روش های عددی در ژئومکانیک

Numerical methods in geotechnics

Meshless methods

Hasan Ghasemzadeh

http://wp.kntu.ac.ir/ghasemzadeh

دانشگاه صنعتی خواجه نصیرالدین طوسی

مقدمه و معرفی

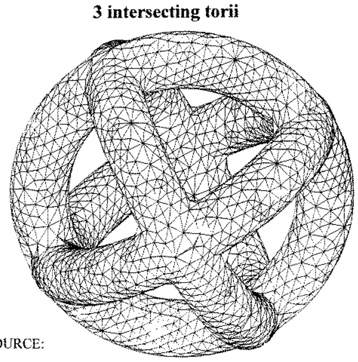
Meshless methods

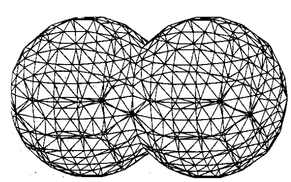
Radial Basis Functions (RBFs)

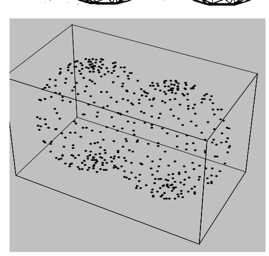
معرفی روشهای بدون شبکه

معرفی توابع پایه شعاعی

3 intersecting torii



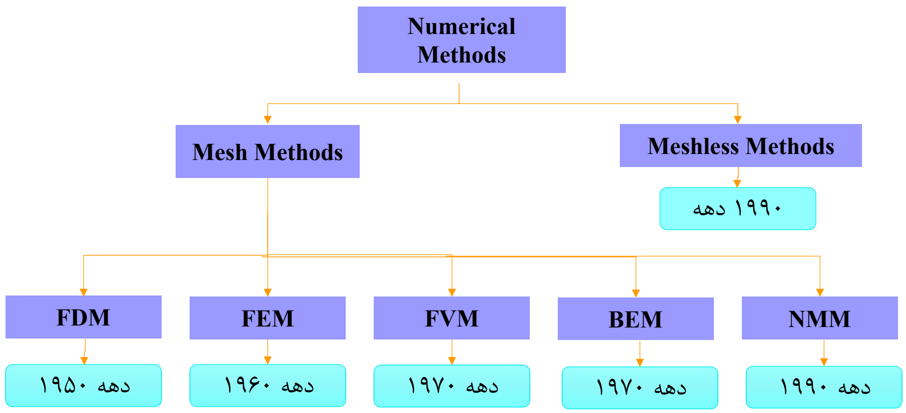




SOURCE:

R. Widman, **an efficient algorithm for the triangulation of surfaces in 3D.** Preprint-Math Dept. Colorado State University,

مقدمه و معرفی



اکنون نرم افزار MIDAS با روش بدون شبکه تجاری شده است

Advantages of Meshless Methods

- It requires neither domain nor surface discretization.
- The formulation is similar for 2D and 3D problems.
- It does not involve numerical integration.
- Ease of learning.
- Ease of coding.
- Cost effectiveness due to the man-power reduction involved for the meshing.

Meshless Methods

- RBF collection Method (Kansa's Method)
- MFS-DRM(Method of Fundamental Solutions -Dual Reciprocity Method)
- SPH(Smooth Particle Hydrodynamics,1977)
- DEM(Diffuse Element Method, 1992)
- DPD(Dissipative Particle Dynamics, 1992)
- RKPM(Reproducing Kernel Particle Method, 1995)
- MLS(Moving Least Squares)
- PIM(Point Interpolation Method)
- EFG(Element Free Galerkin method, 1994)
- FPM(Finite Pointset Method, 1998)

Radial Basis Functions

Let $\varphi : R^+ \to R$ be a continous function with $\varphi(0) \geq 0$. If $\mathbf{x}_i \in \Omega$, let

$$\varphi_i(\mathbf{x}) = \varphi(\|\mathbf{x} - \mathbf{x}_i\|),$$

where $\|\bullet\|$ is the Euclidean norm. Then φ_i is called the RBF.

Linear: r

Cubic: r^3

Multiquadrics: $\sqrt{r^2 + c^2}$ where c is a shape parameter.

Polyharmonic Spines: $\begin{cases} r^{2n} \log r, & n \geq 1, \quad \text{in 2D}, \\ r^{2n-1}, & n \geq 1, \quad \text{in 3D}. \end{cases}$

Gaussian: e^{-cr^2}

Wendland's CS-RBFs

Define

$$(1-r)_+^n = \begin{cases} (1-r)^n, & 0 \le r \le 1, \\ 0, & r > 1. \end{cases}$$

For 1D,

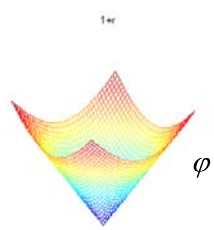
$$\varphi = (1-r) \in C^0$$
$$\varphi = (1-r)^3(3r+1) \in C^2$$
$$\varphi = (1-r)^5(8r^2+5r+1) \in C^4$$

For *2D&3D*,

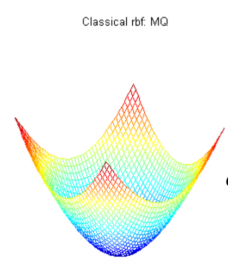
$$\varphi = (1-r)^2 \in C^0$$
$$\varphi = (1-r)^4(4r+1) \in C^2$$
$$\varphi = (1-r)^6(35r^2+18r+3) \in C^4$$
$$\varphi = (1-r)^8(32r^3+25r^2+8r+1) \in C^6$$

Compactly Supported RBFs (CS-RBFs)

Globally Supported RBFs (GS-RBFs)



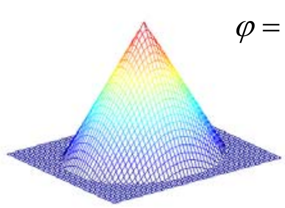
$\varphi = 1 + r$

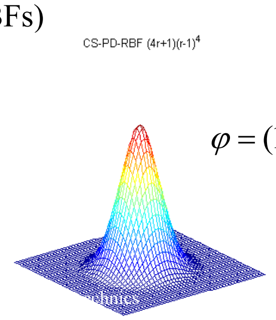


$\varphi = \sqrt{r^2 + c^2}$

Compactly Supported RBFs (CS-RBFs)

$\varphi = (1-r)^2$





$\varphi = (1-r)^4(4r+1)$

Surface Reconstruction Scheme

Assume that $f(\mathbf{x}) \approx \hat{f}(\mathbf{x})$

To approximate f by $\hat{f}$ we usually require fitting the given data set $\{\mathbf{x}_i\}_1^N$ of pairwise distinct centres with the imposed conditions

$$f(\mathbf{x}_i) = \hat{f}(\mathbf{x}_i), \qquad 1 \le i \le N.$$

The linear system

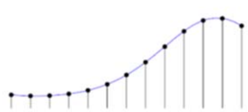
$$\hat{f}(\mathbf{x}_i) = \sum_{i=1}^{N} a_i \varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right), \quad 1 \le i \le N,$$

is well-posed if the interpolation matrix is non-singular

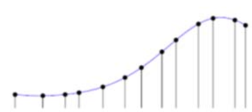
$$A_\varphi = \left[\varphi\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right]_{1 \le i \le N}$$

Surface Reconstruction Scheme

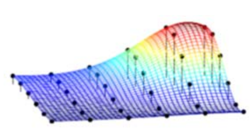
درونیابی داده های منظم و داده های پراکنده در یک و دو بعد



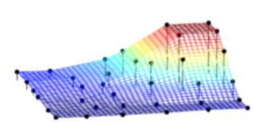
درون یابی داده های منظم در یک بعد



درون یابی داده های پراکنده در یک بعد



درون یابی داده های منظم در دو بعد



درون یابی داده های پراکنده در دو بعد

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Meshless Method I

RBF Collocation Method (Kansa's Method)

Consider the Poisson's equation

$$\nabla^2 u = f(x,y), \qquad (x,y) \in \Omega \qquad (1)$$

$$u = g(x,y), \qquad (x,y) \in \partial\Omega \qquad (2)$$

We approximate u by $\hat{u}$ by assuming

$$\hat{u}(x,y) = \sum_{j=1}^{N} c_j \varphi(r_j) \qquad (3)$$

where

$$r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

For Multiquadrics(MQ): $\varphi(r_j) = \sqrt{r_j^2 + c^2} = \sqrt{(x-x_j)^2 + (y-y_j)^2 + c^2}$

RBF Collocation Method (Kansa's Method)

$$\frac{\partial \varphi}{\partial x} = \frac{x-x_j}{\sqrt{r_j^2+c^2}}, \qquad \frac{\partial \varphi}{\partial y} = \frac{y-y_j}{\sqrt{r_j^2+c^2}}, \qquad (4)$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{(y-y_j)^2 + c^2}{\sqrt{r_j^2+c^2}}, \qquad \frac{\partial^2 \varphi}{\partial y^2} = \frac{(x-x_j)^2 + c^2}{\sqrt{r_j^2+c^2}}. \qquad (5)$$

By substituting (3), (5) into (1)-(2), we have

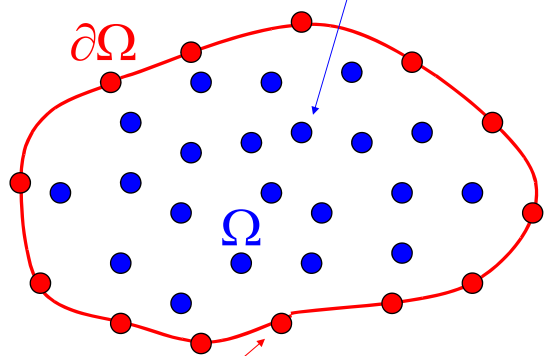
$$\sum_{j=1}^{N} \left(\frac{\partial^2 \varphi(r_j)}{\partial x^2} + \frac{\partial^2 \varphi(r_j)}{\partial y^2} \right) c_j = f(x_i, y_i), \quad i = 1, 2, \cdots N_I, \qquad (6)$$

$$\sum_{j=1}^{N} \varphi(r_j)\, c_j = g(x_i, y_i), \quad i = N_I + 1, N_I + 2, \cdots N. \qquad (7)$$

$\{c_j\}_{j=1}^{N}$ can be obtained by solving N×N system (6)-(7).

RBF Collocation Method (Kansa's Method)

$$\sum_{j=1}^{N}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)c_j = f(x_i, y_i), \quad i=1,2,\cdots,N_I$$



$$\sum_{j=1}^{N}\varphi(r_j)c_j = g(x_i, y_i), \quad i = N_I+1, N_I+2,\cdots,N$$

RBF Collocation Method (Kansa's Method)

For parabolic problems such as heat equation, we have

$$\frac{u^{n+1}}{\delta t}-\left(\frac{\partial^2 u^{n+1}}{\partial x^2}+\frac{\partial^2 u^{n+1}}{\partial y^2}\right)= f(x,y,t_n,u^n,u_x^n,u_y^n) \quad (8)$$

where δt is the time step, and u^n and u^{n+1} are the solutions at time $t_n=n\delta t$ and $t_{n+1}=(n+1)\delta t$.

Similar to elliptic problems, we assume

$$u(x,y,t_{n+1})=\sum_{j=1}^{N}c_j^{n+1}\varphi(r_j) \quad (9)$$

RBF Collocation Method (Kansa's Method)

Substituting (9) into (8) and (2), we have

$$\sum_{j=1}^{N}\left(\frac{\varphi(r_j)}{\delta t}-\frac{\partial^2\varphi(r_j)}{\partial^2 x}-\frac{\partial^2\varphi(r_j)}{\partial^2 y}\right)c_j^{n+1}$$
$$=\frac{u^n}{\delta t}(x_i,y_i)+f(x_i,y_i,t_n,u^n(x_i,y_i),u_x^n(x_i,y_i),u_y^n(x_i,y_i)),\quad i=1,2,\cdots,N_I,$$

$$\sum_{j=1}^{N}\varphi(r_j)c_j^{n+1}=g(x_i,y_i,t_{n+1}),\quad i=N_I+1,N_I+2,\cdots,N.$$

Notice that

$$u^n(x_i,y_i)=\sum_{j=1}^{N}c_j^n\varphi(r_j),$$

$$u_x^n(x_i,y_i)=\sum_{j=1}^{N}c_j^n\frac{\partial\varphi(r_j)}{\partial x},\quad u_y^n(x_i,y_i)=\sum_{j=1}^{N}c_j^n\frac{\partial\varphi(r_j)}{\partial y}$$

RBF Collocation Method (Kansa's Method)

Example I

$$\nabla^2 u=13\exp(-2x+3y)\qquad (x,y)\in\Omega$$
$$u=\exp(-2x+3y)\qquad (x,y)\in\partial\Omega$$

where $\Omega=[0,1]\times[0,1]$

RBFs: MQ, $c = 0.8$; *i.e.*, $\varphi(r)=\sqrt{r^2+c^2}$.

Grid points: 19x19

Maximum error: 8.703E-5

RBF Collocation Method (Kansa's Method)

Approximate Sol. and Maximum Norm Error by Kansa's Method

Example II (Rotating Cone)

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(x, y, t), \qquad (x, y) \in \Omega,$$

$$u|_{\partial\Omega} = g(x, y, t), \qquad (x, y) \in \partial\Omega,$$

$$u|_{t=0} = h(x, y), \qquad (x, y) \in \Omega.$$

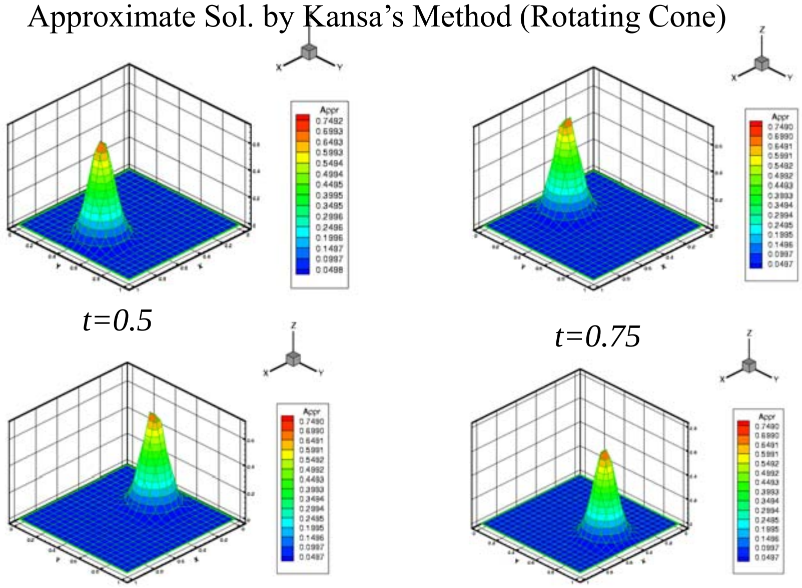
Exact solution: $u(x, y) = 0.8 \exp\left(-80\left[(x - r(t))^2 + (y - s(t))^2\right]\right)$

where $r(t) = \frac{1}{4}(2 + \sin \pi t), \quad s(t) = \frac{1}{4}(2 + \cos \pi t).$

$\delta t = 0.01, \ t \in [0, 2]$

Maximum norm error = 0.001165 with $c = 0.2$ (MQ).

Approximate Sol. by Kansa’s Method (Rotating Cone)



t=0.5 *t=0.75*

t=1.0 *t=1.25*

Maximum Error Norm by Kansa’s Method (Rotating Cone)

t=0.5 *t=0.75*

t=1.0 *t=1.25*

Example III (Burgers' Equation)

$$\frac{\partial u}{\partial t} = \alpha \Delta u - (uu_x + uu_y), \qquad (x, y) \in \Omega,$$
$$u|_{\partial\Omega} = g(x, y, t), \qquad (x, y) \in \partial\Omega,$$
$$u|_{t=0} = h(x, y), \qquad (x, y) \in \Omega.$$

Exact Solution
$$u(x, y, t) = \frac{1}{1 + \exp\left(\frac{x - y - t}{2\alpha}\right)}$$

$\delta t = 0.01$, $t \in [0, 1.25]$, $\alpha = 0.05$

Maximum norm error = 0.0719 with $c = 0.2$ (MQ).

Approximate Sol. of Kansa's Method – Burger's Equation

t=0.5

t=0.75

t=1.0

t=1.25

Maximum Norm Error of Kansa’s Method

t=0.5 *t=0.75*

t=1.0 *t=1.25*