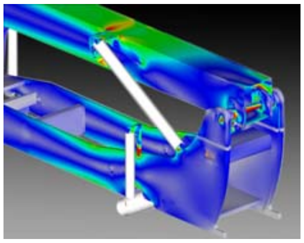
روش های عددی در ژئومکانیک

An Introduction to the Finite Element Analysis



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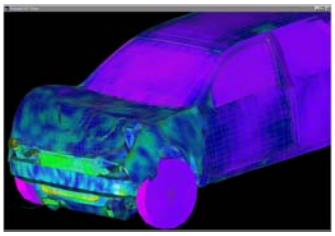

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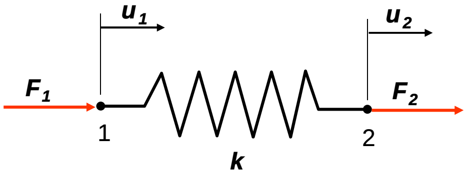
Simple Element Equation Example
Direct Stiffness Derivation



$$\text{Equilibrium at Node 1} \Rightarrow \; F_1 = ku_1 - ku_2$$
$$\text{Equilibrium at Node 2} \Rightarrow \; F_2 = -ku_1 + ku_2$$

or in Matrix Form

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Stiffness Matrix

Nodal Force Vector

$$[K]\{u\} = \{F\}$$

- For Problems in Structural Solid Mechanics, the Appropriate Physics Comes from Either Strength of Materials or Theory of Elasticity

One-Dimensional Bar Element

Approximation: $u=\sum_k \psi_k(x)u_k=[\mathbf{N}]\{u\}$

Strain: $\varepsilon=\frac{du}{dx}=\sum_k \frac{d}{dx}\psi_k(x)u_k=\frac{d[\mathbf{N}]}{dx}\{u\}=[\mathbf{B}]\{u\}$ $[B]=\frac{d[N]}{dx}=[D][N]$ $[D]=\frac{\partial}{\partial x}=\frac{d}{dx}$

Stress-Strain Law: $\sigma=E\varepsilon=E[\mathbf{B}]\{u\}$

$$\int_\Omega \sigma(\delta\varepsilon)\,dV=P_i(\delta u_i)+P_j(\delta u_j)+\int_\Omega f(\delta u)\,dV \Rightarrow$$

$$\{\boldsymbol{\delta}u\}^T\int_0^L A[\mathbf{B}]^T E[\mathbf{B}]dx\{u\}=\{\boldsymbol{\delta}u\}^T\begin{Bmatrix}P_i\\P_j\end{Bmatrix}+\{\boldsymbol{\delta}u\}^T\int_0^L A[\mathbf{N}]^T f dx \Rightarrow$$

$$\int_0^L A[\mathbf{B}]^T E[\mathbf{B}]dx\{u\}=\{\mathbf{P}\}+\int_0^L A[\mathbf{N}]^T f dx$$

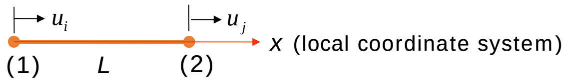
$$\boxed{[\mathbf{K}]\{u\}=\{\mathbf{F}\}}$$

$[K]=\int_0^L A[\mathbf{B}]^T E[\mathbf{B}]dx=$ Stiffness Matrix

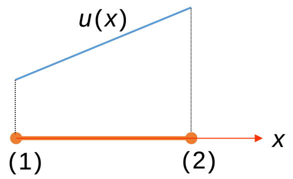
$\{\mathbf{F}\}=\begin{Bmatrix}P_i\\P_j\end{Bmatrix}+\int_0^L A[\mathbf{N}]^T f dx=$ Loading Vector

$\{u\}=\begin{Bmatrix}u_i\\u_j\end{Bmatrix}=$ Nodal Displacement Vector

Linear Approximation Scheme



Approximate Elastic Displacement



$u=a_1+a_2x \Rightarrow \begin{matrix}u_1=a_1\\u_2=a_1+a_2L\end{matrix}$

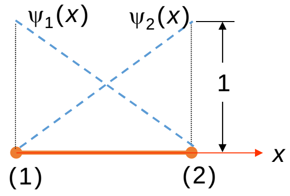
$\Rightarrow u=u_1+\frac{u_2-u_1}{L}x=\left(1-\frac{x}{L}\right)u_1+\left(\frac{x}{L}\right)u_2$

$=\psi_1(x)u_1+\psi_2(x)u_2$

$u=[\psi_1 \quad \psi_2]\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=\left[1-\frac{x}{L} \quad \frac{x}{L}\right]\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=[\mathbf{N}]\{u\}$

$[\mathbf{N}]=$ Approximation Function Matrix

$\{\mathrm{u}\}=$ Nodal Displacement Vector



$\psi_k(x)$ – Lagrange Interpolation Functions

Element Equation

Linear Approximation Scheme, Constant Properties

$$[\mathbf{N}]=\left[1-\frac{x}{L} \quad \frac{x}{L}\right] \qquad [D]=\frac{\partial}{\partial x}=\frac{d}{dx}$$

$$[B(x)]=[D]\{N(x)\}^T=\frac{d}{dx}\left\{1-\frac{x}{L} \quad \frac{x}{L}\right\}=\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}=\frac{1}{L}\{-1 \quad 1\}$$

$$[K]=\int_0^L A[\mathbf{B}]^T E[\mathbf{B}]dx=AE[\mathbf{B}]^T[\mathbf{B}]\int_0^L dx=AE\begin{Bmatrix}-\frac{1}{L}\\ \frac{1}{L}\end{Bmatrix}\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}L=\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\{\mathbf{F}\}=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\int_0^L A[\mathbf{N}]^T f dx=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+Af_o\int_0^L\begin{Bmatrix}-\frac{x}{L}\\ \frac{x}{L}\end{Bmatrix}dx=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

$$\{u\}=\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\text{Nodal Displacement Vector}$$

$$[\mathbf{K}]\{u\}=\{\mathbf{F}\}\Rightarrow\frac{AE}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}P_1\\ P_2\end{Bmatrix}+\frac{Af_oL}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

Example 1 **Deformation and stress in a compound bar made of two different materials**

Use the FEM to determine the displacements at the joint of a compound road made of copper and aluminum induced by a uniaxial force P = 30000 N at of the end of the rod as shown in the Figure A below:

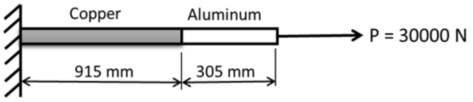


Figure A Compound Rod subjected to a Uniaxial Force

The compound rod has a cross-sectional area A = 650 mm² and the Young's moduli E_{cu} = 10300 MPa and E_{al} =69000 MPa.

Solution:

The situation shown in Figure A indicates that the rod is expected to elongate along the same direction in the x-axis, as shown in Figure B.

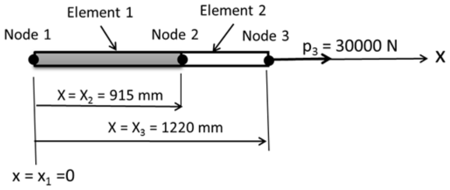
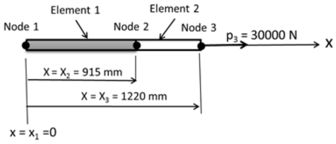


Figure B FE model for a Compound Bar

Example 1 – **Cont'd**



The FE model in Figure B indicates the following:

(1) There are total 3 nodes in the structure
(2) Nodal coordinates:
Node 1 at $x_1 = 0$; Node 2 at $x_2 = 915$ mm, and Node 3 at $x_3 = 1220$ mm
(3) The length of Element 1 = L_1 = 915 mm; the length of Element 2 = L_2 = 305 mm
(4) Both elements have a cross-sectional area: $A_1 = A_2 = A = 650$ mm^2
(5) Displacements at the 3 nodes are: $\{u\}^T = \{u_1 \quad u_2 \quad u_3\}$ with $u_1 = 0$

Our solution begins with developing the "element equations" for both elements in the FE model:

Element 1 made of copper:

Coefficient matrix for Element 1:

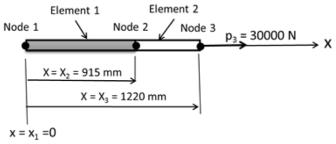
$$\left[K_e^1\right] = \frac{E_1 A_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{10300\times 650}{915\times 10^{-3}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 7.317 & -7.317 \\ -7.317 & 7.317 \end{bmatrix}\times 10^6 \; N/m \qquad \text{(a)}$$

Element equation for Element 1:

$$\begin{bmatrix} 7.317\times 10^6 & -7.317\times 10^6 \\ -7.317\times 10^6 & 7.317\times 10^6 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

where p_1 and p_2 are forces art Node 1 and 2 respectively

Example 1 – Cont'd



Element equation for Element 1:

$$\begin{bmatrix} 7.317\times 10^6 & -7.317\times 10^6 \\ -7.317\times 10^6 & 7.317\times 10^6 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}$$

Element 2 made of aluminum:

Coefficient matrix for element 2:

$$\left[K_e^2\right] = \frac{E_2 A_2}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{69000\times 650}{305\times 10^{-3}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 147.05 & -147.05 \\ -147.05 & 147.05 \end{bmatrix}\times 10^6 \; N/m \qquad \text{(b)}$$

Element equation for Element 2:

$$\begin{bmatrix} 147.05\times 10^6 & -147.05\times 10^6 \\ -147.05\times 10^6 & 147.05\times 10^6 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix}$$

Due to the fact that the present case involves TWO elements with Node 2 being common to both these element, we need to assemble the coefficient matrices by following the established rule by summing up the values of Node 2 from both element coefficient matrices.

Example 1 – Cont'd

We thus assemble the overall stiffness matrix by adding $\left[K_e^1\right]$ In Equation (a) and $\left[K_e^2\right]$ in Equation (b) in the following way:

$$\left[K\right] = 10^6 \begin{bmatrix} 7.317 & -7.317 & 0 \\ -7.317 & \left(\mathbf{7.317}+\mathbf{147.05}\right) & -147.05 \\ 0 & -147.05 & 147.05 \end{bmatrix} = 10^6 \begin{bmatrix} 7.317 & -7.317 & 0 \\ -7.317 & 154.367 & -147.05 \\ 0 & -147.05 & 147.05 \end{bmatrix} \qquad \text{(c)}$$

The numbers in boldface in Equation (c) are those associated with Node 2.

The overall stiffness equation for the bar structure with specified loading/boundary conditions is thus expressed in the following partitioned matrices as:

$$10^6 \left[\begin{array}{c:cc} 7.317 & -7.317 & 0 \\ \hdashline -7.317 & 154.367 & -147.05 \\ 0 & -147.05 & 147.05 \end{array}\right] \left\{\begin{array}{c} u_1 = 0 \\ \hdashline u_2 \\ u_3 \end{array}\right\} = \left\{\begin{array}{c} p_1 = ? \\ \hdashline p_2 = 0 \\ p_3 = 30000 \end{array}\right\} \qquad \text{(d)}$$

The two unknown nodal displacements u2 and u3 may be obtained by the following equations using the partitions in Equation (d):

$$10^6 \begin{bmatrix} 154.367 & -147.05 \\ -147.05 & 147.05 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30000 \end{Bmatrix} \qquad \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 4.100 \\ 4.304 \end{Bmatrix} \times 10^{-3}$$

$$p_1 = -7.317 \times 10^6 \times u_2 = -30000$$

Example 1 – Cont'd

Strain in Elements

Now that we have solved the displacements at the 3 nodes, we may use the train-displacement relations to determine the induced strains in both these elements:

$$\{\varepsilon\}^T = \{\varepsilon_{xx}^1 \quad \varepsilon_{xx}^2\}$$

$$\text{Strain: } \varepsilon = \frac{du}{dx} = [\mathbf{B}]\{u\}$$

$$\text{Stress-Strain Law: } \sigma = E\varepsilon = E[\mathbf{B}]\{u\}$$

where ε_{xx}^1 *and* ε_{xx}^2 *are the strains in Element* 1 *and* 2 *respectively*

$$\left[B(x)\right] = \left[D\right]\{N(x)\}^T = \frac{d}{dx}\left\{1-\frac{x}{L} \quad \frac{x}{L}\right\} = \left\{-\frac{1}{L} \quad \frac{1}{L}\right\} = \frac{1}{L}\{-1 \quad 1\}$$

We have: Node 1 at $x_1 = 0$; Node 2 at $x_2 = 915$ mm, and Node 3 at $x_3 = 1220$ mm, leading to: the length of Element 1 = L_1 = 915 mm; the length of Element 2 = L_2 = 305 mm. We may thus express the [B(x)] for both elements to be:

$$\left[B_1\right] = \frac{1}{915}\{-1 \quad 1\}, \text{ and } \left[B_2\right] = \frac{1}{305}\{-1 \quad 1\}$$

Example 1 – Cont'd

From the [B] matrices for both elements, we may compute the strains in Element 1 and 2 as follows:

$$\varepsilon_{xx}^{1} = \frac{1}{915}\{-1 \quad 1\}\begin{Bmatrix} u_1=0 \\ u_2 \end{Bmatrix} = \frac{1}{915}u_2 = \frac{4.1}{915} = 0.45\%$$ for element 1, and

$$\varepsilon_{xx}^{2} = \frac{1}{305}\{-1 \quad 1\}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = -\frac{u_2}{305} + \frac{u_3}{305} = -\frac{4.100}{305} + \frac{4.304}{305} = \frac{0.204}{305} = 0.067\%$$ for element 2

Stresses in elements

We may use the Hooke's Law to determine the stresses in each of these two elements from their corresponding strains

For element 1 with E_{cu} = 10300 MPa:

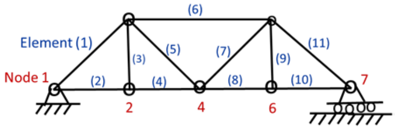
$$\sigma_{xx}^{1} = \left[D_1\right]\varepsilon_{xx}^{1} = E_{cu}\varepsilon_{xx}^{1} = 10300\times 0.0045 = 46.15MPa$$

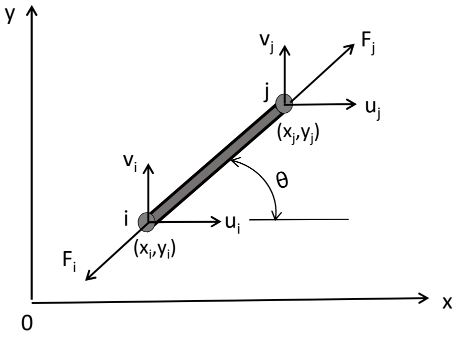
For element 2 with E_{al} = 69000 MPa:

$$\sigma_{xx}^{2} = \left[D_2\right]\varepsilon_{xx}^{2} = E_{al}\varepsilon_{xx}^{2} = 69000\times 0.00067 = 46.15MPa$$

FE Formulation of Inclined Truss Elements Using 1-D Bar Elements

Inclined truss bar element in x-y plane: Element (1), (5), (7) and (11):





The inclined truss bar elements such as shown in the figure at right may have TWO displacement components, but these elements can only elongate or contract in the LONGITUDINAL direction only.

The same applies to the induced strain and stress.

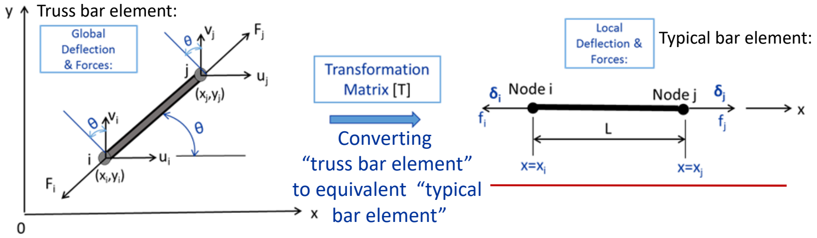
So, These are regarded as "a special bar elements."

These bar elements remain having two nodes: Node i and Node j
Located at specified coordinates (x_i, y_i) and (x_j, y_j) respectively.

Each node has two displacement components:
u_i and v_i = displacement of Node i in respective x- and y- directions
u_j and v_j = displacement of Node j in respective x- and y-directions
F_i and F_j are the forces acting at Node I and Node j respectively

FE Formulation of Truss Elements Using 1-D Bar Elements – Cont'd

Transformation of Nodal displacement components: From the "global coordinates" to "Local coordinates"- The latter is used for the FE formulation as with 1-D bar elements



Converting "truss bar element" to equivalent "typical bar element"

By referring to the above figure, we have the following relationship in transforming the nodal displacement components from The global coordinates (x,y) in the left figure to the local coordinate (x) in the right of the figure:

at Node i: $\delta_i = u_i \cos\theta + v_i \sin\theta = cu_i + sv_i$

at Node j: $\delta_j = u_j \cos\theta + v_j \sin\theta = cu_j + sv_j$

in matrix form: $$\underline{\{\delta\}} = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underline{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

where c = cosθ and s = sinθ

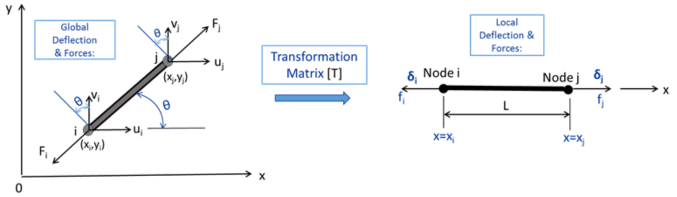
or in a shorthand version of: $\{\delta\} = [T]\{u\}$

$$[T] = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \qquad c = \cos\theta \quad s = \sin\theta$$

with the matrix [T] being the transformation matrix

FE Formulation of Truss Elements Using 1-D Bar Elements – Cont'd

Transformation of Nodal force components: From "global coordinates" to "Local coordinates"



Transformation of nodal forces can be done in a similar way as we did with nodal displacements. However, it would be easier to do it using the work done to the element by these forces, as work done is a scalar quantity, which is easier to transform in space than the vector quantities such as displacements.

Since the work done to the element by nodal forces may be expressed as: $W = \{\delta\}^T \{f\} = \{u\}^T \{F\}$ for the same element in both local and global coordinate systems, or in a long-hand form:

$$W = \{\delta_1 \quad \delta_2\} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \{u_i \quad v_i \quad u_j \quad v_j\} \begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix} \quad or \quad \{\delta\}^T \{f\} = \{u\}^T \{F\}$$

$$\Rightarrow ([T]\{u\})^T \{f\} = \{u\}^T \{F\}$$

$$\Rightarrow \{u\}^T [T]^T \{f\} = \{u\}^T \{F\}$$

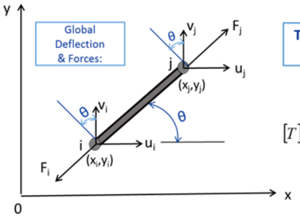
$$[T]^T \{f\} = \{F\} \,,\; \{f\} = \left([T]^T\right)^{-1} \{F\}$$

$$[T]^T = [T]^{-1} \Rightarrow \qquad \{f\} = [T]\{F\}$$

where **{f} = nodal forces in "typical bar element", {F} = nodal forces in "inclined truss element"**

FE Formulation of Truss Elements Using 1-D Bar Elements – Cont’d

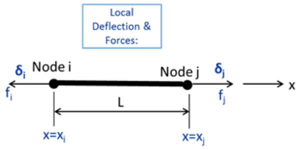
Element equation in Local Coordinate System



Transformation Matrix:

$$[T]=\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

(4.26a)



$$\{N(x)\} = \left\{1-\frac{x}{L} \quad \frac{x}{L}\right\}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} -f_i \\ f_j \end{Bmatrix}$$

$[K_e]_l\{\delta\}=\{f\}$ with $[K_e]_l = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix}$

Relationship of nodal forces between the two coordinate systems: {F} in global coordinate system = $[T]^T\{f\}$ in Local coordinate system .
The element equation in global coordinate system thus have the form:

$$[K_e]_l\{\delta\} = \{f\} \qquad \begin{matrix} \{f\} = [T]\{F\} \\ \{\delta\} = [T]\{u\} \end{matrix} \qquad \Rightarrow \quad [K_e]_l[T]\{u\} = [T]\{F\}$$

For transformation matrix $[T]^T = [T]^{-1} \Rightarrow \quad [T]^{-1}[K_e]_l[T]\{u\} = [T]^{-1}[T]\{F\} \Rightarrow$

The element Equations for inclined truss members: $[T]^{-1}[K_e]_l[T]\{u\} = \{F\}$

$$[T]=\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

$$[T]^T[K_e]_l[T] = [K_e]$$

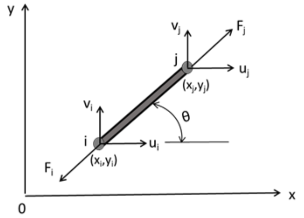
$$[K_e]\{u\} = \{F\}$$

$c = \cos\theta$
$s = \sin\theta$

FE Formulation of Truss Elements Using 1-D Bar Elements – Cont’d

Element equation for Inclined Truss Members

$$[K_e]\{u\}=\{F\}$$

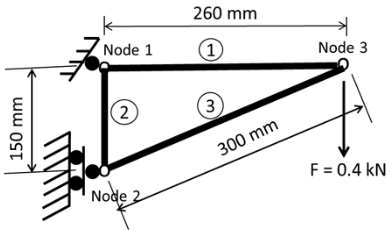


where $[K_e]=\left(\frac{AE}{L}\right)\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \quad \{u\}=\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \quad \{F\}=\begin{Bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \end{Bmatrix}$

The stiffness matrix: $[K_e]=\left(\frac{AE}{L}\right)\begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix}$ $\quad k^e = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$

1) This stiffness matrix represents the stiffness of a single element with incline angle θ
2) It is symmetric about the diagonal line of the square matrix
3) Since there 4 unknown nodal displacements - meaning 4 degree-of-freedom (dof), the matrix is of the size of (4x4)
4) The terms c and s represent the cosine and sine values of the orientation of the element with the horizontal plane, rotated in a counter clockwise direction (+ve direction)

Example 2 **FE Stress Analysis of a Truss Structure:**



A truss with 3 members joined by frictionless hinges as shown in the figure. Element ① and ② are made of aluminum, and element ③ is made of steel.

	Materials	Cross-sectional Area (A) (10^{-6}m^2)	Young's Modulus (E) (10^6 N/m^2)	Yield Strength (σ_y) (10^6 N/m^2)
Member (1)	Aluminum	200	70,000	170
Member (2)	Aluminum	200	70,000	170
Member (3)	Steel	100	200,000	21,000

Solution:

We realize the fact that there are 3 members, each has its own dimension, and material with properties listed in the above table. So, we may conveniently construct the FE model of the truss structure by using 3 elements ① with node pair 1-3, element ② with node pair 2-1, and element ③ with node pair 2-3, as shown in the figure.

We will derive the element equations for all these 3 elements by using the formulations for truss elements

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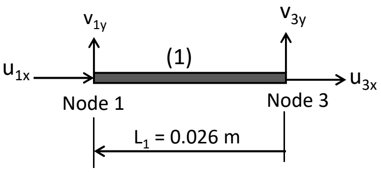
Example 2 FE Stress Analysis of a Truss Structure - cont'd

Derivation of element equations

Element 1: L_1 = 0.26 m, E_1 = 70000 MPa, A_1 =200x10^{-6} m^2:



Being horizontal, we have the incline angle θ = 0, which leads to:
c = cosθ=1, c^2 = 1, s=sinθ = 0, s^2 = 0 and cs = 0

We also calculate the coefficient of the stiffness matrix of element 1 as follows:

$$\frac{E_1A_1}{L_1}=\frac{(70000\times10^6)(200\times10^{-6})}{0.26}=53.84\times10^6 \quad N/m$$

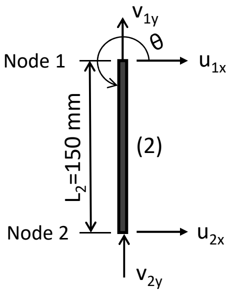
the stiffness matrix for Element 1 is thus:

$$[K_e^1]=\frac{E_1A_1}{L_1}\times\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}=53.84\times10^6\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(a)}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Derivation of element equations

Element 2: L_2 = 0.15 m, E_2 = 70000 MPa, A_2 =200x10^{-6} m^2:



In this Element 2, we have the inclined angle θ=270, which lead to:

c = cos 270° = 0, c^2 = 0; s = sin 270° = -1, s^2 = 1 and cs = 0 , and

$$\frac{E_2 A_2}{L_2}=\frac{\left(70000\times10^{6}\right)\left(200\times10^{-6}\right)}{0.15}=31.06\times10^{6}\quad N/m$$

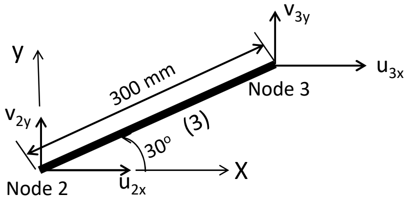
Hence the stiffness matrix for Element 2 is:

$$\left[K_e^2\right]=\frac{E_2 A_2}{L_2}\times\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \qquad \left[K_e^2\right]=31.06\times10^{6}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Derivation of element equations

Element 3: L_3 = 0.3 m, E_3 = 200,000 MPa, A_3 =100x10^{-6} m^2:



Element 3 has an inclined angle θ = 30° leading to:

c= cos 30° = 0.866, c^2 = 0.75, s = sin 30° = 0.5, s^2 = 0.25 and cs = 0.433;

Also the coefficient of the stiffness matrix:

$$\frac{E_3 A_3}{L_3}=\frac{\left(200000\times10^{6}\right)\left(100\times10^{-6}\right)}{0.3}=66.67\times10^{6}\quad N/m$$

The stiffness matrix for element 3 is:

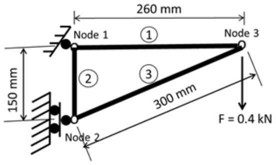
$$\left[K_e^3\right]=\frac{E_3 A_3}{L_3}\times\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}=66.67\times10^{6}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix} \qquad \text{(c)}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Derivation of element equations

From Equation (4.21): $[K_e]\ \{\Phi\} = \{q\}$ with a general expression for element equations, we are now in the position to express the same equations for the 3 elements in the current truss structure as follows.

The 3 Element equations for the truss structure:



Element (1) with Node 1 and 3

$$10^6\begin{bmatrix} 53.84 & 0 & -53.84 & 0 \\ 0 & 0 & 0 & 0 \\ -53.84 & 0 & 53.84 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{1x} \\ v_{1y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

Element (2) with Node 1 and 2

$$10^6\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 31.6 & 0 & -31.6 \\ 0 & 0 & 0 & 0 \\ 0 & -31.6 & 0 & 31.6 \end{bmatrix}\begin{Bmatrix} u_{1x} \\ v_{1y} \\ u_{2x} \\ v_{2y} \end{Bmatrix}=\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

Element (3) with Node 2 and 3

$$10^6\begin{bmatrix} 50 & 28.87 & -50 & -28.86 \\ 28.87 & 16.67 & -28.87 & -16.67 \\ -50 & -28.87 & 50 & 28.87 \\ -28.87 & -16.67 & 28.87 & 16.67 \end{bmatrix}\begin{Bmatrix} u_{2x} \\ v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=\begin{Bmatrix} f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix}$$

in which u_{1x}, u_{2x} and u_{3x} = displacement components in x-direction of Node 1, 2 and 3 respectively
v_{1y}, v_{y2} and v_{3y} = displacement components in y-direction of Node 1, 2 and 3 respectively
f_{1x}, f_{2x} and f_{3x} = Nodal force at Node 1, 2 and 3 respectively
f_{1y}, f_{2y} and f_3y = Nodal force at Node 1, 2 and 3 respectively

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Example 2 FE Stress Analysis of a Truss Structure - cont'd

Assembly of element equations for Overall Stiffness Equation

By following the description on "assembly of element stiffness matrices" in Step 5 with diagram of Chapter 3 Steps in Finite Element Analysis, we may assemble the three (3) truss element stiffness matrices shown above in the following form:

$$\begin{array}{cc} & \begin{array}{cccccc} u_{1x} & v_{1y} & u_{2x} & v_{2y} & u_{3x} & v_{3y} \end{array} \\ [K]=10^6 & \begin{bmatrix} 53.84(1) & 0 & 0 & 0 & -53.84(1) & 0 \\ 0 & 31.6(2) & 0 & -31.6(2) & 0 & 0 \\ 0 & 0 & 50(3) & 28.87(3) & -50(3) & -28.87(3) \\ 0 & -31.6(3) & 28.87(3) & \begin{bmatrix} [31.6(2) \\ +16.67(3)] \\ =48.27 \end{bmatrix} & -28.87(3) & -16.67(3) \\ -53.8(1) & 0 & -50(3) & -28.87(3) & \begin{bmatrix} [53.84(1) \\ +50(3)] \\ =103.84 \end{bmatrix} & 28.87(3) \\ 0 & 0 & -28.87(3) & -16.67(3) & 28.87(3) & 16.67(3) \end{bmatrix} \begin{array}{c} u_{1x} \\ v_{1y} \\ u_{2x} \\ v_{2y} \\ u_{3x} \\ v_{3y} \end{array} \end{array}$$

or in a neat form as shown in the next slide:

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Example 2 FE Stress Analysis of a Truss Structure - cont'd

Element stiffness matrix and Overall Stiffness Equation of the Truss Structure

The overall stiffness matrix of the truss structure:

$$[K]=10^6\begin{bmatrix} 53.84 & 0 & 0 & 0 & -53.84 & 0 \\ 0 & 31.6 & 0 & -31.6 & 0 & 0 \\ 0 & 0 & 50 & 28.87 & -50 & -28.87 \\ 0 & -31.6 & 28.87 & 48.27 & -28.87 & -16.67 \\ -53.8 & 0 & -50 & -28.87 & 103.84 & 28.87 \\ 0 & 0 & -28.87 & -16.67 & 28.87 & 16.67 \end{bmatrix} \qquad \text{(d)}$$

With this overall stiffness matrix, we may establish the overall stiffness equation for the truss structure as shown below:

$$10^6\begin{bmatrix} 53.84 & 0 & 0 & 0 & -53.84 & 0 \\ 0 & 31.6 & 0 & -31.6 & 0 & 0 \\ 0 & 0 & 50 & 28.87 & -50 & -28.87 \\ 0 & -31.6 & 28.87 & 48.27 & -28.87 & -16.67 \\ -53.8 & 0 & -50 & -28.87 & 103.84 & 28.87 \\ 0 & 0 & -28.87 & -16.67 & 28.87 & 16.67 \end{bmatrix}\begin{Bmatrix} u_{1x} \\ v_{1y} \\ u_{2x} \\ v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \end{Bmatrix} \qquad \text{(e)}$$

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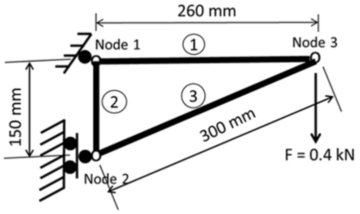
Example 2 FE Stress Analysis of a Truss Structure - cont'd

Boundary and Loading conditions of the Truss Structure



We recognize the following specified conditions for the discretized truss structures:

The boundary conditions (displacement at nodes):

1) With Node 1 being completely fixed: $u_{1x} = v_{1y} = 0$
2) Node 2 is allowed to move in vertical (y) direction: $u_{2x} = 0$

The loading conditions:
All except Node 3 has one force acting in the y-direction:
$f_{1x} = f_{1y} = f_{2x} = f_{2y} = f_{3x} = 0$ and $f_{3y} = 0.4$ kN = 400 N

The overall stiffness equation in Equation (e) after the substitution of the above specified boundary and loading conditions has the form:

$$10^6\begin{bmatrix} 53.84 & 0 & 0 & 0 & -53.84 & 0 \\ 0 & 31.6 & 0 & -31.6 & 0 & 0 \\ 0 & 0 & 50 & 28.87 & -50 & -28.87 \\ 0 & -31.6 & 28.87 & 48.27 & -28.87 & -16.67 \\ -53.8 & 0 & -50 & -28.87 & 103.84 & 28.87 \\ 0 & 0 & -28.87 & -16.67 & 28.87 & 16.67 \end{bmatrix}\begin{Bmatrix} u_{1x}=0 \\ v_{1y}=0 \\ u_{2x}=0 \\ v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=\begin{Bmatrix} f_{1x}=? \\ f_{1y}=? \\ f_{2x}=? \\ f_{2y}=0 \\ f_{3x}=0 \\ f_{3y}=-400 \end{Bmatrix}$$

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Example 2 FE Stress Analysis of a Truss Structure - cont'd

Overall Structure Stiffness Equation with given Boundary and Loading Conditions for the Truss Structure

$$10^6\begin{bmatrix} 53.84 & 0 & 0 & 0 & -53.84 & 0 \\ 0 & 31.6 & 0 & -31.6 & 0 & 0 \\ 0 & 0 & 50 & 28.87 & -50 & -28.87 \\ 0 & -31.6 & 28.87 & 48.27 & -28.87 & -16.67 \\ -53.8 & 0 & -50 & -28.87 & 103.84 & 28.87 \\ 0 & 0 & -28.87 & -16.67 & 28.87 & 16.67 \end{bmatrix}\begin{Bmatrix} u_{1x}=0 \\ v_{1y}=0 \\ u_{2x}=0 \\ v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=\begin{Bmatrix} f_{1x}=? \\ f_{1y}=? \\ f_{2x}=? \\ f_{2y}=0 \\ f_{3x}=0 \\ f_{3y}=-400 \end{Bmatrix} \qquad \text{(f)}$$

we may partition Equation (f) in the following way:

$$\left[\begin{array}{c|c} K_{aa} & K_{ab} \\ \hline K_{ba} & K_{bb} \end{array}\right]\left\{\begin{array}{c} q_a \\ \hline q_b \end{array}\right\}=\left\{\begin{array}{c} R_a \\ \hline R_b \end{array}\right\}$$

where $\{q_a\}$ = specified (known) nodal quantities;
$\{R_b\}$ = specified (known) applied resulting actions, from which we may obtain:

$$\{q_b\} = [K_{bb}]^{-1}\ (\{R_b\} - [K_{ba}]\{q_a\})$$

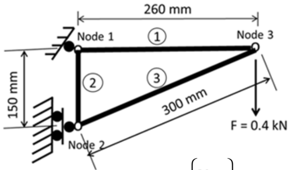
$$10^6\left[\begin{array}{ccc|ccc} 53.84 & 0 & 0 & 0 & -53.84 & 0 \\ 0 & 31.6 & 0 & -31.6 & 0 & 0 \\ 0 & 0 & 50 & 28.87 & -50 & -28.87 \\ \hline 0 & -31.6 & 28.87 & 48.27 & -28.87 & -16.67 \\ -53.8 & 0 & -50 & -28.87 & 103.84 & 28.87 \\ 0 & 0 & -28.87 & -16.67 & 28.87 & 16.67 \end{array}\right]\left\{\begin{array}{c} u_{1x}=0 \\ v_{1y}=0 \\ u_{2x}=0 \\ \hline v_{2y} \\ u_{3x} \\ v_{3y} \end{array}\right\}=\left\{\begin{array}{c} f_{1x}=? \\ f_{1y}=? \\ f_{2x}=? \\ \hline f_{2y}=0 \\ f_{3x}=0 \\ f_{3y}=-400 \end{array}\right\}$$

$$\{q_a\}=\{0\} \qquad \{R_a\}=\{?\}$$

$$\{q_b\}=\begin{Bmatrix} v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix} \qquad \{R_b\}=\begin{Bmatrix} 0 \\ 0 \\ -400 \end{Bmatrix} \qquad [K_{ba}]=10^6\begin{bmatrix} 0 & -31.6 & 2887 \\ -538 & 0 & -50 \\ 0 & 0 & -2887 \end{bmatrix} \qquad [K_{bb}]=10^6\begin{bmatrix} 48.27 & -28.87 & -16.67 \\ -28.87 & 103.84 & 28.87 \\ -16.67 & 28.87 & 16.67 \end{bmatrix}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Solution of 3 unknown nodal displacements in truss structure



We will get the 3 unknown nodal displacements from the portioned over stiffness equation by the expression: $\{q_b\} = [K_{bb}]^{-1}\ \{R_b\}$:

$$\begin{Bmatrix} v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=10^{-6}\begin{bmatrix} 48.27 & -28.87 & -16.67 \\ -28.87 & 103.84 & 28.87 \\ -16.67 & 28.87 & 16.67 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 0 \\ -400 \end{Bmatrix}=10^{-8}\begin{bmatrix} 3.1646 & 867.36\times10^{-19} & 3.1646 \\ 867.36\times10^{-19} & 1.86 & -3.2166 \\ 3.1646 & -3,2166 & 14.734 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -400 \end{Bmatrix}$$

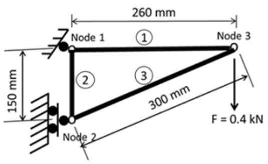
Solve for the unknown displacement components at Node 2 and 3:

$u_{2y} = -12.65\text{x}10^{-6}$ m, $u_{3x} = 12.86\text{x}10^{-6}$ m and $v_{3y} = -58.94\text{x}10^{-6}$ m with negative signs meaning downward direction

$$\begin{Bmatrix} v_{2y} \\ u_{3x} \\ v_{3y} \end{Bmatrix}=10^{-6}\begin{Bmatrix} -12.65 \\ 12.86 \\ -58.94 \end{Bmatrix}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

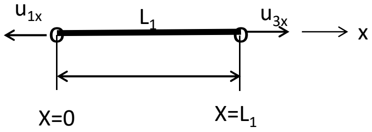
Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure



We will determine the strain components and then stress components in each of the 3 elements in the truss structure.

We should bear in mind that truss members are "two force members," in which only the **in-line displacement components** will produce strains and stresses.

Element 1: with Node 1 and Node 3



We have the relationship between the element displacement U(x) and the corresponding nodal displacement components $\{u_{1x} \quad u_{3x}\}^T$ along the line of the bar element in the figure I the left:

$$U(x)=[N(x)]\{u\}=\left\{1-\frac{x}{L_1} \quad \frac{x}{L_1}\right\}\begin{Bmatrix}u_{1x}\\u_{3x}\end{Bmatrix}=\left(1-\frac{x}{L_1}\right)u_{1x}+\frac{x}{L_1}u_{3x}$$

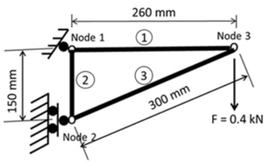
$$\{\varepsilon(x)\}=\varepsilon_{xx}=[B(x)]\{u\}$$

[B(x)] = [*D*][N(x)], with $[D]=\frac{\partial}{\partial x}=\frac{d}{dx}$

$$[B(x)]=\frac{d}{dx}\left\{1-\frac{x}{L_1} \quad \frac{x}{L_1}\right\}$$

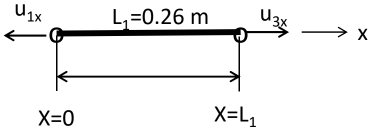
Example 2 FE Stress Analysis of a Truss Structure - cont'd

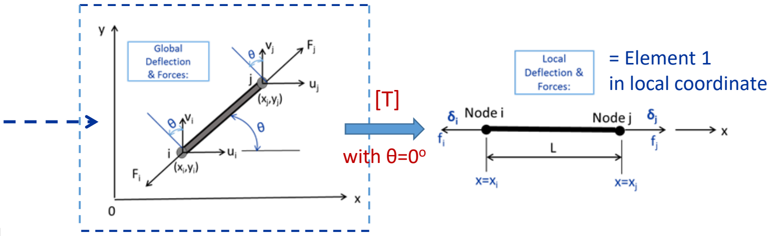
Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure



We should bear in mind that truss members are "two force members," in which only the **in-line displacement components** will produce strains and stresses.

Element 1: with Node 1 and Node 3 in global coordinates





[T]

with θ=0°

= Element 1 in local coordinate

The transformation matrix: $[T]=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}$

The relationship of nodal displacements in Local coordinate system and the corresponding global coordinate systems is: {δ} = [T]{u}, or:

$$\{\delta\}=\begin{Bmatrix}\delta_i\\\delta_j\end{Bmatrix}=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\begin{Bmatrix}u_i\\v_i\\u_j\\v_j\end{Bmatrix}$$

But we obtained the nodal displacements from the previous calculations to be:

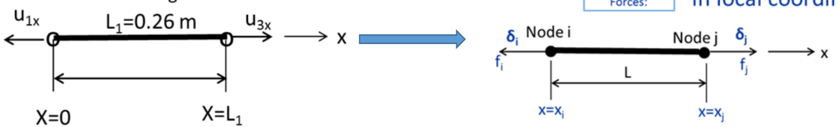
$u_i = u_{1x} =0$, $v_i = v_{1x} =0$, $u_j = u_{3x} = 12.86\text{x}10^{-6}$m, $v_j = v_{3y} = -58.94\text{x}10^{-6}$m

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure

Element 1: with Node 1 and Node 3 in global coordinates

Local Deflection & Forces: = Element 1 in local coordinate



$$\{\delta\}=\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\ 0\\ 12.86\times10^{-6}\\ -58.94\times10^{-6}\end{Bmatrix}=\begin{Bmatrix}0\\ 12.86\times10^{-6}\end{Bmatrix}$$

We have the strain in the element {ϵ} = [B(x)]{Φ}, and [B(x)] = [*D*]{N(x)}

$$[B]=[D]\{N(x)\}=\frac{d}{dx}\left\{1-\frac{x}{L_1}\quad \frac{x}{L_1}\right\}=\left\{-\frac{1}{L_1}\quad \frac{1}{L_1}\right\}$$

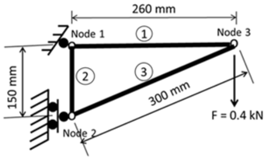
The strain in Element 1 is thus:

$$\{\varepsilon_{xx}\}=\left\{-\frac{1}{L_1}\quad \frac{1}{L_1}\right\}\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\left\{-\frac{1}{0.26}\quad \frac{1}{0.26}\right\}\begin{Bmatrix}0\\ 12.86\times10^{-6}\end{Bmatrix}=49.46\times10^{-6}\ m/m$$

The corresponding stress in Element 1: {σ}=[C]{ϵ}=(70000x10^6)(49.46x10^{-6})=3.43 MPa <170 MPa O.K.

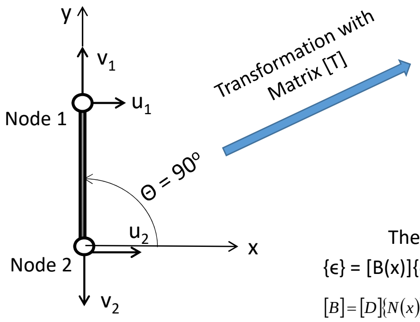
Example 2 FE Stress Analysis of a Truss Structure - cont'd

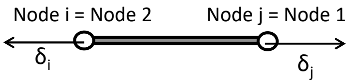
Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure - Cont'd



Element 2: with Node 1 and Node 2

We have computed that $u_1 = v_1 = 0$, $u_2 = 0$ and $v_2 = u_{2y} = -12.65\text{x}10^{-6}$ m





$$\{\delta\}=\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\begin{bmatrix}c & s & 0 & 0\\ 0 & 0 & c & s\end{bmatrix}\begin{Bmatrix}u_i\\ v_i\\ u_j\\ v_j\end{Bmatrix}$$

We have : c = cos 90° =0, and s = sin 90° = 1.0, leading to:

$$\{\delta\}=\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}u_i=u_2=0\\ v_i=v_2=u_{2y}=-12.65\times10^{-6}\\ u_j=u_1=0\\ v_j=v_1=0\end{Bmatrix}$$

The above expression leads to: δ_i = -12x10^{-6} m and δ_j = 0 or: $\{\delta\}=\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\begin{Bmatrix}-12\times10^{-6}\\ 0\end{Bmatrix}$

{ϵ} = [B(x)]{δ}, and [B(x)] = [*D*]{N(x)}

$$[B]=[D]\{N(x)\}=\frac{d}{dx}\left\{1-\frac{x}{L_2}\quad \frac{x}{L_2}\right\}=\left\{-\frac{1}{L_2}\quad \frac{1}{L_2}\right\}$$

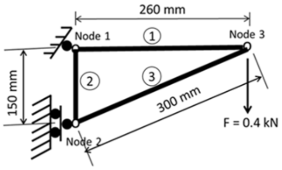
The strain in Element 2 is thus: $\{\varepsilon_{xx}\}=\left\{-\frac{1}{L_2}\quad \frac{1}{L_2}\right\}\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\left\{-\frac{1}{0.15}\quad \frac{1}{0.15}\right\}\begin{Bmatrix}-12\times10^{-6}\\ 0\end{Bmatrix}=80\times10^{-6}\ m/m$

The corresponding stress in Element 2: {σ}=[C]{ϵ}=(70000x10^6)(80x10^{-6})=5.6 MPa <170 MPa O.K.

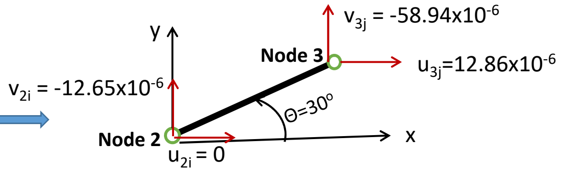
Example 2 FE Stress Analysis of a Truss Structure - cont'd

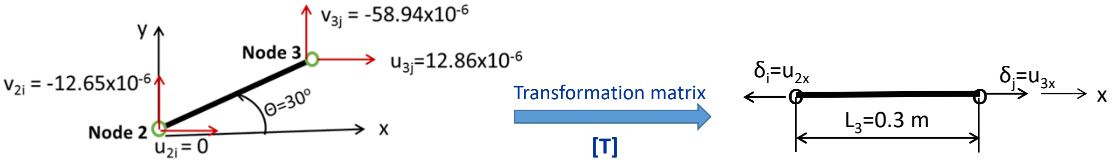
Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure - Cont'd

Element 3: with Node 2 and Node 3



We obtained the nodal Displacements from The previous calculation to be as shown





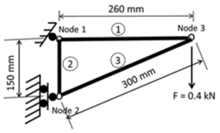
The transformation matrix [T] has the following form with θ = 30°:

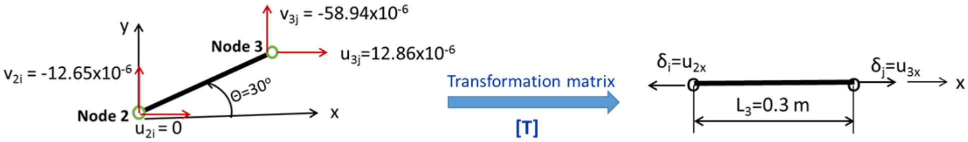
$$[T]=\begin{bmatrix}\cos 30^o & \sin 30 & 0 & 0\\ 0 & 0 & \cos 30^o & \sin 30^o\end{bmatrix}=\begin{bmatrix}0.866 & 0.5 & 0 & 0\\ 0 & 0 & 0.866 & 0.5\end{bmatrix}$$

Example 2 FE Stress Analysis of a Truss Structure - cont'd

Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure - Cont'd

Element 3: with Node 2 and Node 3



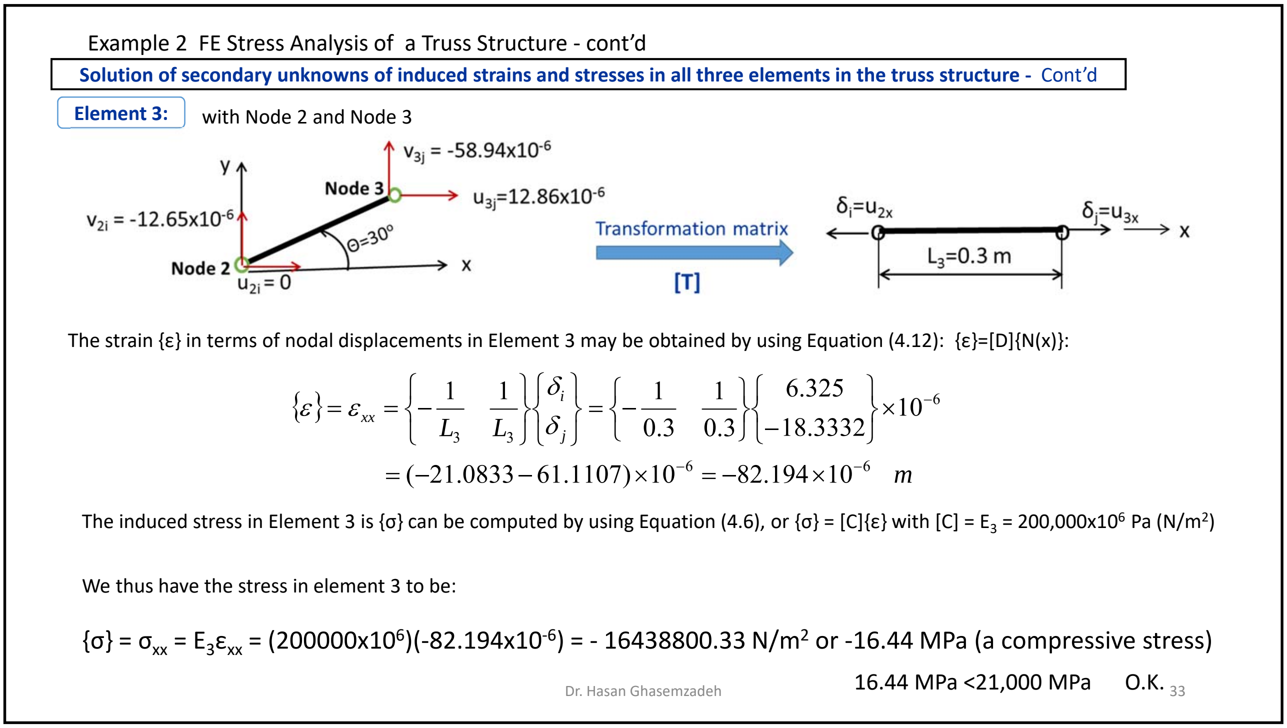


The nodal displacements in the equivalent Element 3 in the Local coordinates can be related to the real Element 3 in the global coordinates by the transformation matrix to be:

$$\{\delta\}=\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\begin{bmatrix}0.866 & 0.5 & 0 & 0\\ 0 & 0 & 0.866 & 0.5\end{bmatrix}\begin{Bmatrix}u_i=u_{2i}=0\\ v_i=v_{2i}=-12.65\times10^{-6}\\ u_j=u_{3j}=12.86\times10^{-6}\\ v_j=v_{3j}=-58.94\times10^{-6}\end{Bmatrix}=\begin{Bmatrix}6.325\times10^{-6}\\ (11.1368-29.47)10^{-6}\end{Bmatrix}=\begin{Bmatrix}6.325\\ -18.3332\end{Bmatrix}x10^{-6}$$

But since the interpolation function of the equivalent Element 3 in the local coordinate is: $\{N(x)\}=\left\{1-\frac{x}{L_3} \quad \frac{x}{L_3}\right\}$

And the matrix [B(x)] = [*D*][N(x,y,z)] : $$[B(x)]=\frac{d}{dx}\left\{1-\frac{x}{L_3} \quad \frac{x}{L_3}\right\}=\left\{-\frac{1}{L_3} \quad \frac{1}{L_3}\right\}$$



Example 2 FE Stress Analysis of a Truss Structure - cont'd

Solution of secondary unknowns of induced strains and stresses in all three elements in the truss structure - Cont'd

Element 3: with Node 2 and Node 3

The strain {ε} in terms of nodal displacements in Element 3 may be obtained by using Equation (4.12): {ε}=[D]{N(x)}:

$$\{\varepsilon\} = \varepsilon_{xx} = \left\{ -\frac{1}{L_3} \quad \frac{1}{L_3} \right\} \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = \left\{ -\frac{1}{0.3} \quad \frac{1}{0.3} \right\} \left\{ \begin{matrix} 6.325 \\ -18.3332 \end{matrix} \right\} \times 10^{-6}$$

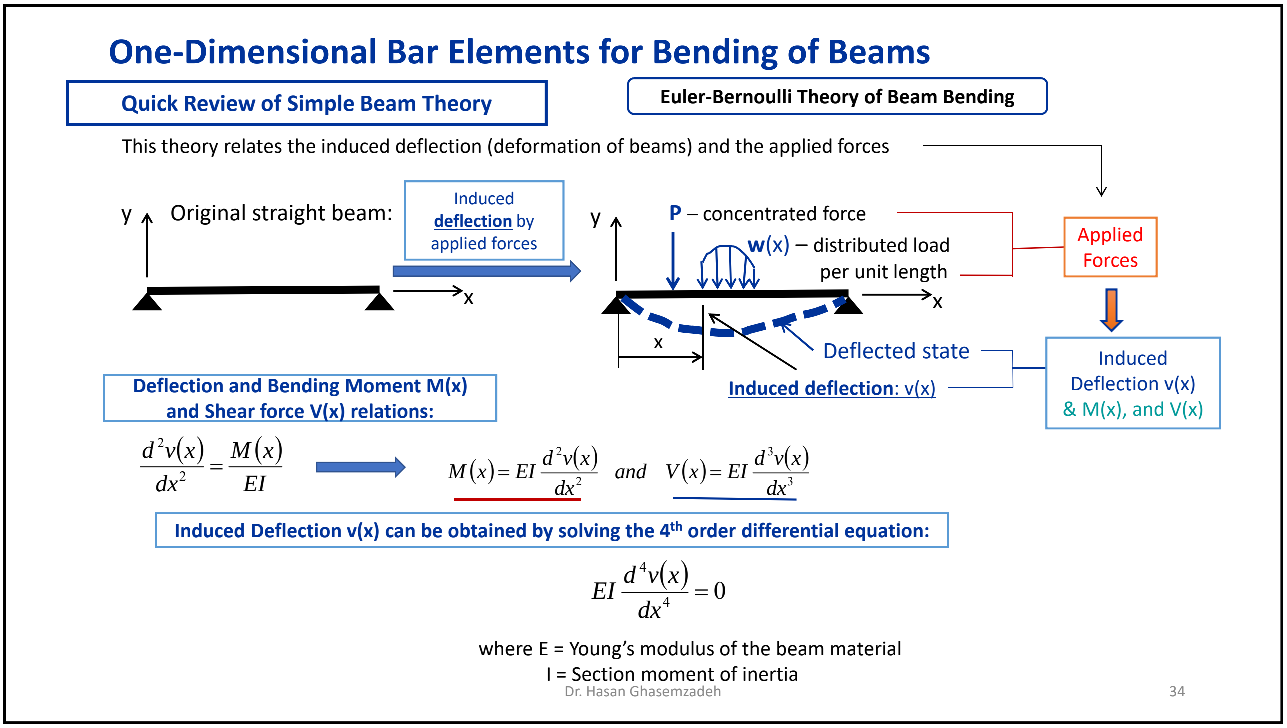
$$= (-21.0833 - 61.1107) \times 10^{-6} = -82.194 \times 10^{-6} \quad m$$

The induced stress in Element 3 is {σ} can be computed by using Equation (4.6), or {σ} = [C]{ε} with [C] = E_3 = 200,000x10^6 Pa (N/m^2)

We thus have the stress in element 3 to be:

{σ} = σ_{xx} = $E_3\varepsilon_{xx}$ = (200000x10^6)(-82.194x10^{-6}) = - 16438800.33 N/m^2 or -16.44 MPa (a compressive stress)

16.44 MPa <21,000 MPa O.K.



One-Dimensional Bar Elements for Bending of Beams

Quick Review of Simple Beam Theory

Euler-Bernoulli Theory of Beam Bending

This theory relates the induced deflection (deformation of beams) and the applied forces

Deflection and Bending Moment M(x) and Shear force V(x) relations:

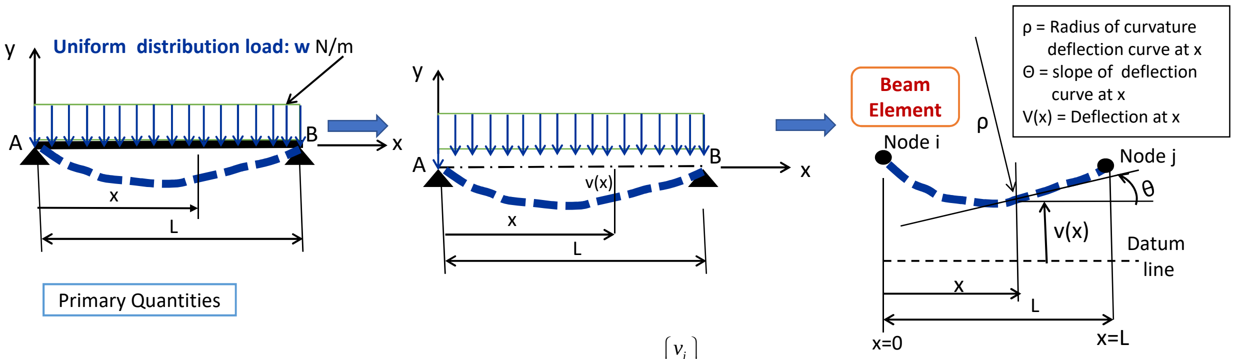
$$\frac{d^2v(x)}{dx^2} = \frac{M(x)}{EI} \longrightarrow M(x) = EI\frac{d^2v(x)}{dx^2} \quad and \quad V(x) = EI\frac{d^3v(x)}{dx^3}$$

Induced Deflection v(x) can be obtained by solving the 4th order differential equation:

$$EI\frac{d^4v(x)}{dx^4} = 0$$

where E = Young's modulus of the beam material
I = Section moment of inertia

FE Formulation of Beam Elements



ρ = Radius of curvature deflection curve at x
Θ = slope of deflection curve at x
V(x) = Deflection at x

Primary Quantities

In the element: The deflection v(x)
At the nodes: The deflections v and slope θ, or as expressed as: $\{d\}=\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$ at Node i and Node j

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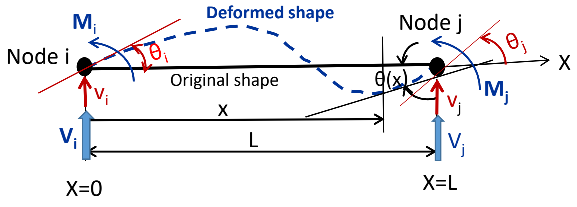
FE Formulation of Beam Elements

Beam elements

Contrary to the “truss elements,” beam elements deforms from its original straight shape into curved bent shape due to lateral (or transverse) forces or applied couples (moments).

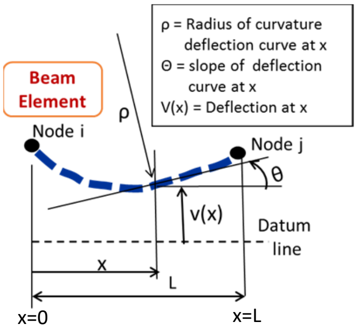


Actions and Induced Reactions in Beam Elements

Actions	Induced Reactions		
	in the element	at Node i	at Node j
Lateral forces V_i and V_j	Linear displacement v(x)	Linear displacement v_i	Linear displacement v_j
Moments M_i and M_j	Rotation θ(x)	Rotation θ_i	Rotation θ_j

FE Formulation of Beam Elements – Cont'd

Derive Interpolation Function



We assume the traverse displacement of the beam element follows a linear polynomial function on the form:

$$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$$

in which a_1, a_2, a_3 and a_4 are constant coefficients

By substituting the coordinates of Node I and Node j into the assumed element deflection in this Equation , we get:

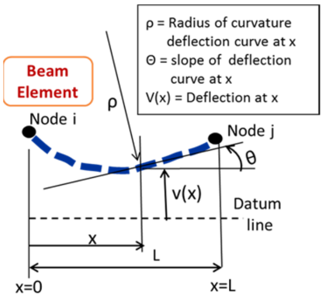
At Node i: $v_i = v(x)\big|_{x=0} = a_4 \qquad \theta_i = \left.\frac{dv(x)}{dx}\right|_{x=0} = a_3$

At Node j: $v_j = v(x)\big|_{x=L} = a_1L^3 + a_2{}^2L^2 + a_3L + a_4$

$$\theta_j = \left.\frac{dv(x)}{dx}\right|_{x=L} = 3a_1L^2 + 2a_2L + a_3$$

FE Formulation of Beam Elements – Cont'd

Derive Interpolation Function



We may thus express the element deflection v(x) in terms of the nodal deflections

$$v(x) = \left[\frac{2}{L^3}(v_i - v_j) + \frac{1}{L^2}(\theta_i + \theta_j)\right]x^3 + \left[-\frac{3}{L^2}(v_i - v_j) - \frac{1}{L}(2\theta_i + \theta_j)\right]x^2 + \theta_i x + v_i$$

The above expression can also be expressed as: $v(x) = \{N_{iv} \quad N_{i\theta} \quad N_{jv} \quad N_{j\theta}\}\begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix}$

where $N_{iv} = \frac{1}{L^3}(2x^3 - 3x^2L + L^3) \qquad N_{i\theta} = \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3)$

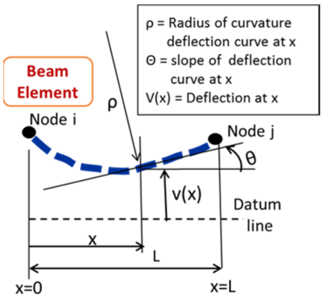
$$N_{jv} = \frac{1}{L^3}(-2x^3 + 3x^2L) \qquad N_{j\theta} = \frac{1}{L^3}(x^3L - x^2L^2)$$

We thus have: $\underbrace{v(x)}_{\text{The element deflections}} = [N(x)]\underbrace{\{d\}}_{\text{The nodal deflections}}$

with **Interpolation function**: $[N(x)] = \{N_{iv} \quad N_{i\theta} \quad N_{jv} \quad N_{j\theta}\}$

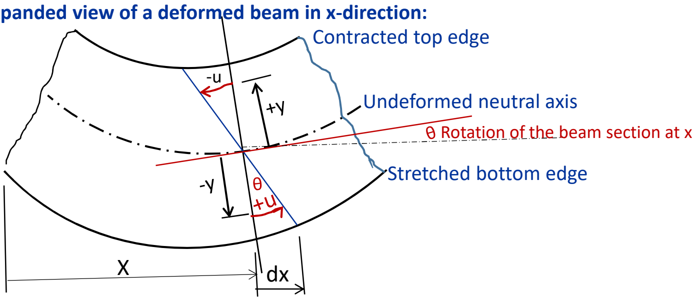
FE Formulation of Beam Elements – Cont'd

Derive Element Strain ε_{xx} – Displacement **v(x)** Relation



We realize the fact that the rotation (or the slope) of the deflected beam is: $\theta = \frac{dv(x)}{dx} \quad for\ 0 \le x \le L$

An expanded view of a deformed beam in x-direction:



We may find from the above diagram of expanded beam that the stretch u can be obtained by: $u(x) = -y\theta = -y\frac{dv(x)}{dx}$

But from theory of elasticity: $\varepsilon_{xx} = \frac{du}{dx} \quad or \quad \underline{\varepsilon_{xx}(x,y)} = -y\frac{d}{dx}\left(\frac{dv(x)}{dx}\right) = \underline{-y\frac{d^2v(x)}{dx^2}}$

FE Formulation of Beam Elements – Cont'd

Derive Element Stress σ_{xx} – strain ε_{xx} Relation

We will have the normal stress $\sigma_{xx} = E\varepsilon_{xx} = -\frac{M(x)y}{I}$

and the shear stress: $\sigma_{yx}(x) = \frac{V(x)}{Ib}Q$ with Q to be the shear moment

The relationships between the bending moment M(x) and Shear force V(x) are expressed in Equation (4.40a) and (4.40b) respectively:

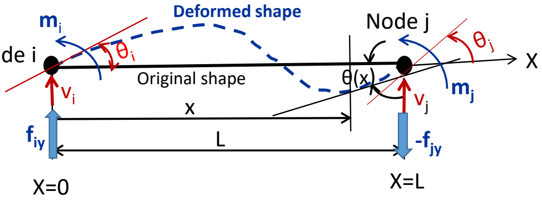
$$M(x) = EI\frac{d^2v(x)}{dx^2} \quad and \quad V(x) = EI\frac{d^3v(x)}{dx^3}$$

FE Formulation of Beam Elements – Cont'd

Derive the Element Stiffness Equations

We may express the applied actions to the beam element in applied forces f_{iy}, f_{jy} and moments m_i, m_j in terms of the element deflections v(x) represented by the assumed polynomial function:

The beam element:



$$v(x)=\left[\frac{2}{L^3}(v_i-v_j)+\frac{1}{L^2}(\theta_i+\theta_j)\right]x^3$$

$$+\left[-\frac{3}{L^2}(v_i-v_j)-\frac{1}{L}(2\theta_i+\theta_j)\right]x^2+\theta_i x+v_i$$

Nodal forces:

$$f_{iy}=V=EI\left.\frac{d^3v(x)}{dx^3}\right|_{x=0}=\frac{EI}{L^3}(12v_i+6L\theta_i-12v_j+6L\theta_j)$$

$$f_{jy}=-V=-EI\left.\frac{d^3v(x)}{dx^3}\right|_{x=L}=\frac{EI}{L^3}(-12v_i-6L\theta_i+12v_j-6L\theta_j)$$

Nodal moments:

$$m_i=-M=-EI\left.\frac{d^2v(x)}{dx^2}\right|_{x=0}=\frac{EI}{L^3}(6Lv_i+4L^2\theta_i-6Lv_j+2L^2\theta_j)$$

$$m_j=M(x)=EI\left.\frac{d^2v(x)}{dx^2}\right|_{x=L}=\frac{EI}{L^3}(6Lv_i+2L^2\theta_i-6Lv_j+4L^2\theta_j)$$

FE Formulation of Beam Elements – Cont'd

Derive the Element Stiffness Equations

The above expressions may be express in the following matrix form for the element stiffness equations:

$$\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}\begin{Bmatrix}v_i\\ \theta_i\\ v_j\\ \theta_j\end{Bmatrix}=\begin{Bmatrix}f_{iy}\\ m_i\\ f_{jy}\\ m_j\end{Bmatrix}$$

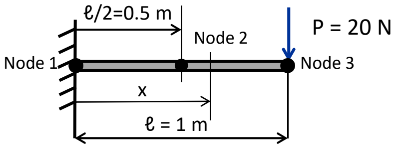
Stiffness matrix [Ke] x Nodal unknown quantities(transverse displacements & rotations = Applied nodal transverse forces & moments

We thus have the stiffness equation of a beam element to be:

$$[k_e]=\frac{EI}{L^3}\begin{bmatrix}12 & 6L & -12 & 6L\\ 6L & 4L^2 & -6L & 2L^2\\ -12 & -6L & 12 & -6L\\ 6L & 2L^2 & -6L & 4L^2\end{bmatrix}$$

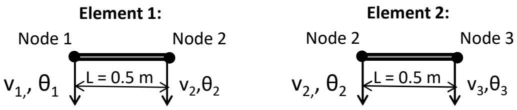
Example 4

Determine the deflection of a cantilever beam at the half span and at the point under the load. Dimensions and applied lateral force are shown in the figure. The beam is made of a material with Young’s modulus E = 10000 MPa

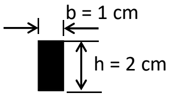


Solution:

We realize the fact that we are seeking solutions at the mid-span and at the point under the applied force. It id reasonable to discretize the beam into to two (2) elements, as shown below:



Beam Cross-section:



Section moment of inertia (I) is:

$$I = \frac{bh^3}{12} = \frac{(10^{-2})(2\times10^{-2})^3}{12} = 0.667\times10^{-8}\ m^4$$

EI = $(10^{10})(0.667\text{x}10^{-8})$ = 66.7 Nm^2

We will first derive the element equations

Example 4

For Element 1 with L_1 = 0.5 m:

$$[k_e^1] = \frac{EI}{L_1^3}\begin{bmatrix} 12 & 6L_1 & -12 & 6L_1 \\ 6L_1 & 4L_1^2 & -6L_1 & 2L_1^2 \\ -12 & -6L_1 & 12 & -6L_1 \\ 6L_1 & 2L_1^2 & -6L_1 & 4L_1^2 \end{bmatrix} = 533.6\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix} \quad \text{(a)}$$

Element equation for Element 1 is:

$$533.6\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \end{Bmatrix} \quad \text{(b)}$$

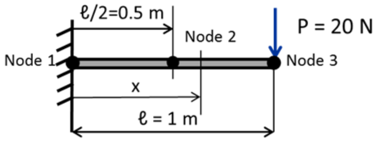
in which v_1,θ_1, v_2, θ_2 are the respective deflections and rotations in Node 1 and Node 2 respectively, whereas f_1, m_1 and f_2, m_2 are the applied lateral forces and moments at Node 1 and Node 2 respectively

Example 4

Due to the fact that Element 2 has the same length and is made of the same material as Element 1, with the only difference of the associated nodes, we can express the same element equation for Element 1 for element 2 as shown below:

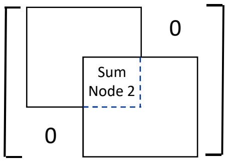
$$533.6\begin{bmatrix} 12 & 3 & -12 & 3 \\ 3 & 1 & -3 & 0.5 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} f_2 \\ m_2 \\ f_3 \\ m_3 \end{Bmatrix} \qquad \text{(c)}$$

Assembly of element equations in (b) and (c) for Overall stiffness equations:



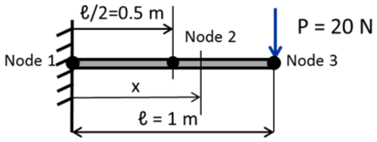
Because Node 2 happens to be common node shared by Element 1 and 2, we need to assemble the element equation:

$$[K]=[K_e^1]+[K_e^2]=$$



Applied loads at Node 2 in Element 1 and 2 should be summed up too in the load matrix

Example 4



We thus have the assembled overall stiffness equations as:

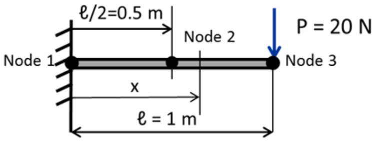
$$533.6\begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & (12+12) & (-3+3) & -12 & 3 \\ 3 & 0.5 & (-3+3) & (1+1) & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \\ f_3 \\ m_3 \end{Bmatrix}$$

The assembled overall stiffness equation for the beam structure is:

$$533.6\begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} f_1 \\ m_1 \\ f_2 \\ m_2 \\ f_3 \\ m_3 \end{Bmatrix} \qquad \text{(d)}$$

We may solve the six unknown responses at the three nodes in the beam structure from Equation (d)

Example 4



We realize that the following boundary and applied loading conditions apply:

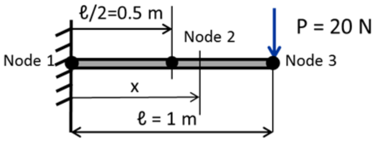
v_1 and θ_1 = 0 for having Node 1 being fixed at the built-in end, and f_1, m_1, f_2,m_2 and m_3 = 0. the only non-zero load is the applied force f_3 = -20 N at Node 3

We thus have the overall stiffness equation of the beam structure with specified boundary and loading conditions take the form:

$$533.6\begin{bmatrix} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} ? \\ ? \\ 0 \\ 0 \\ -20 \\ 0 \end{Bmatrix} \qquad \text{(e)}$$

The four unknown at Node 2 and 3 can be solved from Equation (e) by partitioning the above matrix equations following Step 6 in Chapter 3.

Example 4



$$533.6\left[\begin{array}{cc|cccc} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ \hline -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ \hline v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{array}\right\} = \left\{\begin{array}{c} ? \\ ? \\ \hline 0 \\ 0 \\ -20 \\ 0 \end{array}\right\}$$

In the form of:

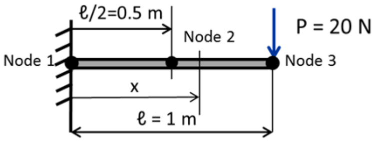
$$\left[\begin{array}{c|c} K_{aa} & K_{ab} \\ \hline K_{ba} & K_{bb} \end{array}\right]\left\{\begin{array}{c} q_a \\ \hline q_b \end{array}\right\} = \left\{\begin{array}{c} R_a \\ \hline R_b \end{array}\right\}$$

where $\{q_a\}$ = specified (known) nodal quantities; $\{R_b\}$ = specified (known) applied resulting actions, from which we may obtain:

$$\{q_b\} = [K_{bb}]^{-1} (\{R_b\} - [K_{ba}]\{q_a\})$$

In the present case, we have $\{q_a\} = \{0 \quad 0\}^T$ We thus have the 4 unknowns obtained by: $\{q_b\} = [K_{bb}]^{-1} \{R_b\}$

Example 4



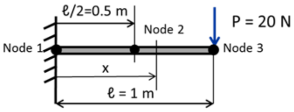
$$533.6\left[\begin{array}{cc|cccc} 12 & 3 & -12 & 3 & 0 & 0 \\ 3 & 1 & -3 & 0.5 & 0 & 0 \\ \hline -12 & -3 & 24 & 0 & -12 & 3 \\ 3 & 0.5 & 0 & 2 & -3 & 0.5 \\ 0 & 0 & -12 & -3 & 12 & -3 \\ 0 & 0 & 3 & 0.5 & -3 & 1 \end{array}\right]\begin{Bmatrix} 0 \\ \underline{0} \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ \underline{0} \\ 0 \\ 0 \\ -20 \\ 0 \end{Bmatrix}$$

$\{q_b\} = [K_{bb}]^{-1}\ \{R_b\}$:

$$\{q_b\}=\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} \qquad [K_{bb}]=\begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & 3 \\ -12 & -3 & 12 & -3 \\ 3 & 0.5 & -3 & 1 \end{bmatrix} \qquad \{R_b\}=\begin{Bmatrix} 0 \\ 0 \\ -20 \\ 0 \end{Bmatrix}$$

One will find that: $$[K_{bb}]^{-1}=\begin{bmatrix} 0.3333 & 1 & 0.8333 & 1 \\ 1 & 4 & 3 & 4 \\ 0.8333 & 3 & 2.6667 & 4 \\ 1 & 4 & 4 & 8 \end{bmatrix}$$

Example 4



Given conditions:
$v_1 = 0$, $\theta_1 = 0$

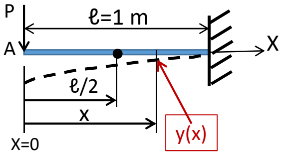
We may thus solve for the 4 primary unknown quantities in $\{q_b\}$ from the following equations:

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix}=\frac{1}{533.6}\begin{bmatrix} 0.3333 & 1 & 0.8333 & 1 \\ 1 & 4 & 3 & 4 \\ 0.8333 & 3 & 2.6667 & 4 \\ 1 & 4 & 4 & 8 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -20 \\ 0 \end{Bmatrix}=\frac{1}{533.6}\begin{Bmatrix} 0.8333\times(-20) \\ 3\times(-20) \\ 2.6667\times(-20) \\ 4\times(-20) \end{Bmatrix}=-\begin{Bmatrix} 0.03123 \\ 0.1124 \\ 0.099995 \\ 0.1499 \end{Bmatrix}$$

From which we obtain the following solutions:

Deflections at Node 2, v_2 = -0.03123 m = -3.123 cm, at Node 3, v_3=-0.09995 m = -9.9995 cm
Slope at Node 2, θ_2 = -0.1124 rad, at Node 3 θ_3=-0.1499 rad.

Check with solutions from classical beam theory:



The induced deflection in the cantilever beam by the applied force P = 20 N is

$$y(x)=-\frac{1}{6}\frac{P}{EI}\left(x^3-3\ell^2x+2\ell^3\right)$$

One may find the deflection at Point A at x=0 (equivalent to Node 3) = -0.0999 m and the deflection at Point B at x = ℓ/2 (equivalent to Node 2) =-0.03123m. Both these values fully agree with the solutions we obtained from the FE analysis.

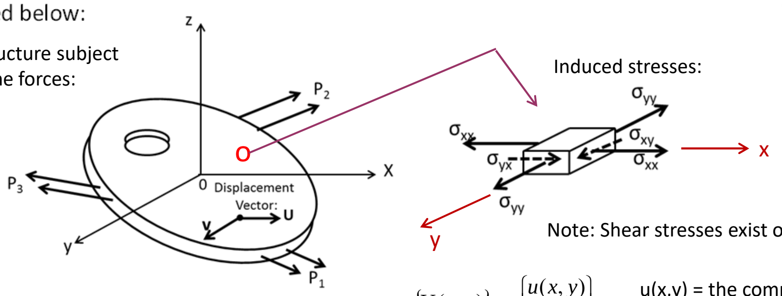
FEM for Plane Stress and Plane Strain Analysis

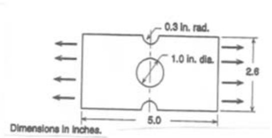
Examples of FE Stress Analysis of Solid Structures

This type of structures are typically thin in their thickness in comparison to the bulk volume of the overall structure. As such only three (3) out of total six (6) independent stress components need to be considered in the analysis, as illustrated below:

Plane structure subject to in-plane forces:

Induced stresses:





Note: Shear stresses exist on the thickness or “edges.”

The in-plane displacement components in the solid: $\{U(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$

u(x,y) = the component along the x-coordinate
V(x,y) = the component along the y-coordinate

The stress components: $\{\sigma(x,y)\}^T = \{\sigma_{xx}(x,y)\ \sigma_{yy}(x,y)\ \sigma_{xy}(x,y)\}^T$

The strain components: $\{\varepsilon(x,y)\}^T = \{\varepsilon_{xx}(x,y)\ \ \varepsilon_{yy}(x,y)\ \ \varepsilon_{xy}(x,y)\}^T$

The strain – displacement relation :

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} \dfrac{\partial u(x,y)}{\partial x} \\ \dfrac{\partial v(x,y)}{\partial y} \\ \dfrac{\partial u(x,y)}{\partial y} + \dfrac{\partial v(x,y)}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

FEM for Plane Stress and Plane Strain Analysis

The stress-strain relation for Plane Stress :

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\{\sigma(x,y)\} = \mathrm{D}\{\varepsilon(x,y)\}$$

D: modulus Matrix

$$\mathrm{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}$$

$$\{\varepsilon(x,y)\} = \mathrm{C}\{\sigma(x,y)\} \qquad \mathrm{C} = \mathrm{D}^{-1}$$

C: Compliance Matrix

FEM for Plane Stress and Plane Strain Analysis

The stress-strain relation for Plane Strain :

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{Bmatrix}$$

$$\{\sigma(x,y)\}=\mathrm{D}\{\varepsilon(x,y)\}$$

D: modulus Matrix

$$\mathrm{D}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

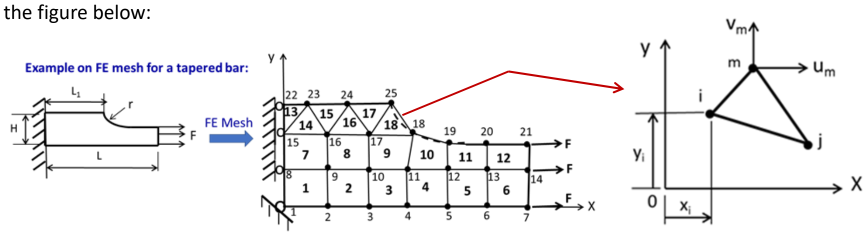
$$\{\varepsilon(x,y)\}=\mathrm{C}\{\sigma(x,y)\}$$

$$\mathrm{C}=\mathrm{D}^{-1}$$

C: Compliance Matrix

FEM for Plane Stress and Plane Strain Analysis

Let us now formulating the FE for plate structures such as with the discretization in a tapered plate as illustrated In the figure below:



We begin our FE formulation of the plate structure with the expression of the "element displacement" components {U(x,y)} in terms of the corresponding "nodal displacements" using an interpolation function {N(x,y)} as follows:

$$\{U(x,y)\}=\begin{Bmatrix}u(x,y)\\ v(x,y)\end{Bmatrix}=\{N(x,y)\}\{u\} \qquad (4.)$$

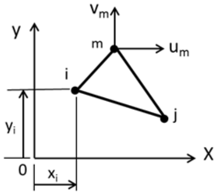
Element displacements

Interpolation function

Nodal displacements

where the nodal displacement components, $\{u\}=\begin{Bmatrix}u_i\\ u_j\\ u_m\\ v_i\\ v_j\\ v_m\end{Bmatrix}$ (4.)

FEM for Plane Stress and Plane Strain Analysis



Derivation of interpolation function {N(x,y)}

We assume the elements used in this FE analysis are the "simplex" elements, meaning that the Element displacements follow linear polynomial functions in relating their nodal displacements:

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$ along the x-coordinate

and $$v(x,y) = \alpha_4 + \alpha_5 x + \alpha_6 y$$ along the y-coordinate

We will thus have: $$\{U(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

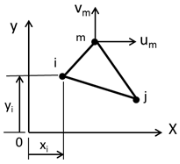
$$= \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6\}^T$$

or in an alternative matrix form: $$\{U(x,y)\} = [R(x,y)]\{\alpha\}$$

where α_1, α_2, α_3, α_4, α_5, α_6 in the matrix $\{\alpha\}^T$ are constants to be determined with specified nodal coordinates later.

The matrix [R(x,y)] in Equation (4.28) has the form: $$[R(x,y)] = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

FEM for Plane Stress and Plane Strain Analysis



Derivation of interpolation function {N(x,y)} - cont'd

With the specified nodal coordinates:

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i, \quad u_j = \alpha_1 + \alpha_2 x_j + \alpha_3 y_j, \quad u_m = \alpha_1 + \alpha_2 x_m + \alpha_3 y_m, \text{ and}$$

$$v_i = \alpha_4 + \alpha_5 x_i + \alpha_6 y_i, \quad v_j = \alpha_4 + \alpha_5 x_j + \alpha_6 y_j, \quad v_m = \alpha_4 + \alpha_5 x_m + \alpha_6 y_m$$

the nodal displacements form :

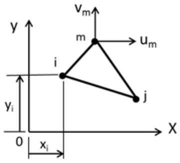
$$\{u\} = \begin{Bmatrix} u_i \\ u_j \\ u_m \\ v_i \\ v_j \\ v_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 1 & x_m & y_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 0 & 0 & 0 & 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = [A]\{\alpha\}$$

Displacement components of 3 nodes in the element

Constant coefficients

Specified coordinates of the 3 nodes in the element

FEM for Plane Stress and Plane Strain Analysis



Derivation of interpolation function {N(x,y)} - cont'd

$$\{u\} = [A]\{\alpha\}$$

in which the matrix [A] has the form:

$$[A] = \begin{bmatrix} 1 & x_i & y_i & 0 & 0 & 0 \\ 1 & x_j & y_j & 0 & 0 & 0 \\ 1 & x_m & y_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_i & y_i \\ 0 & 0 & 0 & 1 & x_j & y_j \\ 0 & 0 & 0 & 1 & x_m & y_m \end{bmatrix}$$

nodal coordinate matrix

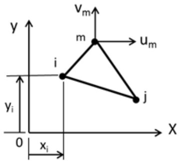
We may obtain the solution of the unknown coefficient matrix as:

$$\{\alpha\} = [A]^{-1}\{u\} = [h]\{u\}$$

where the matrix: $[h] = [A]^{-1}$

with $[A]^{-1}$ = the inverse of the nodal coordinate matrix [A]

FEM for Plane Stress and Plane Strain Analysis



Derivation of interpolation function {N(x,y)} - cont'd

Using

$$\{U(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \{\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6\}^T$$

$$[R(x,y)] = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

We will obtain the following expression:

$$\{U(x,y)\} = [R(x,y)]\,[h]\,\{u\}$$

Displacement components in Element ijm

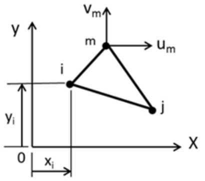
Displacement components of 3 nodes in the element

So we have the interpolation function of this simplex element to be: $[N(x,y)] = [R(x,y)]\,[h]$

FEM for Plane Stress and Plane Strain Analysis

The interpolation function {N(x,y)} - cont'd

$$\{U(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \{N(x,y)\}\{u\} = \{N_i(x,y) \quad N_j(x,y) \quad N_m(x,y)\}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$



An alternative expression:

$$\{U(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \{N(x,y)\}\{u\} = \begin{bmatrix} N_i & N_j & N_m & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i & N_j & N_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \\ v_i \\ v_j \\ v_m \end{Bmatrix}$$

where

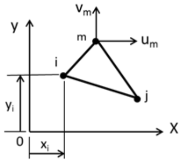
$$N_i(x,y) = \frac{1}{A}\left[\left(x_j y_m - x_m y_j\right) + \left(y_j - y_m\right)x + \left(x_m - x_j\right)y\right]$$

$$N_j(x,y) = \frac{1}{A}\left[\left(x_m y_i - x_i y_m\right) + \left(y_m - y_i\right)x + \left(x_i - x_m\right)y\right]$$

$$N_m(x,y) = \frac{1}{A}\left[j\left(x_i y_j - x_j y_i\right) + \left(y_i - y_j\right)x + \left(x_j - x_i\right)y\right]$$

$$2A = \left(x_i y_j - x_j y_i\right) + \left(x_j y_m - x_m y_j\right) + \left(x_m y_i - x_i y_m\right) \qquad A = the\ area\ of\ the\ element\ made\ of\ triangle\ (ijm)$$

FEM for Plane Stress and Plane Strain Analysis



Derivation of Element equation

Because the element equation is derived by minimizing the Potential energy in the deformed solid, we need to derive the expression of "strain energy" in terms of nodal displacements (the primary quantities in the analysis).

We will first express the element strain vs. nodal displacements as:

$$\{\epsilon(x.y)\} = [B]\{u\}$$

where [B(x,y,z)] = [*D*][N(x,y,z)]

$$[B] = \frac{1}{2A}\begin{bmatrix} (y_j - y_m) & (y_m - y_i) & (y_i - y_j) & 0 & 0 & 0 \\ 0 & 0 & 0 & (x_m - x_j) & (x_i - x_m) & (x_j - x_i) \\ (x_m - x_j) & (x_i - x_m) & (x_j - x_i) & (y_j - y_m) & (y_m - y_i) & (y_i - y_j) \end{bmatrix}$$

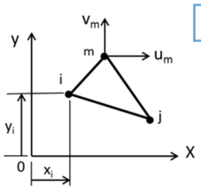
The potential energy in the deformed element is:

$$P(\{u\}) = \frac{1}{2}\int_v \{u\}^T [B]^T [\mathrm{D}][B]\{u\}\,dv - \int_v \{u\}^T \left[N(x,y)\right]^T \{f\}\,dv - \int_s \{u\}^T \left[N(x,y)\right]^T \{t\}\,ds$$

body force

Surface traction

FEM for Plane Stress and Plane Strain Analysis



Derivation of Element equation

The element equation is obtain by minimizing the potential energy :

$$\frac{\partial P(\{u\})}{\partial\{u\}} = 0$$

Leading to the following element equation:

$$[K_e]\{u\} = \{p\}$$

where $[K_e] = Element\ stiffness\ matrix = \int_v [B]^T [\mathrm{D}][B]dv$

and the nodal force matrix

$$\{p\} = Nodal\ \ force\ matrix = \int_v [N(x,y)]^T \{f\}dv + \int_s [N(x,y)]^T \{t\}ds$$

However, if the size of the element is not too large, this integration may be approximated by the following expression without significant error:

$$[K_e] \approx [B]^T [\mathrm{D}][B](WA)$$

Volume of element

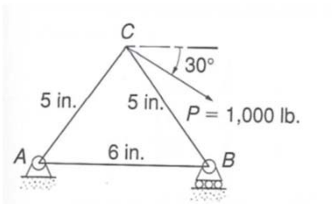
in which W is the thickness of the plane element, and A is the plane area.

$$2A = (x_i y_j - x_j y_i) + (x_j y_m - x_m y_j) + (x_m y_i - x_i y_m) \quad A = the\ area\ of\ \ the\ element\ made\ of\ \ triangle\ (ijm)$$

FEM for Plane Stress and Plane Strain Analysis

Example:

A structure made of a triangular plate defined by three corners at A,B and C. A force P is applied at corner C as shown in the figure.

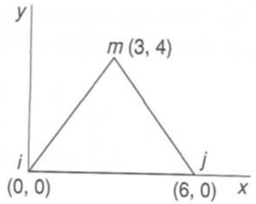


Find the following:

(a) The displacement of the plate at corner C
(b) The displacement in the plate
(c) The stresses and strains in the plate, and
(d) The reactions at the two fixed corners

Solution:

We assume that only ONE element is used for the analysis. This triangular plate element has three nodes I, j and m located as shown at the right:



The coordinates of the 3 nodes are:

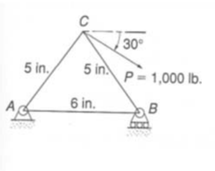
{x_i = 0, y_i=0), (x_j=6, y_j=0) and (x_m=3, y_m=4)

$$A: \ \frac{0}{0}\ \frac{6}{0}\ \frac{3}{4}\ \frac{0}{0} \qquad A = \frac{1}{2}[6\times 4] = 12$$

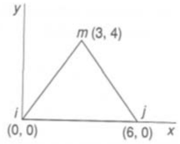
FEM for Plane Stress Analysis

(a) To determine the nodal displacements:

We will first obtain the [A] matrix with the specified nodal coordinates:



$$[A]=\begin{bmatrix}1&0&0&0&0&0\\1&6&0&0&0&0\\1&3&4&0&0&0\\0&0&0&1&0&0\\0&0&0&1&6&0\\0&0&0&1&3&4\end{bmatrix}$$



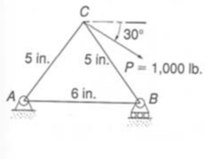
and the [h] matrix:

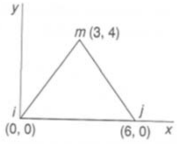
$$[h]=\begin{bmatrix}1&0&0&0&0&0\\-0.167&0.167&0&0&0&0\\-0.125&-0.125&0.25&0&0&0\\0&0&0&1&0&0\\0&0&0&-0.167&0.167&0\\0&0&0&-0.125&-0.125&0.25\end{bmatrix}$$

the interpolation function

$$[N(x,y)]=\begin{bmatrix}(1-0.167x-0.125y)&(0.167x-0.125y)&0.25y&0&0&0\\0&0&0&(1-0.167x-0.125y)&(0.167x-0.125y)&0.25y\end{bmatrix}$$

FEM for Plane Stress Analysis





We are now ready to derive the element matrix for the structure with the newly derived interpolation function. We will obtain first the [B] matrix :

$$[B]=\frac{1}{24}\begin{bmatrix}-4&4&0&0&0&0\\0&0&0&-3&-3&6\\-3&-3&6&-4&4&0\end{bmatrix}$$

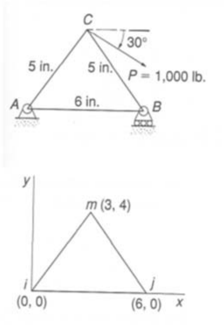
and the [D] matrix

$$[\mathrm{D}]=\frac{E}{1-\nu^2}\begin{bmatrix}1&\nu&0\\\nu&1&0\\0&0&\frac{1-\nu}{2}\end{bmatrix}=10.99\times10^6\begin{bmatrix}1&0.3&0\\0.3&1&0\\0&0&0.35\end{bmatrix}$$

We are now ready to determine the element stiffness matrix:

$$[K_e]=228958.32\begin{bmatrix}19.15&-12.85&-6.3&7.8&-0.6&-7.2\\&19.15&-6.3&0.6&-7.8&7.2\\&&12.6&-8.4&8.4&0\\&SYM&&14.6&3.4&-18\\&&&&14.6&-18\\&&&&&36\end{bmatrix}$$

FEM for Plane Stress Analysis



Constructing the "element equation:"

We realize the following boundary and applied force conditions:

u_i, v_i, v_j = 0 (fixed ends), and the applied nodal forces:

$p_{ix} = p_{iy} = p_{jx} = p_{jy} = 0$, and $p_{mx} = p \cos 30^o = 866$ lb, and $p_{my} = p \sin 30^o = -500$ lb

The "element equation" :

$$228958.32\begin{bmatrix} 19.15 & -12.85 & -6.3 & 7.8 & -0.6 & -7.2 \\ & 19.15 & -6.3 & 0.6 & -7.8 & 7.2 \\ & & 12.6 & -8.4 & 8.4 & 0 \\ & SYM & & 14.6 & 3.4 & -18 \\ & & & & 14.6 & -18 \\ & & & & & 36 \end{bmatrix}\begin{Bmatrix} u_i = 0 \\ u_j \\ u_m \\ v_i = 0 \\ v_j = 0 \\ v_m \end{Bmatrix} = \begin{Bmatrix} p_{ix} = ? \\ p_{jx} = 0 \\ p_{mx} = 866 \\ p_{iy} = ? \\ p_{jy} = ? \\ p_{my} = -500 \end{Bmatrix}$$

Because there is only one element in the structure, the above element equation is also the "overall stiffness equation" of the structure, from which we may solve the "displacement components of ALL nodes after Making necessary interchange of rows and columns in the above equation

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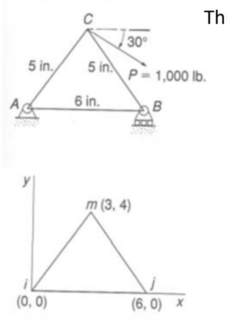
FEM for Plane Stress Analysis



Thus, after the necessary interchanges of rows and columns, we reached the partition of the element equation as:

$$228958.32\left[\begin{array}{ccc|ccc} 19.15 & 7.8 & -0.6 & -12.85 & -6.3 & -7.2 \\ 7.8 & 14.6 & 3.4 & 0.6 & -8.4 & -18 \\ -0.6 & 3.4 & 14.6 & -7.8 & 8.4 & -18 \\ \hline -12.85 & 0.6 & -7.8 & 19.15 & -6.3 & 7.2 \\ -6.3 & -8.4 & 8.4 & -6.3 & 12.6 & 0 \\ -7.2 & -18 & -18 & 7.2 & 0 & 36 \end{array}\right]\left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline u_j \\ u_m \\ v_m \end{array}\right\} = \left\{\begin{array}{c} ? \\ ? \\ ? \\ \hline 0 \\ 866 \\ -500 \end{array}\right\}$$

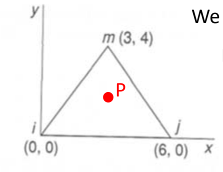
The 3 nonzero nodal displacements can be solve from the above portioned overall stiffness equation by the following simultaneous equations:

$$228958.32\begin{bmatrix} 19.15 & -6.3 & 7.2 \\ -6.3 & 12.6 & 0 \\ 7.2 & 0 & 36 \end{bmatrix}\begin{Bmatrix} u_j \\ u_m \\ v_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 866 \\ -500 \end{Bmatrix} \qquad \Rightarrow \begin{Bmatrix} u_j = 0.16 \\ u_m = 0.38 \\ v_m = -0.093 \end{Bmatrix} \times 10^{-3} \quad \text{inch}$$

The above simultaneous equations may be solved by either matrix inversion method or Gaussian elimination method

FEM for Plane Stress Analysis

(b) The displacements in the element:



We may use the interpolation function {N(x,y)} to determine the displacement components everywhere in the element:

For the displacement in the x-direction:

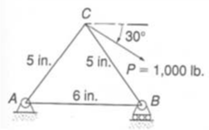
$u(x,y) = (1 –- 0.167x – 0.125y)u_i + (0.167x -0.125y)u_j + 0.25\ u_m = 0 +(0.167x-0.125y)x0.16x10^{-3}+0.25x0.38x10^{-3}y$
$V(x,y) = (1 –- 0.167x – 0.125y)v_i + (0.167x -0.125y)v_j + 0.25\ v_m = -0.09264x0.25x10^{-3}y$

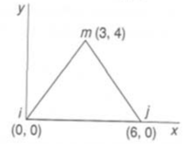
For instance the displacements at Point P(3,2) have the values of:
$u(3,2) = 0 +(0.167x3-0.125x2)x0.16x10^{-3}+0.25x0.38x10^{-3}\ x2 = 0.23x10^{-3}$ inch
$v(3,2) = -0.09264x0.25x10^{-3}x2 = -.04632x10^{-3}$ inch

(c) The strain components in the element considering {ε} = [B]{u}:

$$\{\varepsilon(x,y)\}=\begin{Bmatrix}\varepsilon_{xx}(x,y)\\ \varepsilon_{yy}(x,y)\\ \varepsilon_{xy}(x,y)\end{Bmatrix}=\frac{1}{24}\begin{bmatrix}-4 & 4 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -3 & -3 & 6\\ -3 & -3 & 6 & -4 & 4 & 0\end{bmatrix}\begin{Bmatrix}u_i=0\\ u_j=0.16\\ u_m=0.38\\ v_i=0\\ v_j=0\\ v_m=-0.09\end{Bmatrix}\times10^{-3}=\begin{Bmatrix}26.65\\ -23.16\\ 75\end{Bmatrix}\times10^{-6}$$

FEM for Plane Stress Analysis





The stresses in the element may be obtained by using the generalized Hooke's Law:

$$\{\sigma(x,y)\}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{Bmatrix}=10.99\times10^{6}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}\begin{Bmatrix}26.65\\ -23.16\\ 75\end{Bmatrix}\times10^{-6}=\begin{Bmatrix}216.5\\ -166.67\\ 288.66\end{Bmatrix}\ psi$$

(d) The reactions at all nodes:

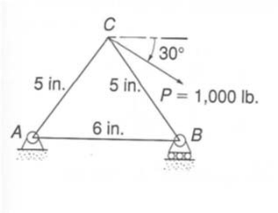
One may derive the following expression for the nodal forces:

$$\{R\}=\int_v [B]^T\{\sigma\}\,dv\approx[B]^T\{\sigma\}(WA)$$

$$\{R\}=\begin{Bmatrix}R_{ix}\\ R_{jx}\\ R_{mx}\\ R_{iy}\\ R_{jy}\\ R_{my}\end{Bmatrix}=\frac{1}{24}\begin{bmatrix}-4 & 0 & -3\\ 4 & 0 & -3\\ 0 & 0 & 6\\ 0 & -3 & -4\\ 0 & -3 & 4\\ 0 & 6 & 0\end{bmatrix}\begin{Bmatrix}216.5\\ -166.67\\ 288.66\end{Bmatrix}\times12\times1=\begin{Bmatrix}-866\\ 0\\ 866\\ -327.3\\ 827.3\\ -500\end{Bmatrix}$$

The reactions at Node i therefore have numerical values at: R_{ix} = 866 lb towards left, and R_{iy}=327.3 lb in the downward direction

FEM for Plane Stress Analysis



Important lesson learned from this numerical example:

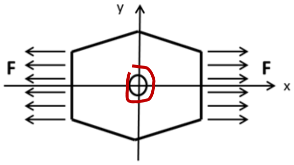
We noticed that the stresses and strains in the element (and thus the triangular plate structure re CONSTANT:

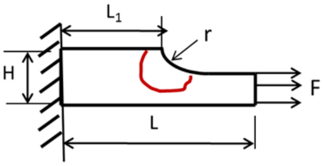
$$\{\varepsilon(x,y)\} = \begin{Bmatrix} \varepsilon_{xx}(x,y) \\ \varepsilon_{yy}(x,y) \\ \varepsilon_{xy}(x,y) \end{Bmatrix} = \begin{Bmatrix} 26.65 \\ -23.16 \\ 75 \end{Bmatrix} \times 10^{-6} \quad \text{and} \quad \{\sigma(x,y)\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} 216.5 \\ -166.67 \\ 288.66 \end{Bmatrix} psi$$

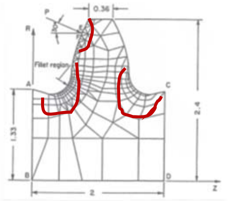
– meaning there is no variation of stresses and strains throughout the entire structure. This is obviously not realistic!!

The reason for what has happened in this (and the other)numerical example is because we used “linear polynomial” in deriving the interpolation function – resulting in using “**SIMPLEX**” element in the FE analysis. “Simplex elements” offers “simple mathematical formulation in FEA, but results in constant stresses and strains in elements.

That was the reason why engineers need to place many more (smaller) elements in the area with conceivable high gradients of primary unknown quantities, such as in the following cases:







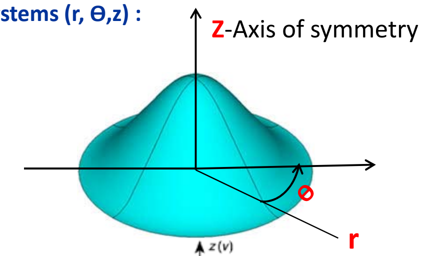
This is good lesson for any intelligent FE user to learn and exercise

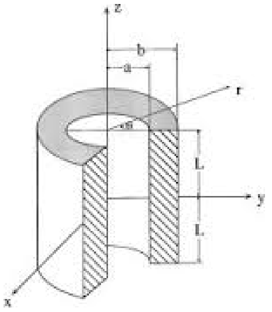
Finite Element Formulation of Stress Analysis of Axisymmetric Solids Structures

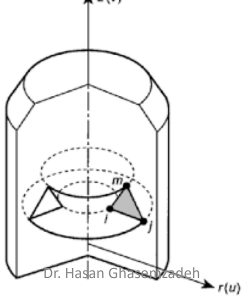
FEM Formulation of Stress Analysis of Axisymmetric Solids Structures

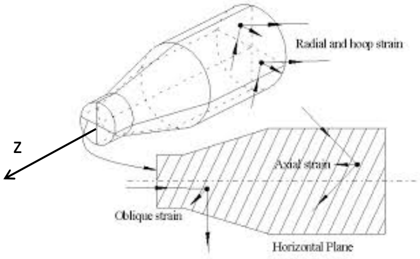
Axisymmetric: Any solid with its geometry symmetric to an axis
Examples: Pile, Circular foundation, Wheels, Cylinders of constant or variable radius along the z-axis such as pressure vessels

Coordinate systems (r, Θ,z) :









Classifications of Axisymmetric solids :

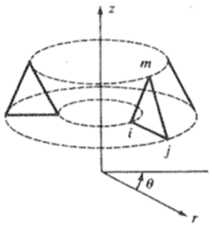
Circular foundation and Wheels and thin circular plates:

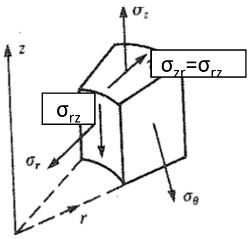
2-D axisymmetric solids with 3 components of stress and strains as defined in the case of thin plates

Piles and Cylinders with constant or variable radius:

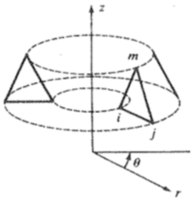
3-D axisymmetric solids with four (4) stress components:

σ_{rr} (r,z) = Stress in the radial direction,
$\sigma_{\theta\theta}$(r,z) = Stress in the tangential θ-direction (or
σ_{zz}(r,z) = Stress in z-direction, and
σ_{rz} (r,z) = Shearing stress on the surface of the solid element that is perpendicular to the r-coordinate but in the z-direction





Formulas Derived from Theory of Elasticity



Displacements:

u(r,z) = displacement component in r-direction
w(r,z) = displacement component in z-direction

Strains:

$$\varepsilon_{rr}(r,z)=\frac{\partial u(r,z)}{\partial r}$$ Normal strain in along r-coordinate

$$\varepsilon_{\theta\theta}(r,z)=\frac{u(r,z)}{r}$$ Normal strain along θ-coordinate

$$\varepsilon_{zz}(r,z)=\frac{\partial w(r,z)}{\partial z}$$ Normal strain along z-coordinate

$$\varepsilon_{rz}(r,z)=\frac{\partial u(r,z)}{\partial z}+\frac{\partial w(r,z)}{\partial r}$$ Shearing strain

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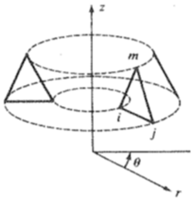
Formulas Derived from Theory of Elasticity - Cont'd

Stress-Strain Relation – derived from Generalized Hooke's law

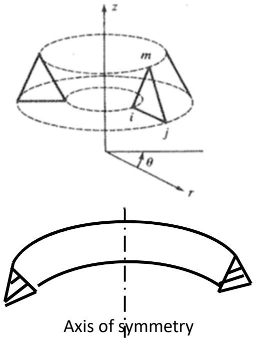


$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{zz}\\ \sigma_{\theta\theta}\\ \sigma_{rz}\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0\\ \nu & 1-\nu & \nu & 0\\ \nu & \nu & 1-\nu & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}$$

$$[\mathrm{D}]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0\\ \nu & 1-\nu & \nu & 0\\ \nu & \nu & 1-\nu & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}$$

Finite Element Formulation of Solid Structures of Axisymmetric Geometry

Step 1 Select Element Type:

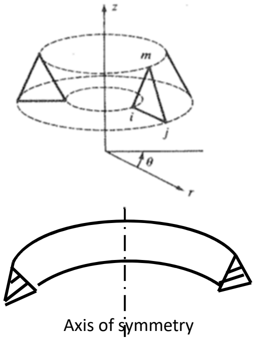


Our FE formulation will be based on a typical triangular element.

The element has three nodes with two degree-of-freedom per node, e.g., u_i and w_i at node i, etc.

Derivation of Stiffness Matrix – cont'd

Step 2 Select Element Displacement Functions:



We will stick to simplex elements. So, the displacements in the element will be of the forms of simple linear polynomial functions as:

Displacement in r-direction: $u(r,z)=\alpha_1+\alpha_2 r+\alpha_3 z$

Displacement in z-direction: $w(r,z)=\alpha_4+\alpha_5 r+\alpha_6 z$

The six nodal displacements are expressed as follow:

Nodal displacements: $$\{d\}=\begin{Bmatrix}\{d_i\}\\ \{d_j\}\\ \{d_m\}\end{Bmatrix}=\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

Displacements in 3 nodes

Displacement components in 3 nodes

For example: $$u_i=u(r_i,z_i)=\alpha_1+\alpha_2 r_i+\alpha_3 z_i$$
$$w_i=w(r_i,z_i)=\alpha_4+\alpha_5 r_i+\alpha_6 z_i$$ at Node i

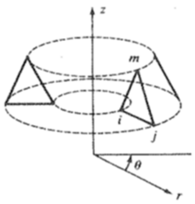
Derivation of Stiffness Matrix – cont'd

Step 3 Derive Interpolation Function:

The general displacement in the triangular torus element can thus be expressed as:

$$\{\phi\}=\begin{Bmatrix}u\\w\end{Bmatrix}=\begin{Bmatrix}\alpha_1+\alpha_2r+\alpha_3z\\\alpha_4+\alpha_5r+\alpha_6z\end{Bmatrix}=\begin{bmatrix}1&r&z&0&0&0\\0&0&0&1&r&z\end{bmatrix}\begin{Bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\alpha_5\\\alpha_6\end{Bmatrix}$$

Substituting the nodal coordinates into Equation (4.47) will result in:



$$\begin{Bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{Bmatrix}=\begin{bmatrix}1&r_i&z_i\\1&r_j&z_j\\1&r_m&z_m\end{bmatrix}^{-1}\begin{Bmatrix}u_i\\u_j\\u_m\end{Bmatrix}=\frac{1}{2A}\begin{bmatrix}\alpha_i&\alpha_j&\alpha_m\\\beta_i&\beta_j&\beta_m\\\gamma_i&\gamma_j&\gamma_m\end{bmatrix}\begin{Bmatrix}u_i\\u_j\\u_m\end{Bmatrix}$$

$$\begin{Bmatrix}\alpha_4\\\alpha_5\\\alpha_6\end{Bmatrix}=\begin{bmatrix}1&r_i&z_i\\1&r_j&z_j\\1&r_m&z_m\end{bmatrix}^{-1}\begin{Bmatrix}w_i\\w_j\\w_m\end{Bmatrix}=\frac{1}{2A}\begin{bmatrix}\alpha_i&\alpha_j&\alpha_m\\\beta_i&\beta_j&\beta_m\\\gamma_i&\gamma_j&\gamma_m\end{bmatrix}\begin{Bmatrix}w_i\\w_j\\w_m\end{Bmatrix}$$

where $\alpha_i=r_jz_m-z_jr_m$ $\quad\alpha_j=r_mz_i-z_mr_i$ $\quad\alpha_m=r_iz_j-z_ir_j$

$\beta_i=z_j-z_m$ $\quad\beta_j=z_m-z_i$ $\quad\beta_m=z_i-z_j$

$\gamma_i=r_m-r_j$ $\quad\gamma_j=r_i-r_m$ $\quad\gamma_m=r_j-r_i$

Derivation of Stiffness Matrix – cont'd

Step 3 Derive Interpolation Function – cont'd:

The interpolation function is:

$$[N(r,z)]=\begin{bmatrix}N_i&0&N_j&0&N_m&0\\0&N_i&0&N_j&0&N_m\end{bmatrix}$$

where $N_i=\frac{1}{2A}(\alpha_i+\beta_ir+\gamma_iz)$ $\quad N_j=\frac{1}{2A}(\alpha_j+\beta_jr+\gamma_jz)$ $\quad N_m=\frac{1}{2A}(\alpha_m+\beta_mr+\gamma_mz)$

with the cross-sectional area A obtained from: $2A=\begin{vmatrix}1&r_i&z_1\\1&r_j&z_j\\1&r_m&z_m\end{vmatrix}$

Element and Nodal displacements relation:

$$\{\phi\}=\begin{Bmatrix}u(r,z)\\w(r,z)\end{Bmatrix}=\begin{bmatrix}N_i&0&N_j&0&N_m&0\\0&N_i&0&N_j&0&N_m\end{bmatrix}\begin{Bmatrix}u_i\\w_i\\u_j\\w_j\\u_m\\w_m\end{Bmatrix}$$

Derivation of Stiffness Matrix – cont'd

Step 4 Define the Strain-displacement and Stress-Strain Relationship:

The strain components:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\begin{Bmatrix}\alpha_2\\ \alpha_6\\ \dfrac{\alpha_1}{r}+\alpha_2+\dfrac{\alpha_3 z}{r}\\ \alpha_3+\alpha_5\end{Bmatrix}$$

We may express the element strains and nodal displacements in the following expression:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\frac{1}{2A}\begin{Bmatrix}\beta_i & 0 & \beta_j & 0 & \beta_m & 0\\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m\\ \dfrac{\alpha_i}{r}+\beta_i+\dfrac{\gamma_i z}{r} & 0 & \dfrac{\alpha_j}{r}+\beta_j+\dfrac{\gamma_j z}{r} & 0 & \dfrac{\alpha_m}{r}+\beta_m+\dfrac{\gamma_m z}{r} & 0\\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m\end{Bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

with

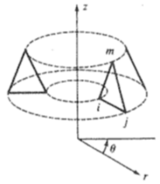
$$\alpha_i=r_j z_m-z_j r_m \quad \alpha_j=r_m z_i-z_m r_i \quad \alpha_m=r_i z_j-z_i r_j$$

$$\beta_i=z_j-z_m \quad \beta_j=z_m-z_i \quad \beta_m=z_i-z_j$$

$$\gamma_i=r_m-r_j \quad \gamma_j=r_i-r_m \quad \gamma_m=r_j-r_i$$

Derivation of Stiffness Matrix – cont'd

Step 4 Define the Strain-displacement and Stress-Strain Relationship:



$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\frac{1}{2A}\begin{Bmatrix}\beta_i & 0 & \beta_j & 0 & \beta_m & 0\\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m\\ \dfrac{\alpha_i}{r}+\beta_i+\dfrac{\gamma_i z}{r} & 0 & \dfrac{\alpha_j}{r}+\beta_j+\dfrac{\gamma_j z}{r} & 0 & \dfrac{\alpha_m}{r}+\beta_m+\dfrac{\gamma_m z}{r} & 0\\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m\end{Bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

Or:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\{[B_i]\quad[B_j]\quad[B_m]\}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

where $[B_i]=\dfrac{1}{2A}\begin{bmatrix}\beta_i & 0\\ 0 & \gamma_i\\ \dfrac{\alpha_i}{r}+\beta_i+\dfrac{\gamma_i z}{r} & 0\\ \gamma_i & \beta_i\end{bmatrix}$

The other components of {B} matrix in the Equation may be obtained by Substituting j for i for $[B_j]$ and m for $[B_m]$

We may thus express the element strains in terms of nodal displacements in the following expression:

$$\{\varepsilon\} = [B]\{d\}$$

in which {d} is nodal displacement matrix. One needs to notice that the [B] matrices are function of (r,z), which makes the non-constant strains in torus element with linear element displacement functions

Derivation of Stiffness Matrix – cont'd

Step 4 Define the Stress-Strain and Stress-nodal displacement Relationships:

We have relate the element stresses and element strains by the generalized Hooke's law for multi-axially loaded solids:

$$\{\sigma\} = [D]\{\varepsilon\}$$

Yet the element strains and nodal displacements are related :

$$\{\varepsilon\} = [B]\{d\}$$

We can thus relate the element stresses and nodal displacements to be:

$$\{\sigma\} = [D]\,[B]\,\{d\}$$

where $[B] = \left[[B_i] \quad [B_j] \quad [B_m] \right]$ with $[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix}$ etc. shown in Equation

---

Derivation of Stiffness Matrix – cont'd

Step 4 Define the Stress-Strain and Stress-nodal displacement Relationships – Cont'd:

The elasticity matrix [C] for torus element can be derived from Generalized Hooke's Law :

$$[\mathrm{D}] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Step 5 The Element stiffness matrix

Following Equation (4.39), we have the element equation for torus element to be:

$$[K_e]\{d\} = \{q\}$$

where $[K_e]$ = element stiffness matrix = $\int_v [B]^T [\mathrm{D}][B]\,dv = 2\pi \int_A [B]^T [\mathrm{D}][B](r\,dr\,dz)$

The following approximate method may be used to compute the $[K_e]$ matrix by letting:

$r \approx \bar{r} = \frac{r_i + r_j + r_m}{3}$ *and* $z \approx \bar{z} = \frac{z_i + z_j + z_m}{3}$, and thus use $[B] \approx [\bar{B}] = [B(\bar{r}, \bar{z})]$ in computing $[K_e]$

We may thus compute the $[K_e]$ matrix by the expression $[k_e] \approx 2\pi \bar{r} A [\bar{B}]^T [\mathrm{D}][\bar{B}]$

Derivation of Element Stiffness Equation

By following the general formulations of element stiffness equation:

$$[K_e]\{\phi\} = \{q\}$$

with $[K_e]$ stiffness matrix for element, $\{\phi\}$ = nodal displacements and $\{q\}$ = applied nodal forces

For triangular torus elements, the element stiffness equation takes the form:

$$[K_e]\{d\} = \{q\}$$

where {d} =nodal displacements and {q} = applied nodal forces

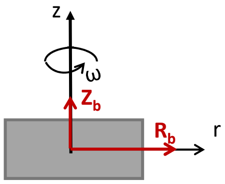
$$\{q\} = \int_v [N(r,z)]^T \{f_b\} dv + \int_s [N(r,z)]^T \{f_s\} ds$$

in which the interpolation function [N(r,z)]. The two types of forces $\{f_b\}$ = body forces, and $\{f_s\}$ = surface tractions appear frequently in axisymmetric solid structures.

Derivation of Element Stiffness Equation

Body forces $\{f_b\}$ in axisymmetric structures

Body forces {f} in finite element formulations means the forces that are **distributed** in the solid structures. Weight is one form of body force. The centrifugal forces generated by the spinning mass, such as spinning wheels (i.e., flywheels), disks or cylinder (rotors) are common place in structures of axisymmetric geometry. We will formulate {f} of the latter types of body forces as follows: Rb



The mathematical expression of the body forces $\{f_b\}$:

$$\{f_b\} = 2\pi \int_A [N(r,z)]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} rdrdz$$

where $R_b = \omega^2 \rho r$ with ω = angular velocity of the spinning mass and ρ = mass density of the material.
Z_b = body force per unit volume of the material along the z-direction

Body force at Node i:

$$\{f_{bi}\} = 2\pi \int_A [N_i]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} rdrdz$$

where $[N_i]^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$

Alternately, with approximation if the triangular torus element is not too large, we may use:

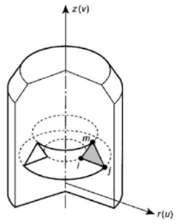
with $\bar{R}_b = \omega^2 \rho \bar{r}$

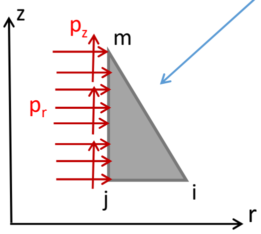
$$\{f_b\} \approx \frac{2\pi \bar{r} A}{3} \begin{Bmatrix} \bar{R}_b \\ \bar{Z}_b \\ \bar{R}_b \\ \bar{Z}_b \\ \bar{R}_b \\ \bar{Z}_b \end{Bmatrix}$$

Derivation of Element Stiffness Equation

Surface traction forces {f$_s$} in axisymmetric structures

Surface traction means the distributed forces acing on the surface of the solid structure, such as pressure loadings.
For structures of axisymmetric geometry, such as pressure vessels of cylindrical geometry there could be pressure loadings
Acting on the inner or outside surface of the pressure vessels. So, we will take a close look at this kind of load and also
Derive mathematical expression for the surface tractions acting at nodes of torus elements.





The surface traction (or forces) can be expressed as:

$$\{f_s\} = \int_s [N_s]^T \{T\} ds$$

where [N$_s$]=the interpolation function evacuated along the surface where the surface traction acts

The radial and axial pressure p$_r$ and p$_z$ may thus expressed as:

$$\{f_s\} = \int_s [N_s]^T \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} ds$$

Surface forces along the surface j-m with r = r$_j$:

For Node j, substituting N$_j$ leads to:

$$\{f_{sj}\} = \int_{z_j}^{z_m} \frac{1}{2A} \begin{bmatrix} \alpha_j + \beta_j + \gamma_j & 0 \\ 0 & \alpha_j + \beta_j + \gamma_j \end{bmatrix} \begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r_j dz$$

evaluated at r=r$_j$, z=z

The total distribution of surface force to node I,j and m is:

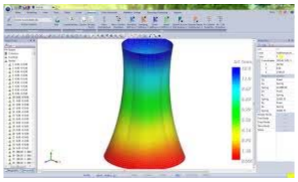
$$\{f_s\} = \frac{2\pi r_j (z_m - z_j)}{2} \begin{Bmatrix} 0 \\ 0 \\ p_r \\ p_z \\ p_r \\ p_z \end{Bmatrix}$$

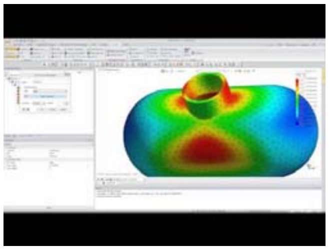
Finite Element Formulation of Solid Structures of Axisymmetric Geometry

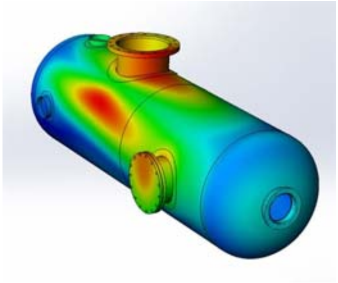
Step 6 Assemble the element equations to Overall Stiffness Equations

Step 7 Solution of primary unknown (nodal displacements) from overall stiffness equations

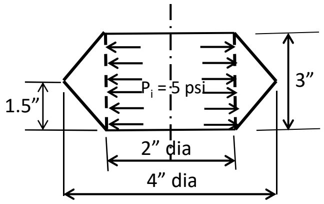
Step 8 Solve secondary unknown quantities of strains and stresses in all elements of the discretized model







Example Stress analysis of a pressure ring



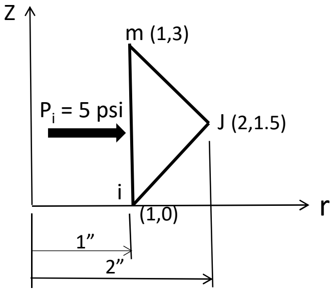
Material: Aluminum with following properties:

Young’s modulus E =70,000 x 10^6 psi
Poisson’s ratio v = 0.3

Finite element model with one element:

Nodal coordinates:

$r_i = 1$	$r_j = 2$	$r_m = 1$
$z_i = 0$	$z_j = 1.5$	$z_m = 3$



Cross-sectional area of the element:

$$2A = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1.5 \\ 1 & 1 & 3 \end{vmatrix} = 3$$

from which the cross-sectional area A = 1.5 in^2

Example Stress analysis of a pressure ring – cont’d

Derive the Interpolation function:

Components of the interpolation function from Equation (4.49):

$$\alpha_i = r_j z_m - z_j r_m = 2\times3-1.5\times1=4.5 \qquad \beta_i = z_j - z_m = 1`.5-3=-1.5 \qquad \gamma_i = r_m - r_j = 1-2=-1$$
$$\alpha_j = r_m z_i - z_m r_i = 1\times0-3\times1=-3 \qquad \beta_j = z_m - z_i = 3-0=3 \qquad \gamma_j = r_i - r_m = 1-1=0$$
$$\alpha_m = r_i z_j - z_i r_j = 1\times1.5-0\times2=1.5 \qquad \beta_m = z_i - z_j = 0-1.5=-1.5 \qquad \gamma_m = r_j - r_i = 2-1=1$$

The interpolation function:

$$[N(r,z)] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

with the components N$_i$, N$_j$ and N$_m$ expressed in Equation (4.51) to be:

$$N_i = \frac{1}{3}(4.5-1.5r-z) \qquad N_j = \frac{1}{3}(-3+3r) \qquad N_m = \frac{1}{3}(1.5-1.5r+z)) \qquad \text{(a)}$$

Example Stress analysis of a pressure ring – cont’d

Element strain and nodal displacements:

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\frac{1}{2A}\begin{Bmatrix}\beta_i & 0 & \beta_j & 0 & \beta_m & 0\\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m\\ \frac{\alpha_i}{r}+\beta_i+\frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r}+\beta_j+\frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r}+\beta_m+\frac{\gamma_m z}{r} & 0\\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m\end{Bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}-1.5 & 0 & 3 & 0 & -1.5 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ \frac{4.5}{r}-1.5-\frac{z}{r} & 0 & -\frac{3}{r}+3+0 & 0 & \frac{1.5}{r}-1.5+\frac{z}{r} & 0\\ -1 & -1.5 & 0 & 3 & 1 & -1.5\end{bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

Example Stress analysis of a pressure ring – cont’d

Elasticity matrix:

$$[\mathrm{D}]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0\\ \nu & 1-\nu & \nu & 0\\ \nu & \nu & 1-\nu & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}=\frac{700000\times10^6}{1.3\times0.4}\begin{bmatrix}0.7 & 0.3 & 0.3 & 0\\ 0.3 & 0.7 & 0.3 & 0\\ 0.3 & 0.3 & 0.7 & 0\\ 0 & 0 & 0 & 0.2\end{bmatrix}.$$

$$=4.03846\times10^{10}\begin{bmatrix}2.33 & 1 & 1 & 0\\ 1 & 2.33 & 1 & 0\\ 1 & 1 & 2.33 & 0\\ 0 & 0 & 0 & 0.67\end{bmatrix}$$

Example Stress analysis of a pressure ring – cont'd

Element stiffness matrix [K$_e$]:

$$[K_e]=\int_v [B]^T[\mathrm{D}][B]dv=2\pi\int_A [B]^T[\mathrm{D}][B](rdrdz)$$

We approximate [Ke] by using the average r and z in lieu of integration.

These average coordinates $\bar{r}$ *and* $\bar{z}$ are defined as:

$$\bar{r}=\frac{r_i+r_j+r_m}{3}=\frac{1+2+1}{3}=1.3333\ \textit{inches, and}\ \bar{z}=\frac{z_i+z_j+z_m}{3}=\frac{0+1.5+3}{3}=1.5\ \textit{inches}$$

The [B] matrix, with {[B$_i$] [B$_j$] [B$_m$]} with [B$_i$], [B$_j$] and [B$_m$] will now be replace by:

$$\left[\bar{B}(\bar{r},\bar{z})\right]\equiv\left\{\left[\bar{B}_i(\bar{r},\bar{z})\right]\ \left[\bar{B}_j(\bar{r},\bar{z})\right]\ \left[\bar{B}_m(\bar{r},\bar{z})\right]\right\} \qquad \text{(b)}$$

where

$$\left[\bar{B}_i(\bar{r},\bar{z})\right]=\frac{1}{3}\begin{bmatrix}-1.5 & 0\\ 0 & -1\\ 1.25 & 0\\ -1 & -1.5\end{bmatrix} \qquad \left[\bar{B}_j(\bar{r},\bar{z})\right]=\frac{1}{3}\begin{bmatrix}3 & 0\\ 0 & 0\\ 0.75 & 0\\ -1 & 3\end{bmatrix} \qquad \left[\bar{B}_m(\bar{r},\bar{z})\right]=\frac{1}{3}\begin{bmatrix}1.5 & 0\\ 0 & 1\\ -0.75 & 0\\ 1 & -1.5\end{bmatrix}$$

Example Stress analysis of a pressure ring – cont'd

Element stiffness matrix [K$_e$]-cont'd:

We may express the $\left[\bar{B}(\bar{r},\bar{z})\right]$ matrix in the form:

$$\left[\bar{B}(\bar{r},\bar{z})\right]=\frac{1}{3}\left\{\begin{bmatrix}-1.5 & 0\\ 0 & -1\\ 1.25 & 0\\ -1 & -1.5\end{bmatrix}\ \begin{bmatrix}3 & 0\\ 0 & 0\\ 0.75 & 0\\ -1 & 3\end{bmatrix}\ \begin{bmatrix}1.5 & 0\\ 0 & 1\\ -0.75 & 0\\ 1 & -1.5\end{bmatrix}\right\} \qquad \text{(c)}$$

The transpose of the $\left[\bar{B}(\bar{r},\bar{z})\right]$ Is:

$$\left[\bar{B}(\bar{r},\bar{z})\right]^T=\frac{1}{3}\left\{\begin{bmatrix}-1.5 & 0 & 1.25 & -1\\ 0 & -1 & 0 & -1.5\end{bmatrix}\ \begin{bmatrix}3 & 0 & 0.75 & -1\\ 0 & 0 & 0 & 3\end{bmatrix}\ \begin{bmatrix}1.5 & 0 & -0.75 & 1\\ 0 & 1 & 0 & -1.5\end{bmatrix}\right\} \qquad \text{(d)}$$

we will have the element stiffness matrix in the following form:

$$[K_e]\approx(2\pi)(1.3333)(1.5)\left[\bar{B}(\bar{r},\bar{z})\right]^T[\mathrm{D}]\left[\bar{B}(\bar{r},\bar{z})\right] \qquad \text{(e)}$$

Example

Element stiffness matrix [K$_e$]-cont'd:

$$[K_e]=\frac{50.72\times10^{10}}{3}\left\{\overset{[B_i]^T}{\begin{bmatrix}-1.5 & 0 & 1.25 & -1\\ 0 & -1 & 0 & -1.5\end{bmatrix}}\ \overset{[B_j]^T}{\begin{bmatrix}3 & 0 & 0.75 & -1\\ 0 & 0 & 0 & 3\end{bmatrix}}\ \overset{[B_m]^T}{\begin{bmatrix}1.5 & 0 & -0.75 & 1\\ 0 & 1 & 0 & -1.5\end{bmatrix}}\right\}$$

$$\underset{[D]}{\begin{bmatrix}2.33 & 1 & 1 & 0\\ 1 & 2.33 & 1 & 0\\ 1 & 1 & 2.33 & 0\\ 0 & 0 & 0 & .67\end{bmatrix}}\left\{\underset{[B_i]}{\begin{bmatrix}-1.5 & 0\\ 0 & -1\\ 1.25 & 0\\ -1 & -1.5\end{bmatrix}}\ \underset{[B_j]}{\begin{bmatrix}3 & 0\\ 0 & 0\\ 0.75 & 0\\ -1 & 3\end{bmatrix}}\ \underset{[B_m]}{\begin{bmatrix}1.5 & 0\\ 0 & 1\\ -0.75 & 0\\ 1 & -1.5\end{bmatrix}}\right\} \qquad \text{(f)}$$

from which we may express the stiffness matrices relating to the three nodes i, j and m:

$$[K_e^i]=\frac{50\times10^{10}}{3}[B_i]^T[\mathrm{D}][B_i] \quad \textit{for Node i}$$

$$[K_e^j]=\frac{50\times10^{10}}{3}[B_j]^T[\mathrm{D}][B_j] \quad \textit{for Node j}$$

$$[K_e^m]=\frac{50\times10^{10}}{3}[B_m]^T[\mathrm{D}][B_m] \quad \textit{for Node m}$$

Example

Element stiffness matrix [K$_e$]-cont'd:

We thus have the following stiffness matrices relating to the 3 nodes for the present example to be:

$$[K_e^i]=\frac{50\times10^{10}}{3}[B_i]^T[\mathrm{D}][B_i]=16.91\times10^{10}\begin{bmatrix}5.0803 & 1.255\\ 1.255 & 3.8375\end{bmatrix} \quad \textit{for Node i} \qquad \text{(g1)}$$

$$[K_e^j]=\frac{50\times10^{10}}{3}[B_j]^T[\mathrm{D}][B_j]=16.91\times10^{10}\begin{bmatrix}27.45 & -2.01\\ -2.01 & 6.03\end{bmatrix} \quad \textit{for Node j} \qquad \text{(g2)}$$

$$[K_e^m]=\frac{50\times10^{10}}{3}[B_m]^T[\mathrm{D}][B_m]=16.91\times10^{10}\begin{bmatrix}4.9731 & -0.255\\ -0.255 & 3.8375\end{bmatrix} \quad \textit{for Node m} \qquad \text{(g3)}$$

Example

Element stiffness equation

We may express the element stiffness equation in the form:

$$\begin{bmatrix} [K_e^i] & 0 & 0 \\ 0 & [K_e^j] & 0 \\ 0 & 0 & [K_e^m] \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} = \begin{Bmatrix} q_{ri} \\ q_{zi} \\ q_{rj} \\ q_{zj} \\ q_{rm} \\ q_{zm} \end{Bmatrix} \qquad \text{(h)}$$

By substituting the 3 sub-stiffness matrices in Equations (g1), (g2) and (g3) into Equation (h), we will have the element equation in the following form:

$$16.91\times10^{10}\begin{bmatrix} 5.0803 & 1.255 & 0 & 0 & 0 & 0 \\ 1.255 & 3.8375 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27.45 & -2.01 & 0 & 0 \\ 0 & 0 & -2.01 & 6.03 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.9731 & -0.255 \\ 0 & 0 & 0 & 0 & -0.255 & 3.8375 \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} = \begin{Bmatrix} q_{ri} \\ q_{zi} \\ q_{rj} \\ q_{zj} \\ q_{rm} \\ q_{zm} \end{Bmatrix} \qquad \text{(j)}$$

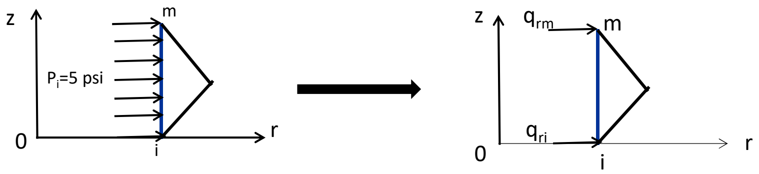
The nodal forces q_{ri}, q_{zi}, q_{rj}, q_{zj}, q_{rm} and q_{zm} in Equations (h) and (j)are the APPLIED FORCES at Node i, j and m respectively.

Example

Overall stiffness equation

The Element stiffness equation in Equation (h) is used as the Overall stiffness equation for the structure because the structure has only one element. We may thus use this equation to compute the six displacements at all 3 nodes.

We observe from the loading to the structure to be the pressure Pi applied to the inner surface in the radial direction. The uniform pressure applied to the inner surface may be converted to the concentrated forces acting at Nodes i and m by the following formula:



The equivalent nodal forces are:

$$q_{ri} = q_{rm} = \frac{2\pi r_i(z_m - z_i)}{2}P_i = \pi r_i(z_m - z_i)P_i = 3.14(1)(3-0)\times5 = 47.1\ lb_f$$

Example

Solution of nodal displacements from overall stiffness equations

We are now ready to solve the displacement components at each of the three nodes in the element by inputting the applied forces at the nodes, as shown in the following equations [K]{d} ={q} with numerical values:

$$16.91\times10^{10}\begin{bmatrix} 5.0803 & 1.255 & 0 & 0 & 0 & 0 \\ 1.255 & 3.8375 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27.45 & -2.01 & 0 & 0 \\ 0 & 0 & -2.01 & 6.03 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.9731 & -0.255 \\ 0 & 0 & 0 & 0 & -0.255 & 3.8375 \end{bmatrix}\begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}=\begin{Bmatrix} q_{ri}=47.1 \\ q_{zi}=0 \\ q_{rj}=0 \\ q_{zj}=0 \\ q_{rm}=47.1 \\ q_{zm}=0 \end{Bmatrix} \qquad \text{(k)}$$

[K] {d} {q}

The nodal displacement {d} in Equation (k) may be solved by matrix inversion:

$$\{d\}=\frac{1}{16.91\times10^{10}}[K]^{-1}\{q\} \qquad \text{(m)}$$

Example

Solution of nodal displacements from overall stiffness equations

The inverse of [K] matrix in Equation (k) is:

$$[K]^{-1}=\begin{bmatrix} 0.2145 & -0.07 & 0 & 0 & 0 & 0 \\ -0.07 & 0.2835 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.03734 & 0.01245 & 0 & 0 \\ 0 & 0 & 0.01245 & 0.17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2018 & 0.01341 \\ 0 & 0 & 0 & 0 & 0.01341 & 0.2615 \end{bmatrix}$$

We thus have the 6 nodal displacements from Equation (m) to be:

$$\{d\}=\begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}=\frac{1}{16.91\times10^{10}}\begin{Bmatrix} 10.0892 \\ -3.297 \\ 0 \\ 0 \\ 9.5048 \\ 0.06316 \end{Bmatrix}=10^{-12}\begin{Bmatrix} 59.66 \\ -19.5 \\ 0 \\ 0 \\ 56.33 \\ 3.74 \end{Bmatrix} \textit{inches}$$

Example

Strains in the structure

The element, and thus structure strains may be obtained from Equation (4.54) with $r=\bar{r}=1.3333", \; z=\bar{z}=1.5" \; and \; 2A=3\,in^2$

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{zz}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{rz}\end{Bmatrix}=\frac{1}{3}\begin{bmatrix}-1.5 & 0 & 3 & 0 & -1.5 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ \frac{4.5}{r}-1.5-\frac{z}{r} & 0 & -\frac{3}{r}+3+0 & 0 & \frac{1.5}{r}-1.5+\frac{z}{r} & 0\\ -1 & -1.5 & 0 & 3 & 1 & -1.5\end{bmatrix}\begin{Bmatrix}u_i\\ w_i\\ u_j\\ w_j\\ u_m\\ w_m\end{Bmatrix}$$

$$=\frac{10^{-12}}{3}\begin{bmatrix}-1.5 & 0 & 3 & 0 & -1.5 & 0\\ 0 & -1 & 0 & 0 & 0 & 1\\ 0.75 & 0 & 0.75 & 0 & 0.75 & 0\\ -1 & -1.5 & 0 & 3 & 1 & -1.5\end{bmatrix}\begin{Bmatrix}59.66\\ -19.5\\ 0\\ 0\\ 56.33\\ 3.74\end{Bmatrix}=\begin{Bmatrix}-48\\ 7.75\\ 29\\ 6.77\end{Bmatrix}10^{-12}$$

Example

Stresses in the structure

Stresses in the structure {σ} = [C]{ε}:

$$\{\sigma\}=\begin{Bmatrix}\sigma_{rr}\\ \sigma_{zz}\\ \sigma_{\theta\theta}\\ \sigma_{rz}\end{Bmatrix}=\left(4.03846\times10^{10}\right)\times\begin{bmatrix}2.33 & 1 & 1 & 0\\ 1 & 2.33 & 1 & 0\\ 1 & 1 & 2.33 & 0\\ 0 & 0 & 0 & 0.67\end{bmatrix}\begin{Bmatrix}-58\\ 7.75\\ 29\\ 6.77\end{Bmatrix}\left(10^{-12}\right)$$

$$=\left(4.03846\times10^{-2}\right)\begin{Bmatrix}-295.2\\ -32.85\\ 51.95\\ 13.6\end{Bmatrix}=\begin{Bmatrix}-11.92\\ -1.3265\\ 2.097\\ 0.55\end{Bmatrix}psi$$

We thus have the induced stresses in the wall of this pressure ring to be:

σ_{rr} = -11.92 psi, σ_{zz} = -1.3265 psi, $\sigma_{\theta\theta}$= 2.097 psi and σ_{rz} = 0.55 psi

SUMMARY

1. A brief overview of key formulations of linear theory of elasticity relevant to the finite element (FE) formulation for stress analysis of deformable elastic solid structures subjected to applied forces, either in the form of body force or as surface tractions.
2. The small deformation of the solids is supposed so stresses are within the elastic limit of the material
3. The FE formulation is based on the theories of linear elasticity
4. General FE formulation is presented for "simplex elements" only. (simplex elements is defined with element displacements follow linear polynomial functions)
5. FE are presented for the following specific types of structures:

(a) 1-D bar elements : Solid bars subject to forces and deformation along the length of the bar
(b) 1-D bar elements : Trusses in 2-D plane with planar displacements at nodes but force and stresses along the length
(c) Beam elements : Beam bending in planes
(d) 2-D triangular plate elements : Thin plates with in-plane loading and deformation
(e) 3-D triangular torus elements : Axisymmetric structures

SUMMARY – cont'd

6. Key FE equations are:

a) Element displacements and nodal displacement:

$$Element\ displacements\ \{\Phi\}=[N(\mathbf{r})]\{\phi\}\quad with\{\phi\}=Nodal\ displacements$$

The interpolation function $[N(\mathbf{r})]$ *are available in :*

Key equation for 1-D bar and truss elements; beam elements; triangular plate elements; triangular torus elements

b) Element strains and element displacements: $\{\varepsilon\}=[D]\{\Phi\}\quad with[D]Operator$

c) Element strains and nodal displacements: $\{\varepsilon\}=[D][N(\mathbf{r})]\{\phi\}=[B]\{\phi\}$

d) Element stresses and element strains and nodal displacements:

$$\{\sigma\}=[\mathrm{D}]\{\varepsilon\}=[\mathrm{D}][B]\{\varphi\}\ with[\mathrm{D}]\ moudulus\ Matrix$$

e) Strain energy in element: $U=\frac{1}{2}\int_v\{\varepsilon\}^T\{\sigma\}dv=\frac{1}{2}\int_v\{\varphi\}^T[B]^T[\mathrm{D}][B]\{\varphi\}\,dv=\frac{1}{2}\{\varphi\}^T\left(\int_v[B]^T[\mathrm{D}][B]dv\right)\{\varphi\}$

f) Work done to the element by applied body forces {f} and surface tractions {t}:

$$W=\int_v[N(\mathbf{r})]^T\{f\}dv+\int_s[N(\mathbf{r})]^T\{t\}ds$$

g) Potential energy in elements: $\Pi(\{\phi\})=U+(-W)=U-W$

SUMMARY – cont'd

h) Element equation with minimization of potential energy:

$$\frac{\partial \Pi(\{\phi\})}{\partial \{\phi\}} = 0 \quad \textit{leading to:}$$

j) Element equations: $[K_e]\{\phi\} = \{q\}$

Element stiffness matrix: $[K_e]$ = element stiffness matrix $= \int_v [B]^T [\mathrm{D}][B] dv$

Applied nodal forces: $\{q\} = \int_v [N(\mathbf{r})]^T \{f\} dv + \int_s [N(\mathbf{r})]^T \{t\} ds$

k) Overall Stiffness equations: $[K]\{\phi\} = \{Q\}$

with Overall stiffness matrix: $[K] = \sum_{M=1}^{n} [K_e^M]$

and overall applied loading matrix: $\{Q\} = \sum_{1}^{M} \{q\}$