# روش های عددی در ژئومکانیک

# Numerical methods in geomechanics

Hasan Ghasemzadeh

http://wp.kntu.ac.ir/ghasemzadeh

دانشگاه صنعتی خواجه نصیرالدین طوسی

## قول الحق

وَ الّذی جَعَلَ الشّمسَ ضیاءً والقَمَرَ نوراً وقَدّرَهُ مَنازِلَ لِتَعلَموا عَدَدَ السّنِینَ والحِسابَ

یونس ۵

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## منابع


+ **A first course in the fem(2010)** Daryl Logan 5th ed.
+ **Fundamentals of engineering numerical analysis(2010)** Parviz Moin 2en ed.
+ **Numerical Methods for Engineers and Scientific (2001)** Joe D. Hoffma.
+ **Numerical Mathematics(2000) ,** *Alfio Quarteroni, Riccardo Sacco, Fausto Saleri*
+ **Finite Elements and Approximation**, (1983) Univ. of Wales, Swansea, U. K. (1983) O. C. Zien kiewicz & K. Morgan, John Wiley & Sons,
+ **The Boundary Element Method For Engineers**, 1980, Brebbia C.A.
+ **Numerical Methods for Elliptic and Parabolic Partial Differential Equations, 2003,** Peter Knabner Lutz Angermann
+ **The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media (1998)** R.W. Lewis and B.W. Schrefler
+ مقدمه ای بر روش اجزاء محدود ،مرحوم سیدامیرالدین صدرنژاد ، ۱۳۸۰
+ مقالات علمی مربوط به روشهای عددی
+ جزوه کلاس


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## فهرست مطالب درس

کلیات روش‌های عددی در ژئومکانیک – معادلات دیفرانسیل

روش تفاوت محدود

روش اجزای محدود

کاربرد روش‌های عددی در حل مسائل ژئومکانیک (نشست، گسترش تنشها، پی‌های سطحی، شمع ها، دیوارهای حائل ، مخازن نفت، چاه‌های نفت ...)

حل عددی محیطهای دو سه فازه

حل حرکت موج در محیطهای متخلخل

الگوهای عددی – معادلات ساختاری- ارزیابی فراسنجها

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## کلیات روش‌های عددی در ژئومکانیک

- معادلات در ژئومکانیک
- کلیات روشهای عددی
- انواع روشهای عددی
- یادآوری حل معادلات
- انتگرال گیری عددی
- اصول انرژی

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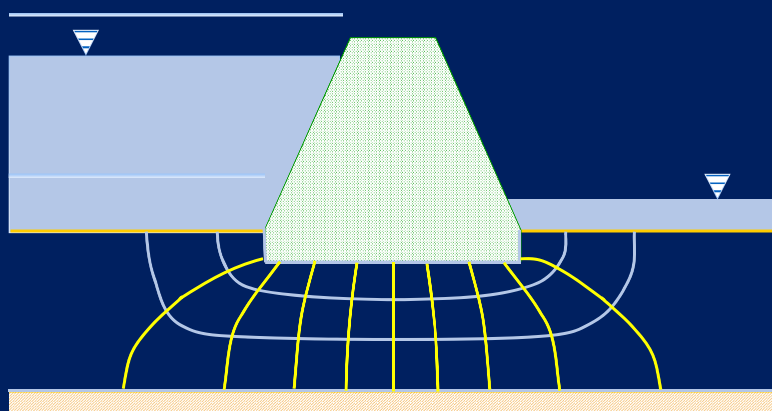
$$k_x \frac{\partial^2 h}{\partial x^2} + k_z \frac{\partial^2 h}{\partial z^2} = 0$$

تراوش در خاک



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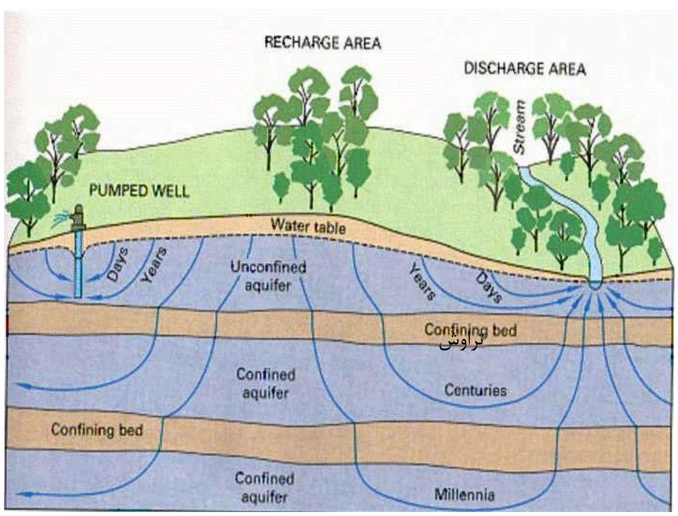


تراوش در خاک

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}(D(\theta)\frac{\partial \theta}{\partial z}) + \frac{\partial K}{\partial z} \qquad D(\theta) = K(\theta)\frac{\partial h}{\partial \theta}$$

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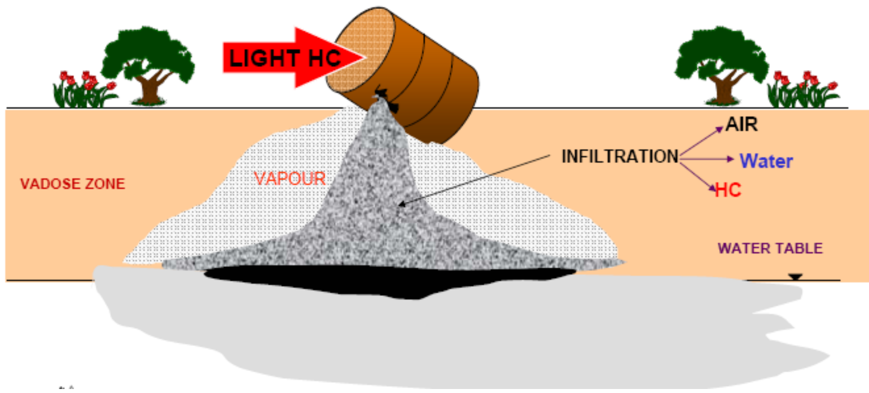
## آلودگی در خاک

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## آلودگی در خاک

**pollution**


$$D\frac{\partial^2 c}{\partial x^2} + C\frac{\partial c}{\partial x} = \frac{\partial c}{\partial t}$$

**groundwater mining**
**subsidence**

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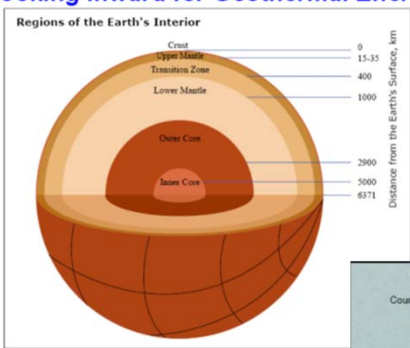
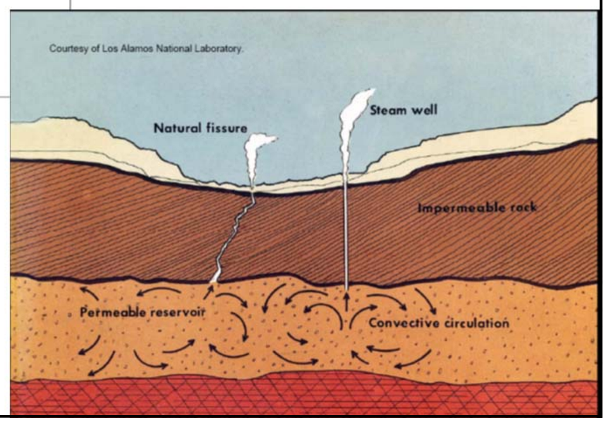
## انتشار حرارت در خاک

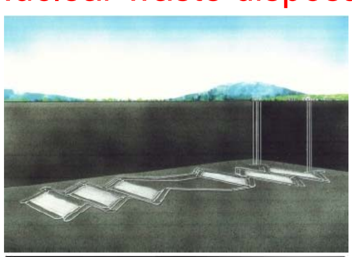
**Looking inward for Geothermal Energy**

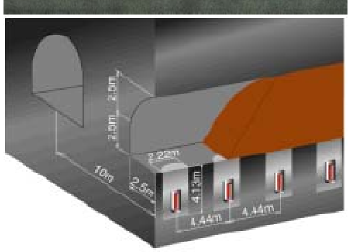


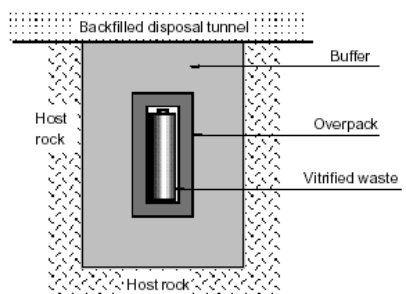




## انتشار حرارت در خاک

**Nuclear waste disposal**



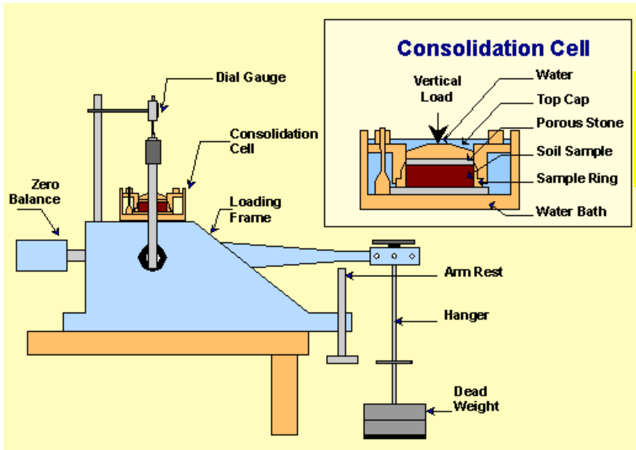



Vertical emplacement concept

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

# تحکیم در خاک




$$\frac{\partial^2 u}{\partial z^2} = c_v \frac{\partial u}{\partial t}$$

$$\begin{cases} \dfrac{\partial u}{\partial t} = C \dfrac{\partial^2 w}{\partial z^2} \\ \dfrac{\partial w}{\partial t} = D \dfrac{\partial^2 u}{\partial z^2} \end{cases}$$



# تحلیل تنش و تغییر شکل در خاک

معادله تعادل (سه معادله)

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{z}}{\partial z} + Z = 0$$

رابطه سازگاری کرنش- شش رابطه

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x} - \left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

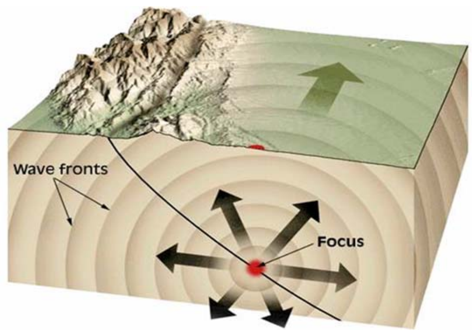
رابطه ساختاری

## انتشار امواج طولی در خاک

$$u_{tt} = c_p^2 u_{xx}$$

$$c_p = \sqrt{E/\rho}$$

معادله موج یک بعدی




معادله موج در محیط سه بعدی

معادلات الاستودینامیک

$$(\lambda+\mu)\frac{\partial \varepsilon_v}{\partial x} + \mu\nabla^2 u_x = \rho\frac{\partial^2 u_x}{\partial t^2}$$

$$(\lambda+\mu)\frac{\partial \varepsilon_v}{\partial y} + \mu\nabla^2 u_y = \rho\frac{\partial^2 u_y}{\partial t^2}$$

$$(\lambda+\mu)\frac{\partial \varepsilon_v}{\partial z} + \mu\nabla^2 u_z = \rho\frac{\partial^2 u_z}{\partial t^2}$$

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# First-Order PDEs

- **First-order linear wave equation (advection eq.)**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

- Propagation of wave with speed c
- Advection of passive scalar with speed c
- **First-order nonlinear wave equation (inviscid Burgers's equation)**

gas dynamics and traffic flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

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# Second-Order PDEs

- Advection-diffusion equation (linear)

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + c \frac{\partial T}{\partial x}$$

- Burger's equation (nonlinear)

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x}$$

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# Other Common PDEs

- Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Nonlinear dispersive wave

waves on shallow water surfaces

- Laplace and Poisson's equations

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad \begin{cases} f = 0 : \text{Laplace} \\ f \neq 0 : \text{Poisson} \end{cases}$$

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## Other Common PDEs

- **Helmholtz equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

**Time-dependent harmonic waves**

**Propagation of acoustic waves**

- **Tricomi equation**

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \begin{cases} y > 0 \ : \ \text{elliptic} \\ y < 0 \ : \ \text{hyperbolic} \end{cases}$$

**Mixed-type**

transonic flow

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## Other Common PDEs

- **Wave equation**

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

- **Fourier equation (Heat equation)**

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

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# Navier-Stokes Equations

- Navier-Stokes equation
- Vorticity / stream function formulation

$$\begin{cases} \nabla^2 \psi = -\omega \\ \dfrac{\partial \omega}{\partial t} + u\dfrac{\partial \omega}{\partial x} + v\dfrac{\partial \omega}{\partial y} = \nu\left(\dfrac{\partial^2 \omega}{\partial x^2} + \dfrac{\partial^2 \omega}{\partial y^2}\right) \end{cases}$$

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# Navier-Stokes Equations

- Navier-Stokes equation
- Primitive variables

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

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# RANS Equations: Turbulent Flows

- Reynolds-Averaged Navier-Stokes equation

$$\begin{cases} \dfrac{\partial U}{\partial x} + \dfrac{\partial V}{\partial y} = 0 \\ \dfrac{\partial U}{\partial t} + U\dfrac{\partial U}{\partial x} + V\dfrac{\partial U}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x} + \nu\left(\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}\right) - \dfrac{\partial \overline{uu}}{\partial x} - \dfrac{\partial \overline{uv}}{\partial y} \\ \dfrac{\partial V}{\partial t} + U\dfrac{\partial V}{\partial x} + V\dfrac{\partial V}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial y} + \nu\left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}\right) - \dfrac{\partial \overline{uv}}{\partial x} - \dfrac{\partial \overline{vv}}{\partial y} \end{cases}$$

$$\begin{cases} \dfrac{\partial k}{\partial t} + U\dfrac{\partial k}{\partial x} + V\dfrac{\partial k}{\partial y} = (\nu + \nu_t)\left(\dfrac{\partial^2 k}{\partial x^2} + \dfrac{\partial^2 k}{\partial y^2}\right) + G - \varepsilon \\ \dfrac{\partial \varepsilon}{\partial t} + U\dfrac{\partial \varepsilon}{\partial x} + V\dfrac{\partial \varepsilon}{\partial y} = (\nu + \nu_t)\left(\dfrac{\partial^2 \varepsilon}{\partial x^2} + \dfrac{\partial^2 \varepsilon}{\partial y^2}\right) + \dfrac{\varepsilon}{k}(C_{\varepsilon 1}G - C_{\varepsilon 2}\varepsilon) \end{cases}$$

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# Classification of PDEs

- **Linear second-order PDE in two independent variables (x,y), (x,t), etc.**

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0$$

- **A, B, C, …, G are constant coefficients (may be generalized)**

- **The equation types are coordinate invariant, i.e., coordinate transformation will not change the type of equations**
- **Physical processes are independent of coordinates**

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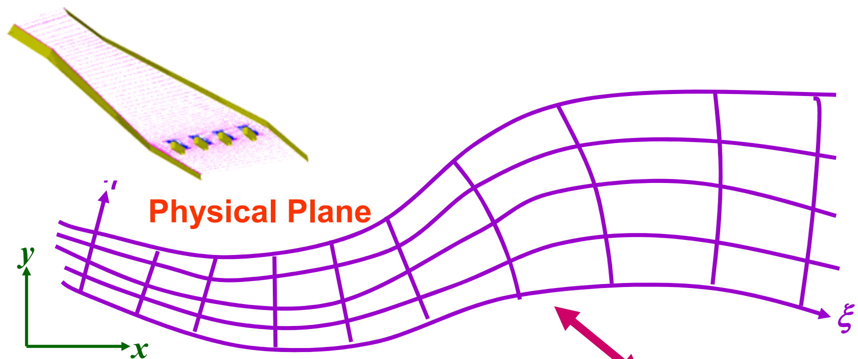
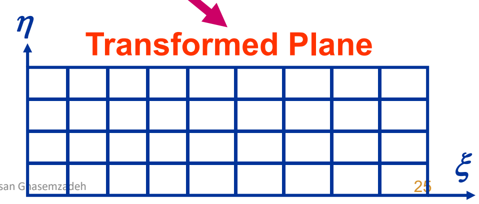
## Coordinate Transformation

- Physical plane ⇔ Transformed plane




$$\begin{cases}\xi = \xi(x,y)\\ \eta = \eta(x,y)\end{cases} \Leftrightarrow \begin{cases}x = x(\xi,\eta)\\ y = y(\xi,\eta)\end{cases}$$



## Classification of PDEs

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

$$\textbf{Classification}\quad \begin{cases} B^2 - 4AC < 0 : \textbf{elliptic}\\ B^2 - 4AC = 0 : \textbf{parabolic}\\ B^2 - 4AC > 0 : \textbf{hyperbolic}\end{cases}$$

- **The classification depends only on the highest-order derivatives (independent of D, E, F, G)**
- **For nonlinear problems [A,B,C = f(x,y,u)], the discriminant can still be used.**

# Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

**(1) Hyperbolic PDEs (Propagation)**

Advection equation

Wave equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0 & \text{(first - order)} \\ \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{(second} - \text{order)} \end{cases}$$

$$B^2 - 4AC = 4c^2 > 0 : \text{hyperbolic}$$



# Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

**(2) Parabolic PDEs (Time- or space-marching)**

Burger's equation

Fourier equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + u\dfrac{\partial \phi}{\partial x} = v\dfrac{\partial^2 \phi}{\partial x^2} \\ \dfrac{\partial \phi}{\partial t} = \alpha\dfrac{\partial^2 \phi}{\partial x^2} \end{cases}$$

Diffusion / dispersion

$$B^2 - 4AC = 0 : \text{parabolic}$$

# Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

**(3) Elliptic PDEs (Diffusion, equilibrium problems)**

$$\begin{cases} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 & \text{Laplace equation} \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y) & \text{Possion's equation} \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + c^2\phi = 0 & \text{Helmholtz equation} \end{cases}$$

$$B^2 - 4AC = -4 < 0 : \text{elliptic}$$

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# Classification of PDEs

- General form of second-order PDEs (2 variables)

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

**(4) Mixed-type PDEs**

Steady, compressible potential flow

$$(1 - M^2)\frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial n^2} = 0 \quad \begin{cases} M < 1 : & \text{subsonic} \\ M > 1 : & \text{supersonic} \end{cases}$$

$$-4(1 - M^2) = -4 + 4M^2 = 0 \Rightarrow M = 1 \quad \begin{cases} M < 1 : \text{elliptic} \\ M = 1 : \text{parabolic} \\ M > 1 : \text{hyperbolic} \end{cases}$$

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# Classification of PDEs

- **General form of second-order PDEs (2 variables)**

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

**(5) System of Coupled PDEs**

**Navier-Stokes Equations**

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) \end{cases}$$

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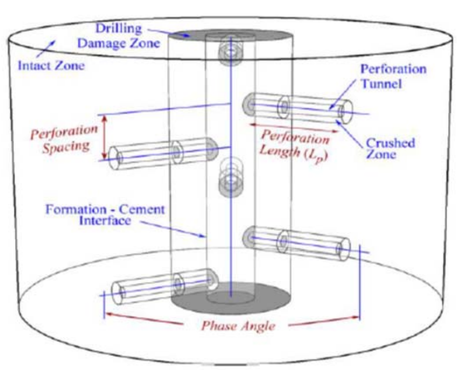
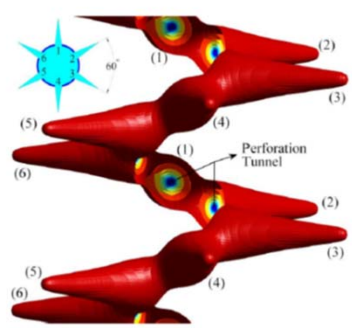
## تراوش اطراف چاه نفت

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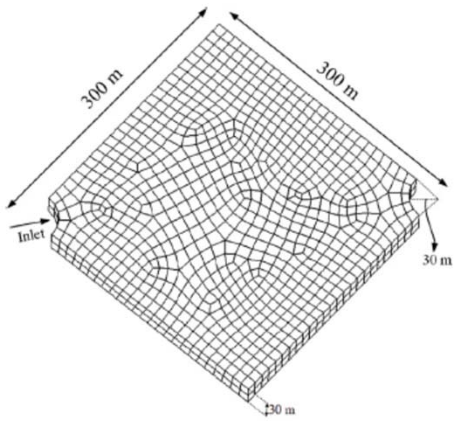
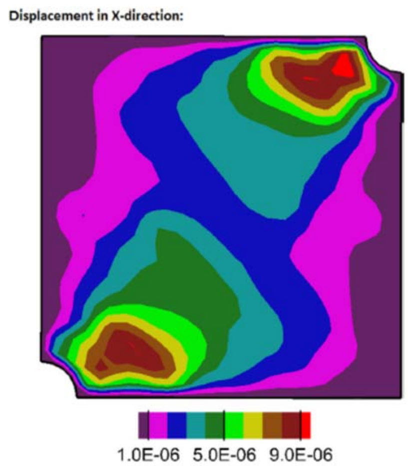
## تغییرشکل اطراف چاه نفت






## روش های حل معادلات

➢ روش های تحلیلی

معادلات ساده و معروف
معادلات خطی
شرایط مرزی ساده
جواب دقیق

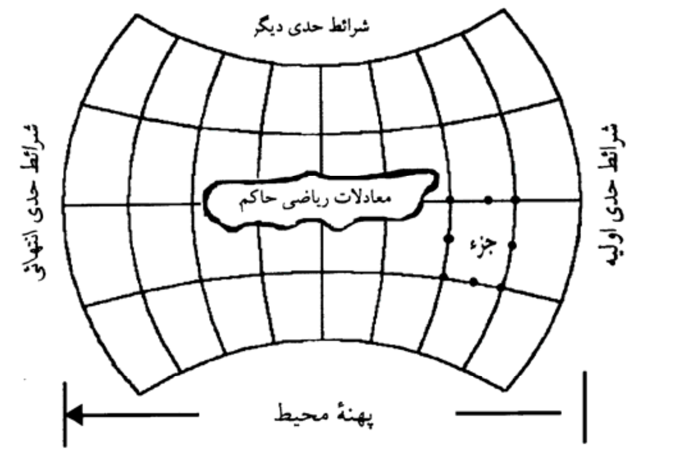
➢ روش های عددی

انواع معادلات
معادلات خطی و غیر خطی
شرایط مرزی پیچیده
جواب تقریبی

➢ نیازهای مهندسی

زمان حل مناسب
دقت مناسب
دانستن حدود تقریب

## نگرش ساده به عملکرد روش های عددی




در روشهای عددی معادلات دیفرانسیل حاکم بر محیط مساله به یکسری معادلات جبری ساده مبدل گردد. با حل عددی معادلات حاصله جواب با قدری خطا حاصل می گردد.

## نگرش ساده به عملکرد روش های عددی

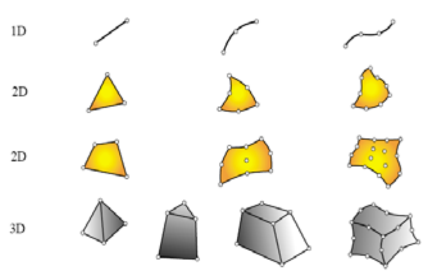
انواع روشها عددی

اختلاف محدود

- اجزا ی محدود
- اجزا ی مرزی
- احجام محدود

- اجزای مجزا

- بدون شبکه



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