روش های عددی در ژئومکانیک

Numerical methods in geotechnic

Ill-posed problems and Regularization Method

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دانشگاه صنعتی خواجه نصیرالدین طوسی

Well-Posedness

Definition due to Hadamard, 1915: Given mapping

$$A : X \to Y$$
$$Ax = y$$

is well-posed provided:

- (Existence) For each $y \in Y$, $\exists x \in X$ such that $Ax = y$;
- (Uniqueness)
- (Stability) A^{-1} is continuous

Equation is ill-posed if it is not well-posed.
Then an arbitrarily small perturbation of the data can cause an arbitrarily large perturbation of the solution.

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Examples of Ill-Posed problems

$$Ax = b \quad \Rightarrow x = (A)^{-1} b$$

$$\begin{pmatrix} 1.0000000000001 & 3 \\ 2 & 4.0000000000001 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.0000 \\ 2.0000 \end{bmatrix}$$

$$\det(A) = -1$$

$$\begin{pmatrix} 1.0000000000001 & 3 \\ 2 & 6.0000000000001 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} \quad \det(A) = 7.5495\text{e-}15$$

$$x = inv(A) * b = 1.0\text{e+}14 * \begin{bmatrix} 7.9475 & -3.9738 \\ -2.6492 & 1.3246 \end{bmatrix} \begin{bmatrix} 7 \\ 14 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.6250 \\ 2.0938 \end{bmatrix}$$

$$x = pinv(A) * b = \begin{bmatrix} 0.0200 & 0.0400 \\ 0.0600 & 0.1200 \end{bmatrix} \begin{bmatrix} 7 \\ 14 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.7000 \\ 2.1000 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$$ جواب دقیق؟

خطاها معمولا در اندازه گیری وارد مساله می شوند.

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Examples of Ill-Posed problems

- **Differentiation**

$$f_1(x) = f(x) \qquad f_2(x) = f(x) + \varepsilon \sin(\omega x) \quad \Rightarrow \quad |f_1(x) - f_2(x)| \le \varepsilon$$

$\Rightarrow$ Edge-detection is ill-posed! $\qquad |f_1'(x) - f_2'(x)| \le \varepsilon\,\omega$

- **Image restoration**

g:blurred image k:response function f:unknown object

$$g(x) = (k * f)(x)$$

k:band-limited: invisible objects exist

Existence if and only if $\int |\hat{g}(\omega) / \hat{k}(\omega)|^2 \, d\omega < \infty$

- **Fredholm integral equation of the first kind**

$$\int_a^b K(x, y)\varphi(x)dx = f(y) \; , \quad c \le y \le d \quad \Rightarrow \varphi(y) = ?$$

f is noisy data and is not the exact data

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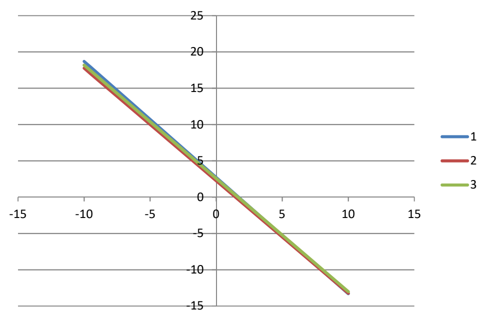
Examples of Ill-Posed problems

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \\ 3.313 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$$



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Examples of Ill-Posed problems

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \\ 3.313 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ?$$

جواب دقیق؟

خطاها معمولا در اندازه گیری وارد مساله می شوند.

جواب دقیق مساله

$$b = \begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} +0.001 \\ -0.002 \\ +0.003 \end{bmatrix}$$

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Examples of Ill-Posed problems

روش اول: جواب‌ها با حل دو معادله

$$\begin{pmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.5167 \\ 0.1833 \end{bmatrix}$$

$$2.02 * 1.5167 + 1.29(0.1833) = 3.313 + (-0.0128)$$

$$\begin{pmatrix} 0.16 & 0.10 \\ 2.02 & 1.29 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 3.313 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.2250 \\ 0.6500 \end{bmatrix}$$

$$\begin{pmatrix} 0.17 & 0.11 \\ 2.02 & 1.29 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.278 \\ 3.313 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2.0034 \\ -0.5690 \end{bmatrix}$$

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Examples of Ill-Posed problems

روش دوم: جواب‌ها به روش حداقل مربعات خطا

$$min\|Ax - b\|^2$$

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \\ 3.313 \end{bmatrix}$$

سه رقم اعشار

$$f(x_1, x_2) = [0.261 - (0.16x_1 + 0.1x_2)]^2 + [0.278 - (0.17x_1 + 0.11x_2)]^2 + [3.313 - (2.02x_1 + 1.29x_2)]^2$$

$$min\ f \begin{cases} \dfrac{\partial f}{\partial x_1} = 0 & \Rightarrow \\ \dfrac{\partial f}{\partial x_2} = 0 & \Rightarrow \end{cases} \qquad \begin{cases} 8.270x_1 + 5.281x_2 = 13.563 \\ 5.281x_1 + 3.372x_2 = 8.661 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.7077 \\ -0.1059 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} 1.7077 \\ -0.1059 \end{bmatrix} + \begin{bmatrix} 0.0016 \\ 0.0007 \\ -0.0002 \end{bmatrix}$$

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Examples of Ill-Posed problems

روش دوم: جواب‌ها به روش حداقل مربعات خطا

$$min\|Ax-b\|^2$$

تعداد اعشار مورد استفاده در برنامه متلب

$$\begin{bmatrix}0.16 & 0.10\\0.17 & 0.11\\2.02 & 1.29\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0.261\\0.278\\3.313\end{bmatrix}$$

$$f(x_1,x_2)=[0.261-(0.16x_1+0.1x_2)]^2+[0.278-(0.17x_1+0.11x_2)]^2+[3.313-(2.02x_1+1.29x_2)]^2$$

$$min\ f\begin{cases}\frac{\partial f}{\partial x_1}=0\Rightarrow\\ \frac{\partial f}{\partial x_2}=0\Rightarrow\end{cases}\begin{cases}8.2698x_1+5.281x_2=13.5626\\ 5.281x_1+3.3724x_2=8.6609\end{cases}\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1.4615\\0.2796\end{bmatrix}$$

$$b=\begin{bmatrix}0.16 & 0.10\\0.17 & 0.11\\2.02 & 1.29\end{bmatrix}\begin{bmatrix}1.4615\\0.2796\end{bmatrix}+\begin{bmatrix}0.0008\\0.0012\\-0.0002\end{bmatrix}$$

Lscov(A,b) $\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1.4615\\0.2796\end{bmatrix}$

دستور متلب برای حل به روش حداقل مربعات خطا

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Examples of Ill-Posed problems

روش سوم: معکوس ماتریس ضرایب

$$\begin{bmatrix}0.16 & 0.10\\0.17 & 0.11\\2.02 & 1.29\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0.261\\0.278\\3.313\end{bmatrix}$$

ماتریس مستطیلی به روش *Moore-Penrose pseudoinverse* شبه معکوس می‌شود

$$A^+ :\ pinv(A)$$ دستور متلب برای شبه معکوس

$$A=\begin{bmatrix}0.16 & 0.10\\0.17 & 0.11\\2.02 & 1.29\end{bmatrix}\rightarrow A^+=\begin{bmatrix}204.1237 & -135.1226 & -4.3015\\-319.5876 & 211.6601 & 7.5009\end{bmatrix}$$

$$Ax=b\rightarrow x=A^+b\quad\Rightarrow\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1.4615\\0.2796\end{bmatrix}$$

توجه $A^+A=I\quad but\quad AA^+\neq I$

left inverse $A^+A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ $AA^+=\begin{bmatrix}0.7010 & -0.4536 & 0.0619\\-0.4536 & 0.3118 & 0.0938\\0.0619 & 0.0938 & 0.9872\end{bmatrix}$

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Examples of Ill-Posed problems

روش چهارم: استفاده از ترنسپوز ماتریس A

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \\ 3.313 \end{bmatrix}$$

ماتریس مستطیلی به روش *Moore-Penrose pseudoinverse* شبه معکوس می‌شود

$$Ax = b \rightarrow A^T Ax = A^T b \Rightarrow x = \left(A^T A\right)^{-1} A^T b$$

$$A = \begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 0.16 & 0.17 & 2.02 \\ 0.10 & 0.11 & 1.29 \end{bmatrix} \Rightarrow A^T A = \begin{bmatrix} 4.1349 & 2.6405 \\ 2.6405 & 1.6862 \end{bmatrix}$$

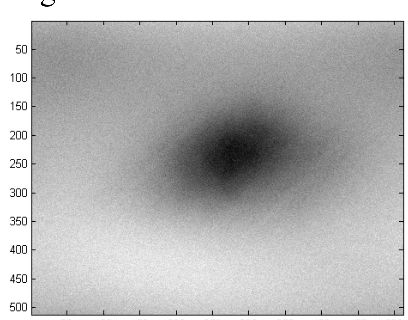
$$\det(A^T A) = 2.8130\text{e-}05$$

$$x = \left(A^T A\right)^{-1} A^T b = \begin{bmatrix} 59940 & -93870 \\ -93870 & 146990 \end{bmatrix} \begin{bmatrix} 6.7813 \\ 4.3304 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.4615 \\ 0.2796 \end{bmatrix}$$

Regularization Mehod

Primary difficulty with the discrete ill-posed problems:

- They are essentially underdetermined due to the cluster of small singular values of *A*.



Original image

Regulirized image

Regularization Mehod

Most of inverse problems end in Ill-posed linear systems:

$$KO \approx G = G_{true} + \delta \quad , \quad G_{true} = KO_{true}$$

$$\delta : noise$$

Which is directly obtained from Fredholm integral:

$$G(x) = \int_a^b K(x,t)O(t)dt$$

Having Kernel function $K(x,t)$ **and** $G(x)$, $O(t)$ **is evaluated:**

$$O(x) = F_{\omega}^{-1}\left[\frac{F_x[G(x)](\omega)}{F_x[K(x)](\omega)}\right] = \int_{-\infty}^{+\infty} \frac{F_x[G(x)](\omega)}{F_x[K(x)](\omega)} e^{2\pi i \omega x} d\omega$$

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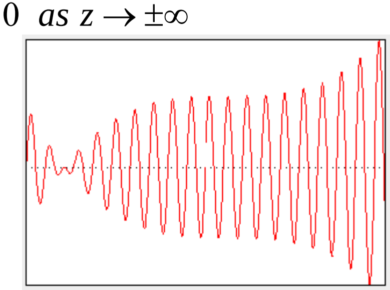
Regularization Mehod

According to *Riemann-Lebesgue lemma,* **Fourier transform of sinusoidal functions which their values fluctuates around zero tends to zero in infinity.**

$$Let \ f : R \to C$$

$$\int_{-\infty}^{+\infty} f(x)e^{izx}dx \to 0 \quad as \ z \to \pm\infty$$



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Regularization Mehod

However, obtaining $O(x)$ **would be difficult using** $G(x)$.

X has a high frequency → $\|KO\| \ll \|O\|$

Tikhonov Regularization Mehod:

$$Ax = b$$

$$\min\left\{\|Ax-b\|_2^2 + \lambda\|Lx\|_2^2\right\} \qquad \lambda = \frac{\|Ax-b\|_2^2}{\|Lx\|_2^2}$$

$$\hat{x} = (A^T A + \lambda L^T L)^{-1} A^T b$$

L **Tikhonov matrix**

$L = I$ In many cases, this matrix is identity matrix(standard form)

$L = 0$ reduces to the unregularized least squares solution

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Regularization Mehod

Morozov's discrepancy principle

Ask for the norm of the residual to be equal to the norm of the noise vector (if it is known)

Choose λ_0 and $0 < d < 1$

Set j = 0 and find $x_0 = (A^T A + \lambda_0 L^T L)^{-1} A^T b$ for λ_0

While $\|Ax - b\| > \delta$

$$j = j+1,\ \lambda_j = d\,\lambda_{j-1}$$

$$x_j = (A^T A + \lambda_j L^T L)^{-1} A^T b$$

While not $\|Ax - b\| = \delta$

$$\lambda_{max} = \lambda_{j-1},\ \lambda_{min} = \lambda_j$$

$$\lambda = (\lambda_{max} + \lambda_{min})/2$$

$$x_\lambda = (A^T A + \lambda L^T L)^{-1} A^T b$$

If $\|Ax_\lambda - b\| > \delta$ then $\lambda_{max} = \lambda$ *else* $\lambda_{min} = \lambda$

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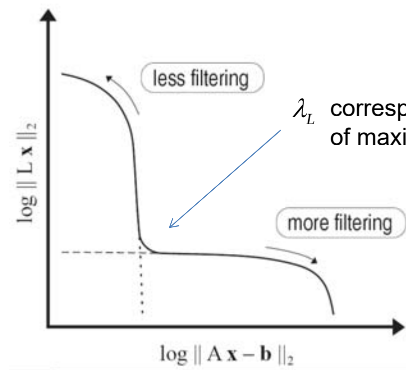
Regularization

plot in log–log scale the curve ($\|Lx_\lambda\|$, $\|Ax_\lambda - b\|$) obtained by varying the value of $\lambda \in [0, \infty)$

In most cases this curve is shaped as an "L"

Lawson and Hanson proposed to choose the value λ_L corresponding to the "corner" of the L–curve



λ_L corresponding to the point of maximal curvature

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Regularization Mehod

روش چهارم: روش منظم سازی تیخونف

$$Ax = b$$

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.261 \\ 0.278 \\ 3.313 \end{bmatrix}$$

$$\min\left\{\|Ax - b\|_2^2 + \lambda \|Lx\|_2^2\right\} \qquad , L = I_n$$

$$\begin{cases} 0.16x_1 + 0.1x_2 = 0.261 \\ 0.17x_1 + 0.11x_2 = 0.278 \\ 2.02x_1 + 1.29x_2 = 3.313 \end{cases}$$

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Regularization Mehod

$$f\left(x_1,x_2\right)=[0.261-\left(0.16x_1+0.1x_2\right)]^2+[0.278-\left(0.17x_1+0.11x_2\right)]^2$$
$$+[3.313-\left(2.02x_1+1.29x_2\right)]^2+\lambda(x_1+x_2)^2$$

$$\min f\begin{cases}\dfrac{\partial f}{\partial x_1}=0\Rightarrow \\ \dfrac{\partial f}{\partial x_2}=0\Rightarrow\end{cases}\quad \begin{cases}8.2698x_1+5.281x_2+2\lambda(x_1+x_2)=13.5626\\ 5.281x_1+3.3724x_2+2\lambda(x_1+x_2)=8.6609\end{cases}$$

$\lambda=0.01$ $\qquad \lambda=0.001$ $\qquad \lambda=0.0001$ $\qquad \lambda=0.00001$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4.5280\\-4.5225\end{bmatrix}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4.4448\\-4.3922\end{bmatrix}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3.8948\\-3.5309\end{bmatrix}\quad\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2.9627\\-2.0713\end{bmatrix}$$

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Regularization Mehod

$\lambda=0.01$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4.5280\\-4.5225\end{bmatrix}\quad b=\begin{bmatrix}0.16&0.10\\0.17&0.11\\2.02&1.29\end{bmatrix}\begin{bmatrix}4.5280\\-4.5225\end{bmatrix}+\begin{bmatrix}0.0112\\-0.0057\\-0.0004\end{bmatrix}\quad \begin{matrix}\log(\|Lx\|) & 1.8563\\ \log(\|Ax-b\|) & -4.3733\end{matrix}$$

$\lambda=0.001$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4.4448\\-4.3922\end{bmatrix}\quad b=\begin{bmatrix}0.16&0.10\\0.17&0.11\\2.02&1.29\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0.0109\\-0.0055\\-0.0004\end{bmatrix}\quad \begin{matrix}\log(\|Lx\|) & 1.8324\\ \log(\|Ax-b\|) & -4.4005\end{matrix}$$

$\lambda=0.0001$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3.8948\\-3.5309\end{bmatrix}\quad b=\begin{bmatrix}0.16&0.10\\0.17&0.11\\2.02&1.29\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0.0091\\-0.0043\\-0.0004\end{bmatrix}\quad \begin{matrix}\log(\|Lx\|) & 1.6596\\ \log(\|Ax-b\|) & -4.6007\end{matrix}$$

$\lambda=0.00001$
$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}2.9627\\-2.0713\end{bmatrix}\quad b=\begin{bmatrix}0.16&0.10\\0.17&0.11\\2.02&1.29\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0.0059\\-0.0022\\-0.0003\end{bmatrix}\quad \begin{matrix}\log(\|Lx\|) & 1.2851\\ \log(\|Ax-b\|) & -5.0670\end{matrix}$$

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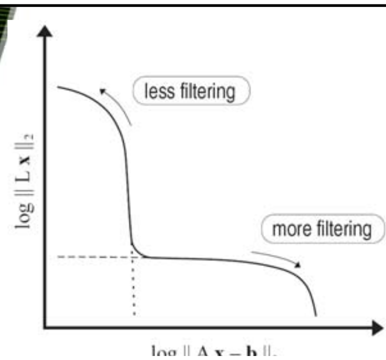
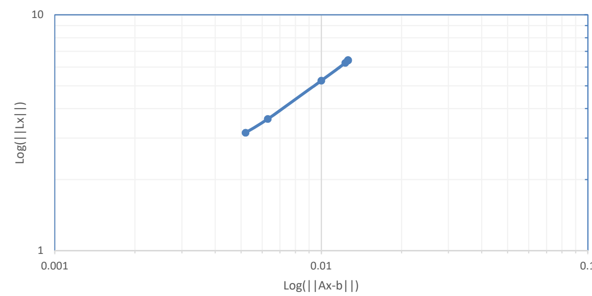
Regularization



λ	$\|Lx\|$	$\|Ax-b\|$
10	6.4174	0.0126
1	6.4172	0.0126
.1	6.4156	0.0126
.01	6.3997	0.0126
.001	6.2489	0.0123
.0001	5.2571	0.0100
.00001	3.6150	0.0063
.000001	3.1622	0.0052



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Examples o

•Fredholm integral equation of the first kind

$$\int_a^b K(x,y)\varphi(x)dx = f(y)\ ,\ c \le y \le d \quad \Rightarrow \varphi(y) = ?$$

Simpson's rule $\quad \sum_{j=1}^{n} K_{ij}\Delta x_j \varphi_j = f(x_i),\quad i = 1,\ldots,n$

inverting the matrix A^n will fail $\quad A_{ij}^n = K_{ij}\Delta x_j$

$$\int_0^1 \frac{1}{1+100(x-y)^2}\varphi(x)dx = f(y)\ ,\ -2 \le y \le 2$$

the data f is chosen such that:

$$\varphi(x) = e^{-100(x-.25)^2} + e^{-100(x-0.75)^2}$$

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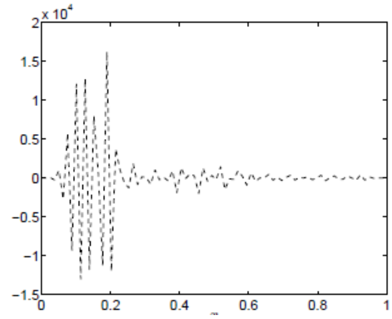
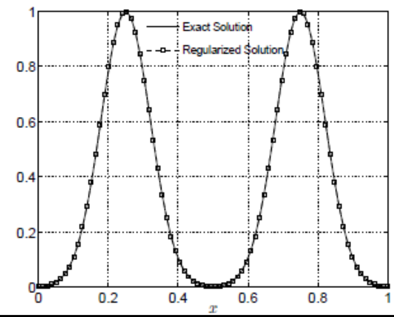
•Fredholm integral equation of the first kind

inverting the matrix A^n



exact solution and the regularized solution



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