روش های عددی در ژئومکانیک

Numerical methods in geomechnics

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کلیات روش‌های عددی در ژئومکانیک

– معادلات در خاک و پی
- کلیات روشهای عددی
- انواع روشهای عددی
- یادآوری محاسبات عددی
- میزان خطا
- قضیه تیلر
- اصول انرژی

Numerical Methodes in Geomechnics 2

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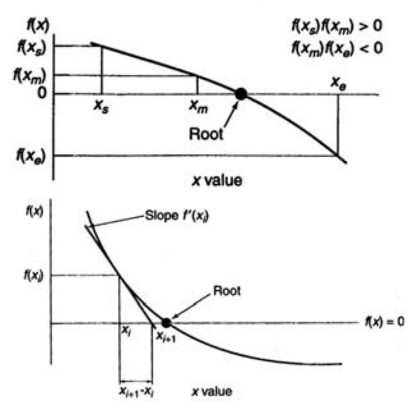
Numerical Methodes in Geomechnics 3

Methods for solving Nonlinear Equations

$$\left.\begin{array}{c} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{array}\right\} \text{No analytic solution}$$

- Bisection Method
- Newton-Raphson Method
- Secant Method
- Fixed-Point Method



Numerical Methodes in Geomechnics 4

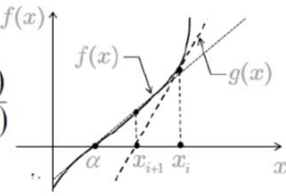
Methods for solving Nonlinear Equations

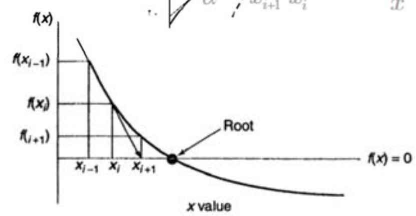
$$\left.\begin{aligned} x^9 - 2x^2 + 5 = 0 \\ x = e^{-x} \end{aligned}\right\} \text{No analytic solution}$$

- Bisection Method
- Newton-Raphson Method $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$
- Secant Method $x_{i+1} = x_i - \dfrac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$
- Fixed-Point Method

$$f(x) = 0 \quad \to g(x) = x$$

$$x_{i+1} = g(x_i)$$





Solution of Systems of Linear Equations

Cramer's Rule can be used to solve the system $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 3 \\ 5 \end{vmatrix}$

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

But Cramer's Rule is not practical for large problems.

To solve N equations in N unknowns we need $(N+1)(N-1)N!$ multiplications.

To solve a 30 by 30 system, 2.3×10^{35} multiplications are needed.

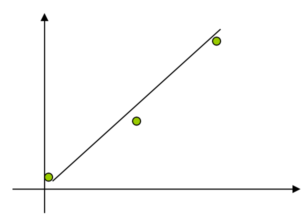
Methods for solving Systems of Linear Equations

- Naive Gaussian Elimination
- Gaussian Elimination with Scaled Partial pivoting
- Algorithm for Tri-diagonal Equations

Curve Fitting

- Given a set of data

x	0	1	2
y	0.5	10.3	21.3

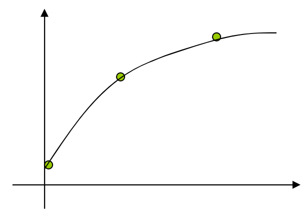


- Select a curve that best fit the data. One choice is find the curve so that the sum of the square of the error is minimized.

Interpolation

- **Given a set of data**

x_i	0	1	2
y_i	0.5	10.3	15.3



- **find a polynomial *P(x)* whose graph passes through all tabulated points.**

$$y_i = P(x_i) \quad if \ x_i \text{ is in the table}$$

Methods for Curve Fitting

- **Least Squares**
  - **Linear Regression**
  - **Nonlinear least Squares Problems**
- **Interpolation**
  - **Newton polynomial interpolation**
  - **Lagrange interpolation**

Methods for Numerical Integration

$$\int_0^a e^{-x^2} dx = ?$$ no analytical solutions

- **Upper and Lower Sums**
- **Trapezoid Method**
- **Romberg Method**
- **Gauss Quadrature**

Solution of Partial Differential Equations

Partial Differential Equations are more difficult to solve than ordinary differential equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} + 2 = 0$$

$$u(0,t) = u(1,t) = 0,\ u(x,0) = \sin(\pi x)$$

Numerical Solution ?

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Normalized Floating Point Representation

- Standard Representations

$$\pm \; 3\;1\;2\;1\;2 \; . \; 4\;5\;1\;5$$

sign integral part fraction part

- Normalized Floating Point Representation

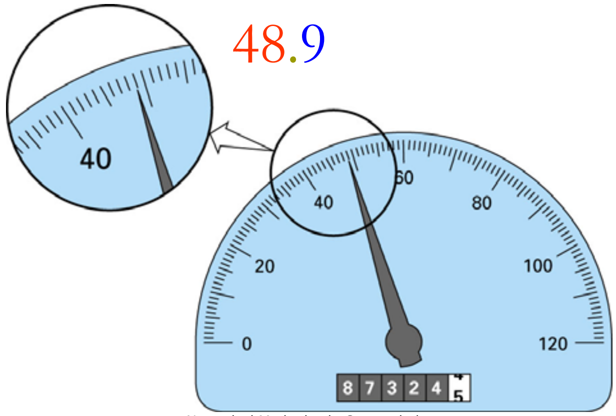
$$\pm \; \underline{0.\,d_1\,d_2\,d_3\,d_4} \times 10^n$$

sign mantissa exponent

- Advantage Efficient in representing very small or very large numbers

Significant Digits

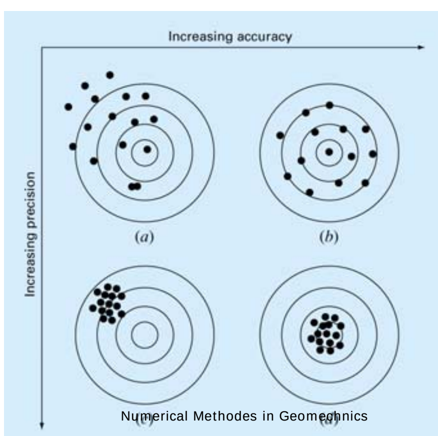
- Significant digits are those digits that can be used with confidence.



Numerical Methodes in Geomechnics 15

Accuracy and Precision

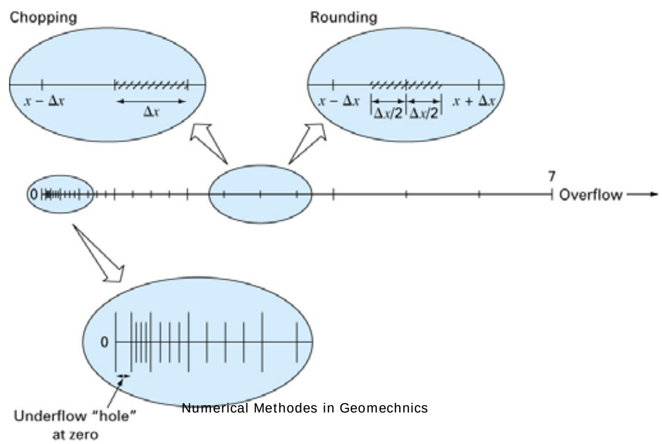
- **Accuracy** is related to closeness to the true value
- **Precision** is related to the closeness to other estimated values



Numerical Methodes in Geomechnics 16

Rounding and Chopping

- **Rounding**: Replace the number by the nearest machine number
- **Chopping**: Throw all extra digits



Error Definitions

True Error if the true value is known

Absolute True Error

$$E_t = \left| \text{true value} - \text{approximation} \right|$$

Absolute Percent Relative Error

$$\varepsilon_t = \left| \frac{\text{true value} - \text{approximation}}{\text{true value}} \right| * 100$$

Error Definitions

Estimated error

When the true value is not known

Estimated Absolute Error

$$E_a = \left| \text{current estimate} - \text{prevoius estimate} \right|$$

Estimated Absolute Percent Relative Error

$$\varepsilon_a = \left| \frac{\text{current estimate} - \text{prevoius estimate}}{\text{current estimate}} \right| * 100$$

Notation

the estimate is correct to n decimal digits if

$$\left| \text{Error} \right| \le 10^{-n}$$

the estimate is correct to n decimal digits **rounded** if

$$\left| \text{Error} \right| \le \frac{1}{2} \times 10^{-n}$$

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Numerical Methodes in Geomechnics 21

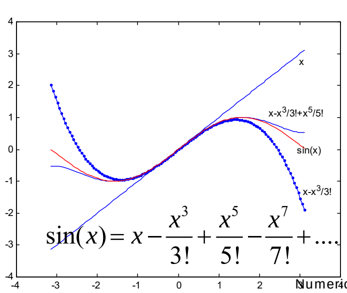
Taylor Series

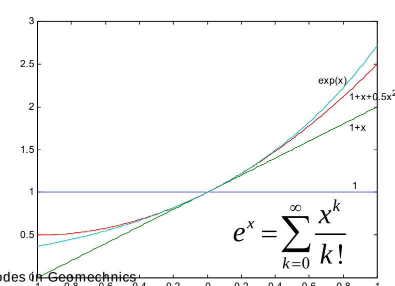
$$f(x) = e^x \Rightarrow f(\alpha) = ?$$

$$Taylor\ \ Series = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\ (x - x_0)^k$$

if the series converge we can write

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\ (x - x_0)^k$$





Numerical Methodes in Geomechnics 22

Convergence of Taylor Series

(Observations, Example 1)

- The Taylor series converges fast (few terms are needed) when *x* is near the point of expansion. If |x-c| is large then more terms are needed to get good approximation.
- How many terms are needed to get good approximation???

These questions will be answered using Taylor Theorem

Taylor Theorem

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}f(x-c)^{n+1}$$

where ξ is between x and c

$x \to x+h, \quad c \to x$

Error

$$f(x+h)=\sum_{k=0}^{n}\frac{f^{(k)}(x)}{k!}h^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

where ξ is between x and $x+h$

Error Term for
Example 1

How large is the error if we replaced $f(x)=e^x$ by the first 4 terms (n = 3) of its Taylor series expansion about $c=0$ *when* $x=0.2$?

$$f^{(k)}(x)=e^x \qquad f^{(k)}(\xi)\le e^{0.2} \quad for\ k\ge 1$$

$$E_{n+1}=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

$$\left|E_{n+1}\right|\le \frac{e^{0.2}}{(n+1)!}(0.2)^{n+1}\Rightarrow \left|E_4\right|\le 8.14268E-05$$