lambda-3 \end{vmatrix}=0 \;\Rightarrow\; \begin{aligned} &(\lambda+2)(\lambda-3)-(\lambda+2)(2\lambda+3)=0 \\ &(\lambda+2)(-\lambda-6)=0 \;\rightarrow\; \lambda_1=-2,\;\; \lambda_2=-6 \end{aligned}$$

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حل به روش اپراتور

$$\begin{bmatrix} \lambda+2 & \lambda+2 \\ 2\lambda+3 & \lambda-3 \end{bmatrix}\begin{Bmatrix} c_{11} \\ c_{12} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \lambda=\lambda_1=-2$$

$$\begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix}\begin{Bmatrix} c_{11} \\ c_{12} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \;\Rightarrow\; \begin{aligned} &-c_{11}-5\,c_{12}=0 \\ &c_{11}=1 \;\Rightarrow\; c_{12}=-1/5 \end{aligned}$$

$\begin{Bmatrix} 1 \\ -1/5 \end{Bmatrix}$ یک بردار ویژه بازاء مقادیر ویژه $\lambda_1=-2$ می باشد

$$\lambda=\lambda_2=-6 \;\Rightarrow\; \begin{bmatrix} -4 & -4 \\ -9 & -9 \end{bmatrix}\begin{Bmatrix} c_{12} \\ c_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \;\Rightarrow\; \begin{aligned} &c_{21}=1 \\ &c_{22}=-1 \end{aligned} \qquad \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

بردار ویژه

$$\begin{Bmatrix} x \\ y \end{Bmatrix}=c_1\begin{Bmatrix} 1 \\ -1/5 \end{Bmatrix}e^{-2t}+c_2\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}e^{-6t}$$

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مثال

$$\begin{cases} x_1' + x_1 + x_2' + 2x_2 = 0 \\ 5x_1' + x_1 + 6x_2' + 3x_2 = 0 \end{cases}$$

$$\begin{bmatrix} D+1 & D+2 \\ 5D+1 & 6D+3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{vmatrix} \lambda+1 & \lambda+2 \\ 5\lambda+1 & 6\lambda+3 \end{vmatrix} = 0$$

$$(\lambda+1)(6\lambda+3) - (5\lambda+1)(\lambda+2) = 0 \Rightarrow \lambda = 1,1 \Rightarrow \lambda_1 = \lambda_2 = 1$$

$$\begin{bmatrix} \lambda_1+1 & \lambda_1+2 \\ 5\lambda_1+1 & 6\lambda_1+3 \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \begin{Bmatrix} c_{11} \\ c_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

اگر $c_{11} = 1$ باشد $c_{12} = -2/3$

$\begin{Bmatrix} 1 \\ -2/3 \end{Bmatrix}$ یک بردار مشخصه است

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مثال

- **Consider two coupled first-order PDEs**

$$\begin{cases} u_t + 8v_x = 0 \\ v_t + 2u_x = 0 \end{cases} \qquad \begin{cases} -\infty < x < \infty \\ 0 < t < \infty \end{cases}$$

Initial condition

$$\begin{cases} u(x,0) = f(x) \\ v(x,0) = g(x) \end{cases} \qquad -\infty < x < \infty$$

Matrix Form

$$\begin{Bmatrix} u_t \\ v_t \end{Bmatrix} + \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix} \begin{Bmatrix} u_x \\ v_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{or} \quad U_t + A\,U_x = 0$$

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مثال

- **Changing variable**

$$U = P\bar{U} \qquad U = \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \bar{U} = \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} \qquad P = \text{eigen vectors of } A$$

$$U_t = P\bar{U}_t$$
$$U_x = P\bar{U}_x$$

PDE $\mathrm{P}\bar{U}_t + A\,\mathrm{P}\bar{U}_x = 0$

$$\mathrm{P}^{-1}\mathrm{P}\bar{U}_t + \mathrm{P}^{-1}A\,\mathrm{P}\bar{U}_x = 0 \Rightarrow \bar{U}_t + \Lambda\bar{U}_x = 0 \quad \Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\begin{cases} \bar{u}_t + 4\bar{u}_x = 0 \\ \bar{v}_t - 4\bar{v}_x = 0 \end{cases}$$ Uncoupled system of PDE

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$$\begin{cases} \bar{u}(x,t) = \phi(x-4t) \\ \bar{v}(x,t) = \psi(x+4t) \end{cases} \quad U = P\bar{U} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{v} \end{Bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \phi(x-4t) \\ \psi(x+4t) \end{Bmatrix}$$

$$\begin{cases} u(x,t) = 2\phi(x-4t) - 2\psi(x+4t) \\ v(x,t) = \phi(x-4t) + \psi(x+4t) \end{cases}$$

$$\begin{cases} \phi(z) = \sin(z) \\ \psi(z) = z^2 \end{cases}$$ Arbitrary functions

$$\begin{cases} u(x,t) = 2\sin(x-4t) - 2(x+4t)^2 \\ v(x,t) = \sin(x-4t) + (x+4t)^2 \end{cases}$$

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ICS $\begin{cases} u(x,0) = f(x) \\ v(x,0) = g(x) \end{cases}$ $\begin{cases} 2\phi(x) - 2\psi(x) = f(x) \\ \phi(x) + \psi(x) = g(x) \end{cases}$

$$\begin{cases} \phi(x) = 0.25(f(x) + 2g(x)) \\ \psi(x) = 0.25(2g(x) - f(x)) \end{cases}$$

$$\begin{cases} u(x,t) = 2\phi(x-4t) - 2\psi(x+4t) \\ \qquad = 0.5(f(x-4t) + 2g(x-4t)) - 0.5(2g(x+4t) - f(x+4t)) \\ v(x,t) = \phi(x-4t) + \psi(x+4t) \\ \qquad = 0.25(f(x-4t) + 2g(x-4t)) + 0.25(2g(x+4t) - f(x+4t)) \end{cases}$$

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دستگاه معادلات دیفرانسیل جزئی

- دینامیک مایعات

$$\begin{cases} u_t + uu_x + vu_y + (1/\rho)P_x = 0 \\ v_t + uv_x + vv_y + (1/\rho)P_x = 0 \\ P_t + (\rho u)_x + (\rho v)_y = 0 \\ \left(\dfrac{\rho}{\rho^\gamma}\right)_t + u\left(\dfrac{\rho}{\rho^\gamma}\right)_x + v\left(\dfrac{\rho}{\rho^\gamma}\right)_y = 0 \end{cases}$$

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حل دستگاه معادلات دیفرانسیل جزئی

- **Consider two coupled first-order PDEs**

$$\begin{cases} A_{11}u_x + B_{11}u_y + A_{12}v_x + B_{12}v_y = E_1 \\ A_{21}u_x + B_{21}u_y + A_{22}v_x + B_{22}v_y = E_2 \end{cases}$$

- **In matrix form**

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{Bmatrix} u_x \\ v_x \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{Bmatrix} u_y \\ v_y \end{Bmatrix} = \begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix}$$

or $\overline{\overline{A}}\,\vec{q}_x + \overline{\overline{B}}\,\vec{q}_y = \vec{E}, \quad \vec{q} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

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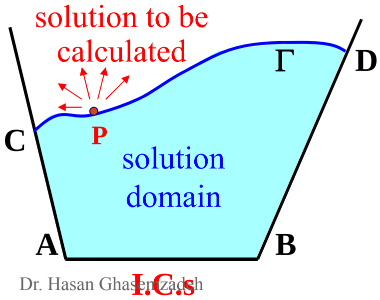
- Questions: Is the behavior of the solution just above P uniquely determined by the information below and on the curve Γ?
- Are the data sufficient to determine the directional derivatives at P in directions that lie above the curve Γ?



$$\begin{cases} du = u_x dx + u_y dy \\ dv = v_x dx + v_y dy \end{cases}$$

Find dy/dx (characteristic direction) along which only the total differentials du and dv appear

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- Under what conditions are the derivatives (u_x, u_y, v_x, v_y) uniquely determined at P by values of (u,v) on Γ?

$$\begin{cases} A_{11}u_x + B_{11}u_y + A_{12}v_x + B_{12}v_y = E_1 \\ A_{21}u_x + B_{21}u_y + A_{22}v_x + B_{22}v_y = E_2 \\ u_x dx + u_y dy = du \\ v_x dx + v_y dy = dv \end{cases}$$

Determinant ≠ 0: unique solution (linearly independent)
Determinant = 0: multiple or no solutions

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- Determinant ≠ 0: unique solution

$$\begin{bmatrix} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ dx & 0 & dy & 0 \\ 0 & dx & 0 & dy \end{bmatrix} \begin{Bmatrix} u_x \\ v_x \\ u_y \\ v_y \end{Bmatrix} = \begin{Bmatrix} E_1 \\ E_2 \\ du \\ dv \end{Bmatrix} \Leftrightarrow \overline{\overline{C}} = \begin{bmatrix} \overline{\overline{A}} & \overline{\overline{B}} \\ \overline{\overline{I}}dx & \overline{\overline{I}}dy \end{bmatrix}$$

Characteristic equation det [C] = 0

$$\det \overline{\overline{C}} = \det\left[\overline{\overline{A}}\,dy - \overline{\overline{B}}\,dx\right] = \begin{vmatrix} A_{11}dy - B_{11}dx & A_{12}dy - B_{12}dx \\ A_{21}dy - B_{21}dx & A_{22}dy - B_{22}dx \end{vmatrix}$$

$$= (A_{11}A_{22} - A_{12}A_{21})dy^2 + (A_{12}B_{21} + A_{21}B_{12} - A_{11}B_{22} - A_{22}B_{11})dxdy$$
$$+ (B_{11}B_{22} - B_{12}B_{21})dx^2$$
$$= A'dy^2 + B'dxdy + C'dx^2 = 0$$

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- Characteristic directions along which (u_x, u_y, v_x, v_y) are not defined uniquely
- Multiple solutions possible, discontinuity may occur

$$A'\left(\frac{dy}{dx}\right)^2 + B'\left(\frac{dy}{dx}\right) + C' = 0$$ Characteristic equation

$$\frac{dy}{dx} = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'}$$ Characteristic directions

Classification of equation types -- depends on the discriminant of the characteristic equation

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- Discriminant DIS

$$DIS = B'^2 - 4A'C' = \left(A_{12}B_{21} + A_{21}B_{12} - A_{11}B_{22} - A_{22}B_{11}\right) - 4(A_{11}A_{22} - A_{12}A_{21})(B_{11}B_{22} - B_{12}B_{21})$$

$$B'^2 - 4A'C' \begin{cases} > 0, & 2 \text{ real roots, } 2 \text{ characteristics directions; Hyperbolic} \\ = 0, & 1 \text{ real root, } 1 \text{ characteristics direction; Parabolic} \\ < 0, & 2 \text{ complex roots, no real characteristics; Elliptic} \end{cases}$$

- DIS < 0 (elliptic), cannot identify characteristic directions along which discontinuity may occur across Γ
- Elliptic equation - continuous (smooth) solution

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حل دستگاه معادلات دیفرانسیل جزئی

- Transformation of higher-order PDE to first-order PDEs

$$A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + H = 0$$

$$let \begin{cases} u = \phi_x \\ v = \phi_y \end{cases} \quad (i.e., \vec{V} = \nabla\phi) \Rightarrow \begin{cases} u_x = \phi_{xx}, & u_y = \phi_{xy} \\ v_x = \phi_{xy}, & v_y = \phi_{yy} \end{cases}$$

- Convert to first-order PDEs

$$\begin{cases} Au_x + Bu_y + Cv_y + H = 0 \\ -u_y + v_x = 0 \end{cases}$$

$$\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_x \\ v_x \end{Bmatrix} + \begin{bmatrix} B & C \\ -1 & 0 \end{bmatrix}\begin{Bmatrix} u_y \\ v_y \end{Bmatrix} = \begin{Bmatrix} -H \\ 0 \end{Bmatrix}$$

$$\overline{\overline{A}}\vec{q}_x + \overline{\overline{B}}\vec{q}_y = \vec{E}$$

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حل دستگاه معادلات دیفرانسیل جزئی

- Transformation of second-order PDE to two first-order PDEs

$$\overline{\overline{A}}\vec{q}_x + \overline{\overline{B}}\vec{q}_y = \vec{E}$$

$$\det\overline{\overline{C}} = \det\left|\overline{\overline{A}}dy - \overline{\overline{B}}dx\right| = \begin{vmatrix} Ady - Bdx & -Cdx \\ dx & dy \end{vmatrix}$$

$$= Ady^2 - Bdxdy + Cdx^2 = 0$$

Characteristics equation and characteristic directions

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \Rightarrow \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\begin{cases} Hpberbolic: & B^2 - 4AC > 0; & \text{2 characteristics} \\ Parabolic: & B^2 - 4AC = 0; & \text{1 characteristics} \\ Elliptic: & B^2 - 4AC < 0; & \text{no real characteristics} \end{cases}$$

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حل دستگاه معادلات دیفرانسیل جزئی

- **One variable, n = 1**

$$\overline{\overline{A}}\vec{q}_x + \overline{\overline{B}}\vec{q}_y = \vec{E} \quad \Rightarrow Au_x + Bu_y = C$$

$$\begin{bmatrix} A & B \\ dx & dy \end{bmatrix} \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{Bmatrix} C \\ du \end{Bmatrix}$$

$$\det\left[\overline{\overline{A}}\,dy - \overline{\overline{B}}\,dx\right] = Ady - Bdx = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{\lambda_x}{\lambda_y} = \frac{B}{A}$$

One real root ⇒ hyperbolic

$$d\lambda = \lambda_x dx + \lambda_y dy = Ady - Bdx = 0$$

$$\lambda = Ay - Bx = const \quad (\text{or } \xi = Ay - Bx)$$

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خطوط مشخصه

Hyperbolic PDEs

- Two real roots, two characteristic directions
- **Two propagation (marching) directions**
- **Domain of dependence**
- **Domain of influence**
- (u_x,u_y,v_x,v_y) are not uniquely defined along the characteristic lines, discontinuity may occur
- Boundary conditions must be specified according to the characteristics

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خطوط مشخصه

Parabolic PDEs

- One real (double) root, one characteristic direction (typically $t = const$)
- **The solution is marching in time (or spatially) with given initial conditions**
- The solution will be modified by the boundary conditions (time-dependent, in general) during the propagation
- Any change in boundary conditions at t_1 will not affect solution at $t < t_1$, but will change the solution after $t = t_1$
- **Irreversible: You can control your future, but not changing what already happened (history!)**

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خطوط مشخصه

Elliptic PDEs

- **det [C] ≠ 0 in every direction**
- The derivatives (u_x, u_y, v_x, v_y) can always be uniquely determined at every point in the solution domain
- **No marching or propagation direction !**
- Boundary conditions needed on all boundaries
- The solution will be continuous (smooth) in the entire solution domain
- **Jury problem** - all boundary conditions must be satisfied simultaneously

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System of Equations

- Two variables, n = 2

$$\bar{\bar{A}}\vec{q}_x + \bar{\bar{B}}\vec{q}_y = \vec{E}$$

$$\det\left[\bar{\bar{A}}dy - \bar{\bar{B}}dx\right] = A'dy^2 + B'dxdy + C'dx^2 = 0$$

$$\frac{dy}{dx} = -\frac{\lambda_x}{\lambda_y} = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'} = \begin{cases}\alpha \\ \beta\end{cases} \Rightarrow \begin{cases} dy - \alpha dx = 0 \\ dy - \beta dx = 0\end{cases}$$

$$d\lambda = \lambda_x dx + \lambda_y dy = 0$$

$$\begin{cases}\lambda_1 = \xi = y - \alpha x = const \\ \lambda_2 = \eta = y - \beta x = const\end{cases}$$

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System of N first-order PDEs

- Two-dimensional (two variables x, y)

$$\begin{cases}\bar{\bar{A}}\vec{q}_x + \bar{\bar{B}}\vec{q}_y = \bar{\bar{E}} \\ dx\vec{q}_x + dy\vec{q}_y = d\vec{q}\end{cases} \Rightarrow \begin{bmatrix}\bar{\bar{A}} & \bar{\bar{B}} \\ \bar{\bar{I}}dx & \bar{\bar{I}}dy\end{bmatrix}\begin{Bmatrix}\vec{q}_x \\ \vec{q}_y\end{Bmatrix} = \begin{Bmatrix}\vec{E} \\ d\vec{q}\end{Bmatrix}; \quad \vec{q} = \begin{Bmatrix}u_1 \\ u_2 \\ \vdots \\ u_n\end{Bmatrix}$$

Characteristic equation $\det\bar{\bar{C}} = \det\left[\bar{\bar{A}}dy - \bar{\bar{B}}dx\right] = 0$

Characteristic lines $\lambda = \text{const}, \quad d\lambda = \lambda_x dx + \lambda_y dy = 0$

Characteristic directions $\frac{dy}{dx} = -\frac{\lambda_x}{\lambda_y}, \quad \det\left[\bar{\bar{A}}\lambda_x + \bar{\bar{B}}\lambda_y\right] = 0$

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Example

- Steady 2D incompressible Navier-Stokes equations

$$\begin{cases} u_x + v_y = 0 \\ uu_x + vu_y + p_x - (u_{xx} + u_{yy})/Re = 0 \\ uv_x + vv_y + p_y - (v_{xx} + v_{yy})/Re = 0 \end{cases}$$

3 coupled PDEs for (u, v, p)

$$let \quad R = v_x, \quad S = v_y = -u_x, \quad T = u_y, \; then$$

$$\begin{cases} -R_y + S_x = -v_{xy} + v_{xy} = 0 \\ S_y + T_x = -u_{xy} + u_{xy} = 0 \\ -uS + vT + p_x - (-S_x + T_y)/Re = 0 \\ uR + vS + p_y - (R_x + S_y)/Re = 0 \end{cases}$$

Convert to 4 first-order PDEs for 4 unknowns (R, S, T, p)

Example

- Steady 2D incompressible Navier-Stokes equations

$$\bar{\bar{A}}\vec{q}_x + \bar{\bar{B}}\vec{q}_y = \vec{E}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/Re & 0 & 1 \\ -1/Re & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} R_x \\ S_x \\ T_x \\ p_x \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/Re & 0 \\ 0 & -1/Re & 0 & 1 \end{bmatrix} \begin{Bmatrix} R_y \\ S_y \\ T_y \\ p_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ uS - vT \\ -uR - vS \end{Bmatrix}$$

- Characteristic equation

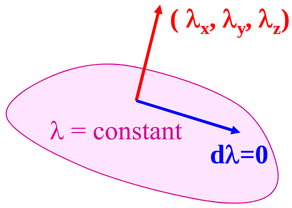
$$\det\left[\bar{\bar{A}}\lambda_x + \bar{\bar{B}}\lambda_y\right] = \begin{vmatrix} -\lambda_y & \lambda_x & 0 & 0 \\ 0 & \lambda_y & \lambda_x & 0 \\ 0 & \lambda_x/Re & -\lambda_y/Re & \lambda_x \\ -\lambda_x/Re & -\lambda_y/Re & 0 & \lambda_y \end{vmatrix} = \frac{1}{Re}\left(\lambda_x^2 + \lambda_y^2\right)^2 = 0$$

Elliptic system (all complex roots, no real roots)

System of Equations

- **Three variables (Fourier analysis)**

$$\begin{cases} \bar{\bar{A}}\vec{q}_x + \bar{\bar{B}}\vec{q}_y + \bar{\bar{C}}\vec{q}_z = \bar{\bar{E}} \\ dx\vec{q}_x + dy\vec{q}_y + dz\vec{q}_z = d\vec{q} \end{cases} \quad ; \quad \vec{q} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix}$$



Characteristic equation $\det \bar{\bar{C}} = det\left[\bar{\bar{A}}\lambda_x + \bar{\bar{B}}\lambda_y + \bar{\bar{C}}\lambda_z\right] = 0$

Characteristic surface $d\lambda = \lambda_x dx + \lambda_y dy + \lambda_z dz = 0 \quad (\lambda = const)$

Characteristic directions $(\lambda_x, \lambda_y, \lambda_z)$

Normal to characteristic surface λ = constant

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System of N Equations

- Classification for first-order equations in n variables
- **Hyperbolic:** n real roots
- **Parabolic:** m real roots, $1 \leq m < n$, and no complex roots
- **Elliptic:** no real roots
- **Mixed:** some real and some complex roots, assumed to be elliptic if any complex roots occur

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Canonical Forms of PDEs

- Any PDE can also be transformed into Canonical form $(x,y) \rightarrow (\xi,\eta)$

$$\begin{cases} A\phi_{xx} + B\phi_{xy} + C\phi_{yy} + D\phi_x + E\phi_y + F\phi = G \\ A'\phi_{\xi\xi} + B'\phi_{\xi\eta} + C'\phi_{\eta\eta} + D'\phi_{\xi} + E'\phi_{\eta} + F'\phi = G' \end{cases}$$

- Cl ... $A' = C' = 0$

$$\begin{cases} A' = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \\ C' = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0 \end{cases}$$

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Canonical Forms of PDEs

- **Characteristic equations:** $A' = C' = 0$

$$\begin{cases} A\left(\dfrac{\xi_x}{\xi_y}\right)^2 + B\left(\dfrac{\xi_x}{\xi_y}\right) + C = 0 \;\Rightarrow\; \dfrac{\xi_x}{\xi_y} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ A\left(\dfrac{\eta_x}{\eta_y}\right)^2 + B\left(\dfrac{\eta_x}{\eta_y}\right) + C = 0 \;\Rightarrow\; \dfrac{\eta_x}{\eta_y} = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A} \end{cases}$$

- **Characteristic directions:** $\xi =$ **const,** $\eta =$ **const** $(d\xi, d\eta = 0)$

$$\left.\begin{cases} \xi = c_1, & d\xi = \xi_x dx + \xi_y dy = 0 \\ \eta = c_2, & d\eta = \eta_x dx + \eta_y dy = 0 \end{cases}\right\} \Rightarrow \frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = -\frac{\eta_x}{\eta_y} = \frac{B \pm \sqrt{B^2 - 4AC}}{2a}$$

$$A' = C' = 0 \;\Rightarrow A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$ **Characteristic equation**

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Canonical Forms

- Hyperbolic PDE: $B'^2 - 4A'C' > 0$, 2 real characteristics

$$\begin{cases} \phi_{\xi\eta} = h_1(\xi,\eta,\phi,\phi_\xi,\phi_\eta), & A' = C' = 0, B' = 1 \\ \phi_{\xi\xi} - \phi_{\eta\eta} = h_1'(\xi,\eta,\phi,\phi_\xi,\phi_\eta), & A' = 1, B' = 0, C' = -1 \end{cases} \qquad \begin{cases} \bar{\xi} = \xi + \eta \\ \bar{\eta} = \xi - \eta \end{cases}$$

- Parabolic PDE: $B'^2 - 4A'C' = 0$, 1 real characteristics

$$\phi_{\xi\xi} = h_2(\xi,\eta,\phi,\phi_\xi,\phi_\eta), \quad A' = 1, B' = C' = 0$$

- Elliptic PDE: $B'^2 - 4A'C' < 0$, no real characteristics

$$\phi_{\xi\xi} + \phi_{\eta\eta} = h_3(\xi,\eta,\phi,\phi_\xi,\phi_\eta), \quad A' = 1, B' = 0, C' = 1$$

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2.1.5 Classification by Fourier Analysis

Characteristic polynomial

- The root determine the characteristic surfaces (λ = const) or directions (λ_x, λ_y, λ_z, …)

Fourier Analysis

- The roots have a different physical interpretation
- Produces the same characteristic polynomial from the "principal part" of the governing equation. However, (λ_x, λ_y, λ_{z}, …) also determine the solution of PDE, e.g., oscillatory, exponential growth, wavelike, …
- Avoids the construction of intermediate first-order PDEs

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Classification by Fourier Analysis

- Consider second-order PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} = 0$$

- General solution in Fourier series

$$u(x,y) = \frac{1}{4\pi^2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{u}_{jk} \exp[i(\sigma_x)_j x] \exp[i(\sigma_y)_k y]$$

$$\begin{cases} \hat{u}_{jk} & - amplitude \\ \sigma_x, \sigma_y & - exponents \end{cases}$$

Determined by boundary conditions

Determine the nature of solutions

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Classification by Fourier Analysis

- Take derivatives of the general solution

$$\begin{cases} u_x = \frac{1}{4\pi^2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} i(\sigma_x)_j \hat{u}_{jk} \exp[i(\sigma_x)_j x] \exp[i(\sigma_y)_k y] \\ u_y = \frac{1}{4\pi^2} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} i(\sigma_y)_k \hat{u}_{jk} \exp[i(\sigma_x)_j x] \exp[i(\sigma_y)_k y] \end{cases}$$

- If A, B, C are not function of u (linear), then the relation between σ_x and σ_y is the same for all modes
- "Superposition" gives all possible functional forms for linear equation

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Classification by Fourier Analysis

- Consider one single model only $(\sigma_x)_j=\sigma_x, (\sigma_y)_k=\sigma_y$

$$u(x,y)=\frac{1}{4\pi^2}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\hat{u}_{jk}\exp(i\sigma_x x)\exp(i\sigma_y y)$$

$$\begin{cases} u_x = i\sigma_x u, \quad u_{xx}=(i\sigma_x)^2 u = -\sigma_x^2 u, \quad u_{xy}=-\sigma_x\sigma_y u \\ u_y = i\sigma_y u, \quad u_{yy}=(i\sigma_y)^2 u = -\sigma_y^2 u \end{cases}$$

$$\therefore\ Au_{xx}+Bu_{xy}+Cu_{yy}=0 \iff -A\sigma_x^2 - B\sigma_x\sigma_y - C\sigma_y^2 = 0$$

- Characteristic polynomial

$$A\left(\frac{\sigma_x}{\sigma_y}\right)^2 + B\left(\frac{\sigma_x}{\sigma_y}\right) + C = 0$$

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Characteristic Equation

- Fourier analysis (discrete mode)

$$A\left(\frac{\sigma_x}{\sigma_y}\right)^2 + B\left(\frac{\sigma_x}{\sigma_y}\right) + C = 0 \Rightarrow \frac{\sigma_x}{\sigma_y} = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$$

- Discriminant $B^2 - 4AC$

$$B^2 - 4AC \begin{cases} \geq 0, \quad real\left(\frac{\sigma_x}{\sigma_y}\right) \\ < 0, \quad complex\left(\frac{\sigma_x}{\sigma_y}\right) \end{cases}$$

Propagation/marching problems (sin/cos functions)

Growth/decay/propagation

Exponential Growth/decay – pure imaginary (σ_x/σ_y)

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Characteristic Equation

- Fourier Transform (continuous spectrum)

$$\begin{cases} \hat{u}(\sigma_x,\sigma_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)\exp(-i\sigma_x x)\exp(-i\sigma_y y)dxdy = Fu \\ i\sigma_x\hat{u}(\sigma_x,\sigma_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,y)\exp(-i\sigma_x x)\exp(-i\sigma_y y)dxdy = F\frac{\partial u}{\partial x} \\ i\sigma_y\hat{u}(\sigma_x,\sigma_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u}{\partial y}(x,y)\exp(-i\sigma_x x)\exp(-i\sigma_y y)dxdy = F\frac{\partial u}{\partial y} \end{cases}$$

- Integration by parts

$$F\frac{\partial u}{\partial x} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}(x,y)e^{-i\sigma_x x}e^{-i\sigma_y y}dxdy$$

Fourier transform of $\partial u/\partial x$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(ue^{-i\sigma_x x}e^{-i\sigma_y y}\right)dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-i\sigma_x)ue^{-i\sigma_x x}e^{-i\sigma_y y}dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ue^{-i\sigma_x x}e^{-i\sigma_y y}\Big|_{-\infty}^{\infty} dy + i\sigma_x Fu = i\sigma_x\hat{u}$$

= 0, if u = 0 at x→±∞

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Characteristic Equation

- Classification by Fourier transform

$$\hat{u} = Fu \quad \Rightarrow \begin{cases} i\sigma_x\hat{u} = F\frac{\partial u}{\partial x}, \quad (i\sigma_x)^2\hat{u} = F\frac{\partial^2 u}{\partial x^2} \\ i\sigma_y\hat{u} = F\frac{\partial u}{\partial y}, \quad (i\sigma_y)^2\hat{u} = F\frac{\partial^2 u}{\partial y^2} \end{cases}$$

$$\therefore Au_{xx} + Bu_{xy} + Cu_{yy} = 0 \Rightarrow$$

$$\left[A(i\sigma_x)^2 + B(i\sigma_x)(i\sigma_y) - C(i\sigma_y)^2\right]\hat{u} = 0$$

- Characteristic polynomial

$$A\left(\frac{\sigma_x}{\sigma_y}\right)^2 + B\left(\frac{\sigma_x}{\sigma_y}\right) + C = 0$$

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Fourier Analysis

- Particularly usseful for system of equations with higher-order PDEs (avoid the construction of first-order PDEs)
- Example: Navier-Stokes equations

$$\begin{cases} u_x + v_y = 0 \\ uu_x + vu_y + p_x - (u_{xx} + u_{yy}) / Re = 0 \\ uv_x + vv_y + p_y - (v_{xx} + v_{yy}) / Re = 0 \end{cases}$$

$$\begin{cases} i\sigma_x \hat{u} + i\sigma_y \hat{v} = 0 \\ iu\sigma_x \hat{u} + iv\sigma_y \hat{u} + i\sigma_x \hat{p} - \left[(i\sigma_x)^2 \hat{u} + (i\sigma_y)^2 \hat{u} \right] / Re = 0 \\ iu\sigma_x \hat{v} + iv\sigma_y \hat{v} + i\sigma_y \hat{p} - \left[(i\sigma_x)^2 \hat{v} + (i\sigma_y)^2 \hat{v} \right] / Re = 0 \end{cases}$$

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Fourier Analysis

- Characteristic equations

$$\begin{bmatrix} i\sigma_x & i\sigma_y & 0 \\ i(u\sigma_x + v\sigma_y) + (\sigma_x^2 + \sigma_y^2)/Re & 0 & i\sigma_x \\ 0 & i(u\sigma_x + v\sigma_y) + (\sigma_x^2 + \sigma_y^2)/Re & i\sigma_y \end{bmatrix} \begin{Bmatrix} \hat{u} \\ \hat{v} \\ \hat{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- Determinant = 0

$$-i(\sigma_x^2 + \sigma_y^2)\left[i(u\sigma_x + v\sigma_y) + \frac{1}{Re}(\sigma_x^2 + \sigma_y^2) \right] = 0$$

first-derivative second-derivative

- Consider only the highest-order derivatives

$$\frac{1}{Re}\left(\sigma_x^2 + \sigma_y^2\right)^2 = 0$$

Elliptic system (complex roots)

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