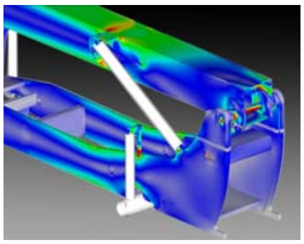
روش های عددی در ژئومکانیک

An Introduction to the Finite Element Analysis



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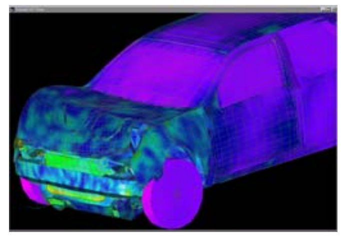

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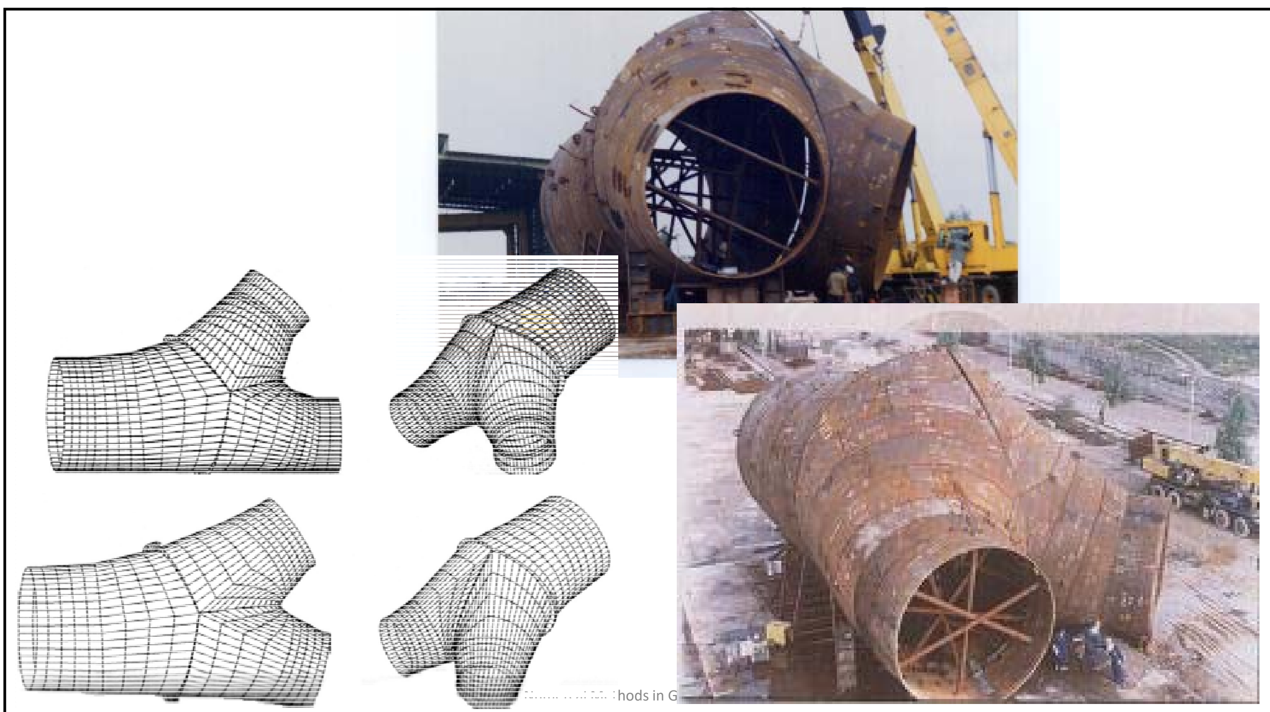
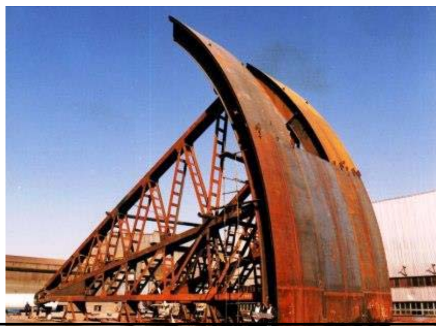
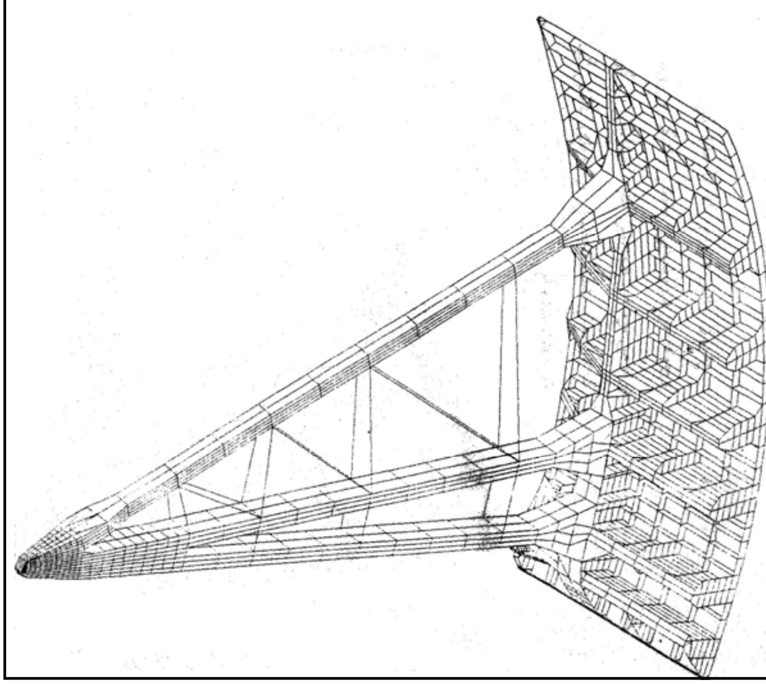
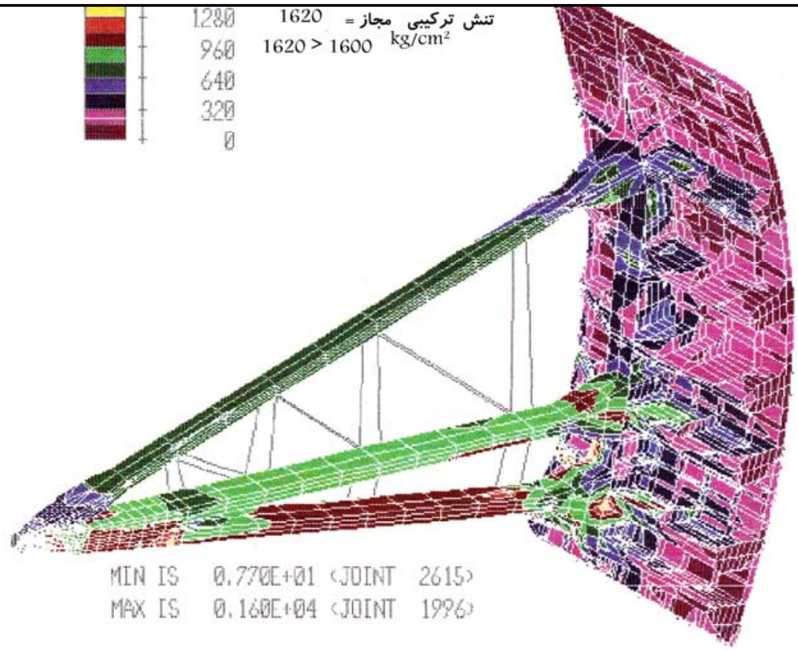








ε

What defines elastic solids?

A solid deforms in response to external actions (e.g., forces, heat, etc.). The deformation is completely reversible – meaning the solid returns to its original shape after the removal of the external actions.

Two Types of Elastic Solids

Type 1: Linear elasticity of materials:

This type of elasticity occurs to solids undergoing small deformations, such as springs that exhibit linear relationships between the applied force (F) and the induced elongation (x) that can be represented by a mathematical formula as: $F = kx$ where k is a constant known as the *rate* or *spring constant*. Many metallic materials fall into the category of linear elastic solids

It can also be stated as a linear relationship between stress (σ) and strain (ε) in stretching or compressing a thin rod by The expression: $\sigma = E\varepsilon$ where E Is known as the elastic modulus or Young's modulus

Type 2: Nonlinear elasticity of materials:

This type of solids behaves as elastic materials as described above, but can exhibit large deformations, such as rubbers and polymers

The FE formulation presented in this course will be based on linear elasticity theory

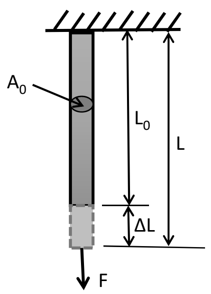
Linear Elastic Behavior of Solids

Fundamental assumptions

(1) The material is treated as a continuous medium (or a continuum). In the words, the material is homogeneous with no internal defects or voids of significant sizes
(2) The material is isotropic – meaning its properties are uniform in all directions
(3) The material has no memory
(4) The material exhibits the same properties in tension and compression

Definitions of stress and strain

Uniaxial elongation of a thin rod:



(1) Engineering strain (e):

It is defined with reference to the original shape of the solid. Mathematically it is expressed as:

$= \Delta L/L_0 = L/L_0 - 1$

(2) True strain (ε):

It is measured on the basis of the immediate proceeding length of the rod sample. Mathematically it is expressed as:

$$\varepsilon = \sum\left(\frac{L_1 - L_0}{L_0} + \frac{L_2 - L_1}{L_1} + \ldots\ldots\ldots + \frac{L - L_n}{L_n}\right) = \int_{L_0}^{L} \frac{dL}{L} = \ell n\left(\frac{L}{L_0}\right)$$

(3) Relationship between ε and e: $\varepsilon = \ell n(1+e)$

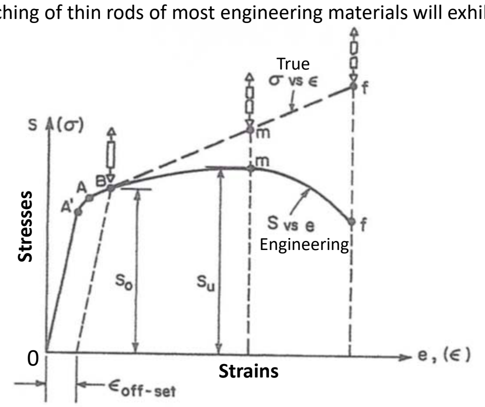
(4) Engineering stress (S): $S = \frac{\text{Instantaneous load}}{\text{Original cross-sectional area}} = \frac{P}{A_0}$

(5) True stress (σ): $\sigma = P/A$

(6) Relationship between S and σ: $\sigma = (1 + e)\, S$

Stress-Strain Curves of Materials

Stretching of thin rods of most engineering materials will exhibit the stress vs. strain relations illustrated in the figure below:

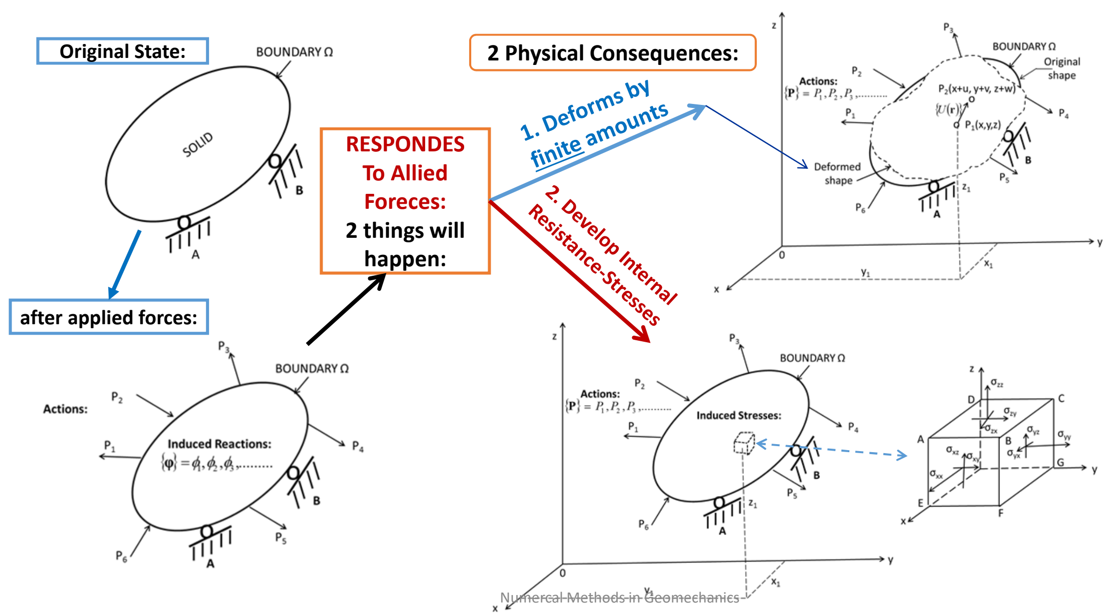


Designations:
A' = proportional limit
A = elastic limit
B = yield point
m =necking point
f = rupture point
S_0 = Yield strength of material
S_u = ultimate tensile strength of material

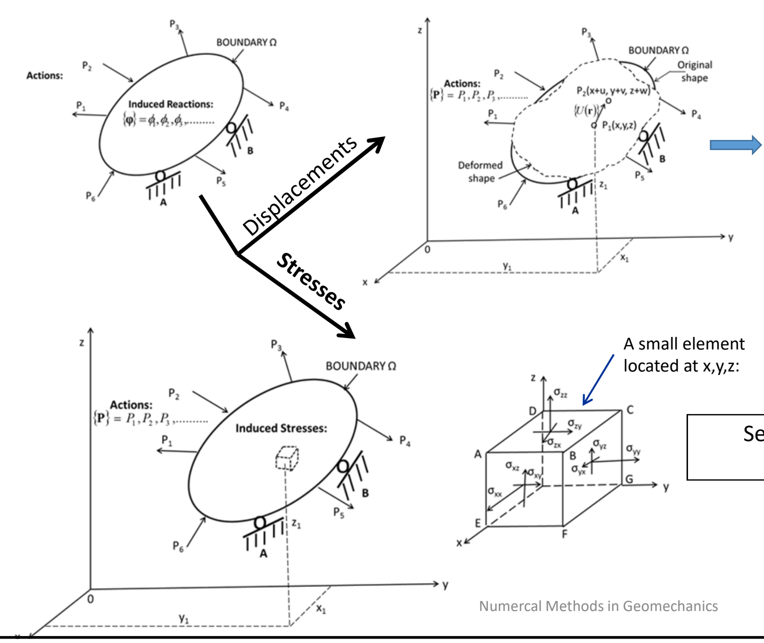
Typical elastic deformation of engineering materials:

(1) Very small deformation with strain up to 0.1%
(2) Straight linear relationship between the stress and strain, resulting in a constant stiffness of the .
(3) The slope of line OA' is called the Young's modulus (E), representing the stiffness of the material
(4) Completely recoverable strain (or deformation) after the applied load is removed
(5) The yield stress (S_o) or σ_y is defined to be the interception of ϵ_{offset}= 0.2% of the σ-ϵ curve. It is a measure of materials exceeding the elastic limit, and undergoing plastic deformation (an irreversible deformation)

Physics of Deformable Solids subjected to External Forces



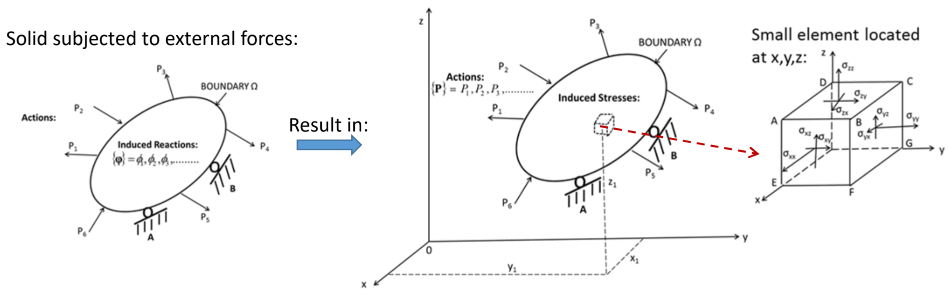
The Many Components of Displacements and Stresses:



(Element) Displacements:
$\{U\}^T$: $\{U_x, U_y, U_z\}$
in a deformed solid

See detailed definitions
on the next slide:

Induced Stress Components in Deformable Solids subjected to External Forces



Because the forces applied to a general 3-D solid, the induced stresses are MULTI-directional designated by $\sigma_{\alpha\beta}$:

σ = magnitude, subscript α = the axis normal to the plane of action, subscript β = the direction the stress component points to

So, stress component σ_{yy} = stress component acting on the plane normal to the y-axis and pointing to the y-direction, whereas stress component σ_{xy} = stress components acting on the face normal to the x-axis but points to the y-direction.

In theory, there are nine (9) stress components everywhere inside the solid: $[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$

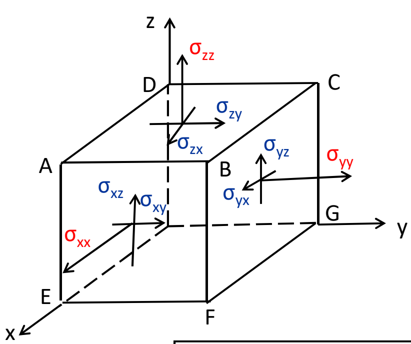
But in reality there are only **six** (6) independent stress components with: $[\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ SYM & & \sigma_{zz} \end{bmatrix}$

$\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, $\sigma_{yz} = \sigma_{zx}$

For FE formulation:
$\sigma_1 = \sigma_{xx}$, $\sigma_2 = \sigma_{yy}$, $\sigma_3 = \sigma_{zz}$, $\sigma_4 = \sigma_{xy}$, $\sigma_5 = \sigma_{yz}$, $\sigma_6 = \sigma_{xz}$

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

Induced Stresses in Deformable Solids subjected to External Forces – Cont'd

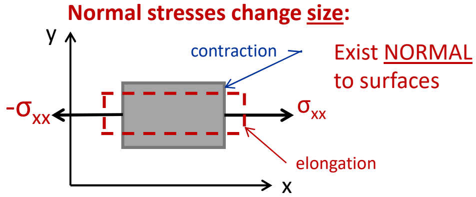


Stress components with same subscripts such as: σ_{xx}, σ_{yy}, σ_{zz} are called NORMAL stress components, with unit: psi, or Pascal (Pa) = N/m^2 .

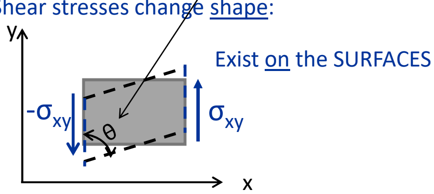
Stress components with different subscripts such as: σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx}, σ_{zy} are SHEARING stress components. Shear stress has the unit of change of angle from the original right angle to the angle with the stress, i.e $(\pi/2) - \theta$. The unit is thus "angle change" in radians (rad)

Effects of Normal and Shear Stresses to Solid Deformation

Normal stresses change size:



Shear stresses change shape:



Relationships between Primary unknown and Secondary Unknowns in FE analyses

Primary Unknown:
Element displacements: {U(x,y,z)} and
Nodal displacements: $[N(x,y,z)]\{u\}^T$
in FE formulation: {U} = [N]{u}
where {N(x,y,z)} = Interpolation function

Strain-displacement relations

Linkage to the 1st secondary unknown:
Element strains: {ε(x,y,z)}=[B(x,y,z)]{U(x,y,z)} →
corresponding to element stresses: {σ(x,y,z)}

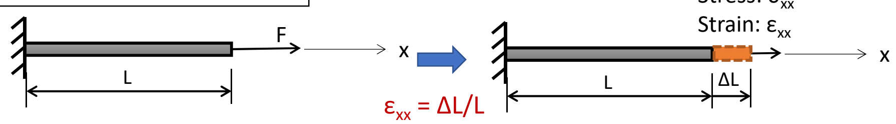
Generalized Hook's Law

2nd Secondary Unknown:
Element stresses {σ(x,y,z)}=[D]{ε}

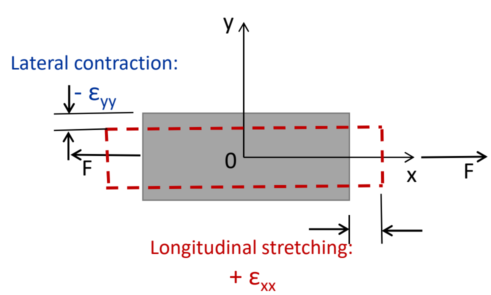
Element strains and stresses, and nodal displacement are the solutions of FE analyses.

Multiaxial Deformations of Elastic Solids

Uniaxial stress with uniaxial deformation:



Uniaxial stress with biaxial deformation with Poisson's ratio:



For solids with significant cross-section, a uniaxial stress may produce biaxial deformation such as illustrate in the figure in the left.

The lateral contraction deformation is represented by: - ε_{yy}

Poisson's ratio defines as:

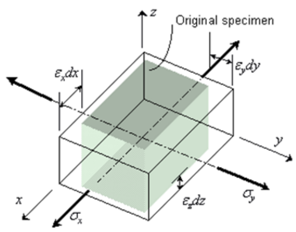
$$\nu = \frac{|Lateral\ contraction\ strain|}{Longitudinal\ stretching\ strain} = \frac{\varepsilon_{yy}}{\varepsilon_{xx}}$$

- **Poisson ratio** exists in most multi-axially loaded elastic solids, and this effect needs to be included in all FE formulations. This effect also leads to the use of Generalize Hooke's Law for the formulations

Generalized Hooke's Law for Solids with Multi-axial Stresses:

Stress vs. strain relationship for deformed solids with 3-D deformation

Elongation in one-direction causes contractions in other directions, and vise versa.



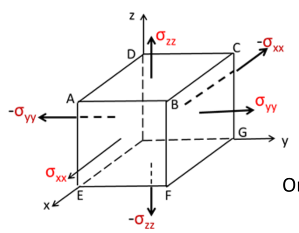
Total strains in three directions induced by the three normal stresses

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \quad \text{in x-direction}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \quad \text{in y-direction}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \quad \text{in z-direction}$$

in which E = Young's modulus, and γ = Poisson's ratio of the material



The following expression express the stresses in terms of strains – the Hooke's law:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{Bmatrix}(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\\ \nu\varepsilon_x + (1-\nu)\varepsilon_y + \nu\varepsilon_z\\ \nu\varepsilon_x + \nu\varepsilon_y + (1-\nu)\varepsilon_z\end{Bmatrix}$$

One may derive the uniaxial stress situation as a special case from the above expression to obtain:

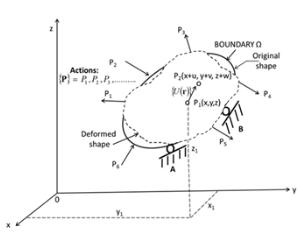
$$\sigma_{xx} = E\varepsilon_x$$

by substituting: $\sigma_{yy} = \sigma_{zz} = 0$, and $\varepsilon_y = -\gamma\varepsilon_x$ and $\varepsilon_z = -\gamma\varepsilon_x$ in the generalized Hooke's Law

Element Strain-Displacement Relations for FE Formulation

There are six (6) strain components corresponding to the stress components in interior of the deformed solid. These strain components are related to the displacements of the solid induced by the external forces. Because deformation of the solid CONTINUOUSLY varying throughout the solid, the following relationship exits:

By theory of elasticity:



$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\end{Bmatrix} = \begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}U_x(x,y,z)\\ U_y(x,y,z)\\ U_z(x,y,z)\end{Bmatrix}$$

Element Strains Corresponding to Element stresses

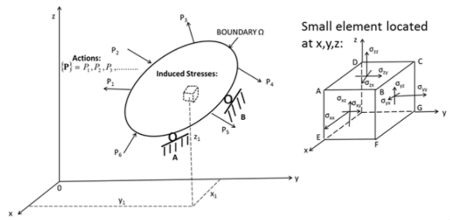
Element Displacements U(x,y,z)

or: **{ε(x,y,z)} = [*D*]{U(x,y,z)}**

$$[D] = \begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\end{bmatrix}$$

Example: For uni-axial elongation or contraction of a rod:
Element displacement {U} = {U_x(x)}, The corresponding strain in element is: $\varepsilon_{xx} = \frac{\partial U_x(x)}{\partial x} = \frac{dU_x(x)}{dx}$

Element Stress-Strain Relations – the Generalized Hooke's Law



According to generalized Hooke's law for MULTI-Axial stress state, the following relationship between the element stress and strain exists:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{xy}\\ \sigma_{yz}\\ \sigma_{xz}\end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & \frac{1-2\nu}{2} & 0 & 0\\ & & & & \frac{1-2\nu}{2} & 0\\ & SYM & & & & \frac{1-2\nu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\end{Bmatrix}$$

or: $\{\sigma\} = [D]\{\varepsilon\}$

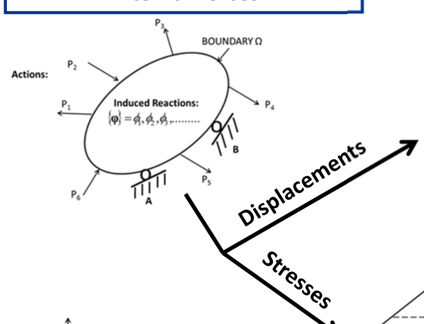
where [D] is the elasticity matrix with:

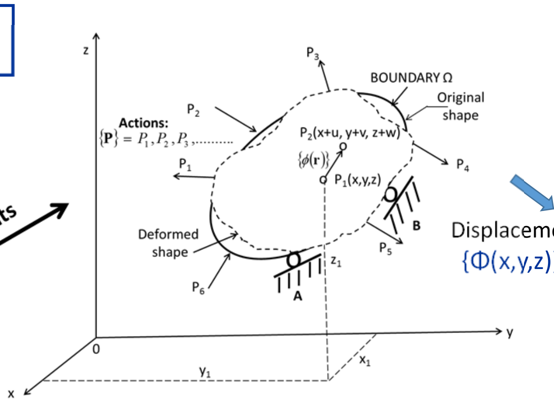
$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & \frac{1-2\nu}{2} & 0 & 0\\ & SYM & & & \frac{1-2\nu}{2} & 0\\ & & & & & \frac{1-2\nu}{2}\end{bmatrix}$$

where E = Young's modulus, and γ = Poisson's ratio

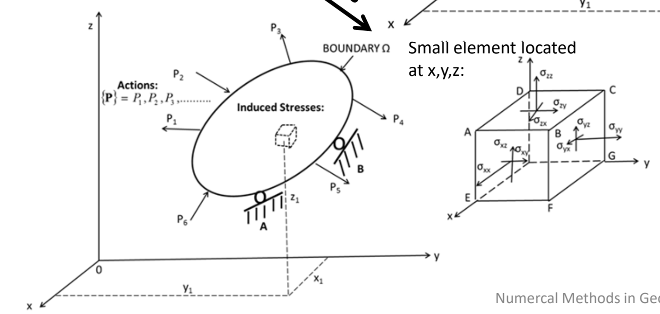
Example: For uni-axially loaded rod: $\sigma_{xx} = E\varepsilon_{xx} = E\frac{dU_x(x)}{dx}$ where $U_x(x)$ is the displacement in the rod

Physics of Solid Deformation by External Forces





Displacements:
$\{\Phi(x,y,z)\}^T$: $\{\Phi_x(x,y,z,\Phi_y(x,y,z,\Phi_z(x,y,z\}$



Total 9 stress components at (x,y,z)

$$[\sigma] = \begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}$$

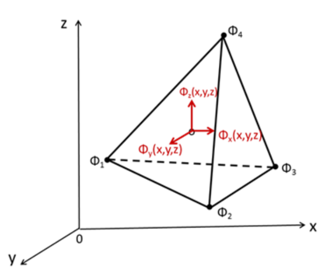
Part 2

FE Formulation of Deformable Elastic Solids

Derivation of Element Equations

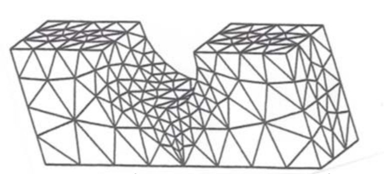
In Step 4, Chapter 3, we derived the element equation using the Rayleigh-Ritz method to take a form:

We will select **tetrahedron elements** as the basis for FE formulation for general **3-D solid structures**



$$[K_e]\{q\} = \{Q\}$$

where $[K_e]$ = Element matrix
$\{q\}$ = Vector of primary unknown quantities at the nodes of the element
$\{Q\}$ = Vector of element nodal actions (e.g., forces)



FE Model for s Structure made of Tetrahedron Elements

Element equations for each tetrahedron element in the FE model for a structure are then assembled to establish "overall stiffness equation" for determining nodal displacements of all nodes in the structure.

$$[K]\{\Phi\} = \{R\}$$

where $[K] = \sum_{1}^{m} [K_e^m]$ m=total number of elements in the FE model

Derivation of Element Equations-Cont'd Principle of deriving element equation using Rayleigh-Ritz variational principle

From Chapter 2: Let us determine a suitable "functional" to derive the element equation.

A general form of functional: $$\chi(\phi) = \int_v f\left(\{\phi\}, \frac{\partial\{\phi\}}{\partial \mathbf{r}}, \ldots\ldots\ldots\right)dv + \int_s g\left(\{\phi\}, \frac{\partial\{\phi\}}{\partial \mathbf{r}}, \ldots\ldots\ldots\right)ds$$ v = volume, s = surface

and then apply the Variational principle on:

$$\frac{\partial\chi(\phi)}{\partial\{\phi\}} = \begin{Bmatrix} \frac{\partial\chi}{\partial\phi_1} \\ \frac{\partial\chi}{\partial\phi_2} \\ \bullet \\ \bullet \\ \bullet \end{Bmatrix} = 0$$

from which equations of each element are derived:

$$\frac{\partial\{\chi\}}{\partial\phi_1} = 0, \quad \frac{\partial\{\chi\}}{\partial\phi_2} = 0, \quad \frac{\partial\{\chi\}}{\partial\phi_3} = 0, \ldots\ldots\ldots\ldots$$

The "functional" for a deformed solid subjected to external forces is "**POTENTIAL ENERGY**" (**P**) in the situation.

The potential energy associated with a deformed solid can be defined as:

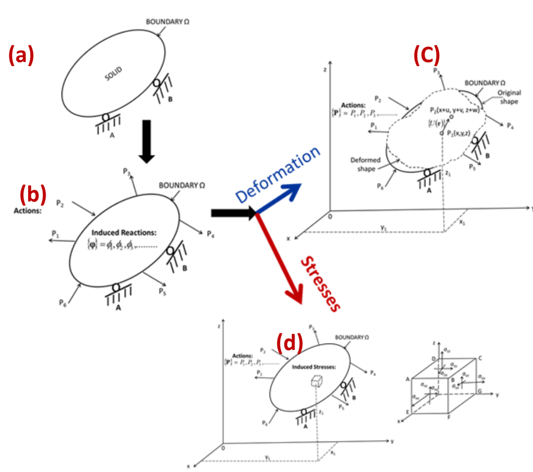
P = U - W

where U = the strain energy in a deformed solid, and W = the work done to the deformed solid by external forces acting on the solid body in the volume and surface of the solid

Derivation of Element Equations-Cont'd **Strain energy in a deformed solid:**

As we mentioned early in this Chapter that a solid in (a) deforms into a new shape in (c) – but not indefinitely.

- It stopped further deformation after deformed by certain amount.
- It reaches a new state of equilibrium. WHY???



Imagine the following phenomenon:
Stretch a free-hung spring by a weight W.
The spring will elongate, but only by a finite amount.
Ask yourself: **WHY**?
Answer: the elongation of the spring develops a "resistance," which increases as the spring elongates. The spring ceases further elongation when the "resistance" in the spring balances the applied weight (force). We say the spring – and the applied weight reaches a new state of equilibrium, which stop the spring From further elongation.
Next: what will happen to the spring after the weight is removed? You will say that the spring returns to its original length, but WHY??
Answer: because a form of ENERGY was stored in the stretched spring when it is elongated. This ENBERGY is released to restore the spring to its original shape after the external force (the weight) was removed.

Now, you know why the solid in (a) ceases to deform further after the application of the system of external forces {p} has been applied to the solid in (b). And you would know that there is such ENERGY associated with the solid deformation developed in the solid called "**STRAIN ENERGY**." which is responsible for restoring the solid to its original shape after the applied forces are removed in "ELASTIC" solids.

Mathematical expression of strain energy in State (c) is: $$U = \frac{1}{2}\int\{\varepsilon\}^T\{\sigma\}dv$$ (3.9) textbook

Derivation of Element Equations-Cont'd

Potential energy in a deformed solid subjected to external forces:

The potential energy in a deformed solid is: P = **U** - W

Strain energy:

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T\{\sigma\}dv = \frac{1}{2}\int_v \left(\{\varepsilon_{xx} \quad \varepsilon_{yy} \quad \varepsilon_{zz} \quad \varepsilon_{xy} \quad \varepsilon_{yz} \quad \varepsilon_{xz}\} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \right) dv$$

$$= \frac{1}{2}\int_v \left(\varepsilon_{xx}\sigma_{xx} + \varepsilon_{yy}\sigma_{yy} + \varepsilon_{zz}\sigma_{zz} + \varepsilon_{xy}\sigma_{xy} + \varepsilon_{yz}\sigma_{yz} + \varepsilon_{xz}\sigma_{xz}\right)dv$$

Both the strain and stress components are function of (x,y,z), and dv = (dx)(dy)(dz) = the volume of given points in the deformed solid.

Strain energy is a scalar quantity.

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Derivation of Element Equations – cont'd

Potential energy in a deformed solid subjected to external forces:

Work done to deform the solid: Definition of "work": **Work** (W) = **Force** x **Displacement** (deformation)

Two kinds of forces: (1) **body forces** (uniformly distributed throughout the volume of the solid (**v**)), e.g., the weight,
(2) **surface tractions**, e.g., the pressure or concentrated forces acting on the boundary surface (**s**)

Mathematical expression of work: $W = \int_v \{\phi(x,y,z)\}^T\{f\}dv + \int_s \{\phi(x,y,z)\}^T\{t\}ds$

$$= \int_v \{\phi_x(x,y,z) \quad \phi_y(x,y,z) \quad \phi_z(x,y,z)\} \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} dv + \int_s \{\phi_x(x,y,z) \quad \phi_y(x,y,z) \quad \phi_z(x,y,z)\} \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} ds$$

where {Φ(x,y,z)} = the displacement of the solid at (x,y,z), {f} = body forces, and {t} = the surface tractions , and ds = the part of the surface boundary on which the surface tractions apply

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Derivation of Element Equations – cont'd

Potential energy in a deformed solid subjected to external forces:

So, the potential energy stored in a deformed solid is: **P = U – W**, or:

$$P = U - W = \frac{1}{2}\int_v \{\varepsilon(x,y,z)\}^T \{\sigma(x,y,z)\} dv - \left(\int_v \{\phi(x,y,z)\}^T \{f\} dv + \int_s \{\phi(x,y,z)\}^T \{t\} ds\right)$$

Following the Rayleigh-Ritz Variational principle, the equilibrium condition for the deformed solid should satisfy the following conditions:

$$\frac{\partial P(\phi)}{\partial\{\phi\}} = 0$$

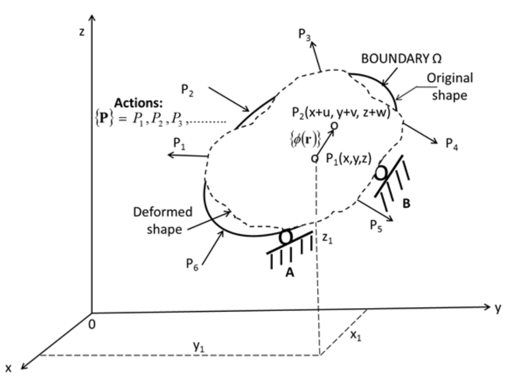
From which, equations for each element may be derived from:

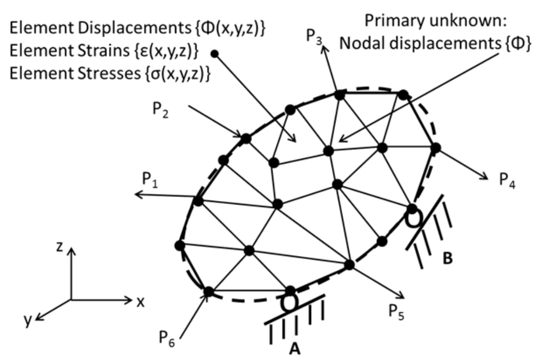
$$\frac{\partial P(\phi)}{\partial \phi_1} = 0, \quad \frac{\partial P(\phi)}{\partial \phi_2} = 0, \quad \frac{\partial P(\phi)}{\partial \phi_3} = 0, \ldots\ldots\ldots\ldots.$$

Derivation of Element Equations – cont'd

for FE mesh of discretized solids

What we had formulated was for continuum solids. We will now derive the ELEMENT EQUATION for discretized solids in FE mesh:





We need to make distinction between the ELEMENT quantities and the NODAL quantities.

The primary quantity in FE analysis is DISPLACEMENTS. We need teo make distinction between the Element displacements and the Nodal displacements.

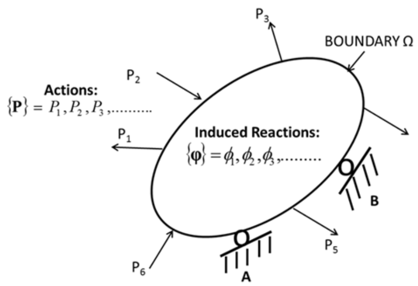
The element displacement is: {Φ(x,y,z)} with three components:

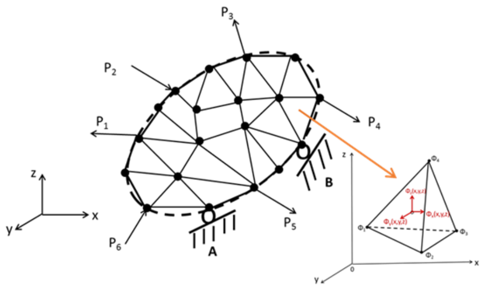
$\Phi_x(x,y,z)$ = the element displacement component along the x-direction

$\Phi_y(x,y,z)$ = the element displacement component along the y-direction, and

$\Phi_z(x,y,z)$ = the element displacement component along the z-direction

Derivation of Element Equations – cont'd for FE mesh of discretized 3-D solids with tetrahedron elements





We realize that TETRAHEDRON and HEXAHEDRN elements are used in FE models for general 3-D solid structures.
The tetrahedron elements are the "basic elements" for this type of structures, because hexahedron elements are made up by 4 or more tetrahedron elements.
Our FE formulation for general 3-D solid structures will thus be based on TETRAHEDRON elements

We notice that tetrahedron elements has four (**4**) associate **nodes**: Φ_1, Φ_2, Φ_3, and Φ_4 with FIXED (specified) COORDINATES.
Each node has three (**3**) displacement components too. These nodal displacement components are:

$$\{\phi\}^T = \{\phi_{1x} \quad \phi_{1y} \quad \phi_{1z} \quad \phi_{2x} \quad \phi_{2y} \quad \phi_{2z} \quad \phi_{3x} \quad \phi_{3y} \quad \phi_{3z} \quad \phi_{4x} \quad \phi_{4y} \quad \phi_{4z}\}^T$$

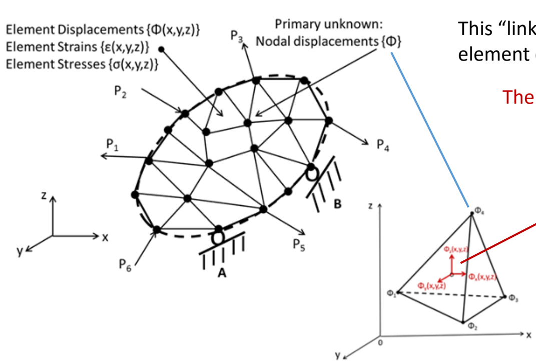
where Φ_{1x}, Φ_{1y}, Φ_{1z} = displacements in Node 1 in 3 directions; Φ_{2x}, Φ_{2y}, Φ_{2z} = displacements in Node 2 in 3 directions; Φ_{3x}, Φ_{3y}, Φ_{3z} = displacements in Node 3 in 3 directions; Φ_{4x}, Φ_{4y}, Φ_{4z} = displacements in Node 4 in 3 directions

Derivation of Element Equations – cont'd for FE mesh of discretized 3-D solids with tetrahedron elements

We mentioned previously that the functional that we will use to derive the element equations in FE formulation of solid structures Is the potential function in the solid as show below:

$$P = U - W = \frac{1}{2}\int_v \{\varepsilon(x,y,z)\}^T \{\sigma(x,y,z)\} dv - \left(\int_v \{\phi(x,y,z)\}^T \{f\} dv + \int_s \{\phi(x,y,z)\}^T \{t\} ds\right)$$

Now because the ELEMENTS in discretized solid) Also, these elements are interconnected by the NODES to simulate the original solid structures. This "link" requires the FE formulation involves the functional with both the ELEMENT and NODAL quantities in the formulation.



This "link" is established using the INTERPOLATION FUNCTION that relates the element quantities with the corresponding nodal quantities such s:

The 3 element displacements:

$$\{\phi(x,y,z)\} = \begin{Bmatrix} \phi_x(x,y,z) \\ \phi_y(x,y,z) \\ \phi_z(x,y,z) \end{Bmatrix} = [N(x,y,z)]\{\phi\}$$

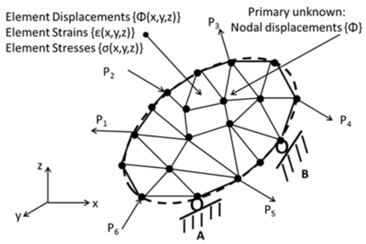
The 12 nodal displacements:

The INTEERPOLATION function

$$\{\phi\}^T = \{\phi_{1x} \quad \phi_{1y} \quad \phi_{1z} \quad \phi_{2x} \quad \phi_{2y} \quad \phi_{2z} \quad \phi_{3x} \quad \phi_{3y} \quad \phi_{3z} \quad \phi_{4x} \quad \phi_{4y} \quad \phi_{4z}\}^T$$

Derivation of Element Equations – cont'd

Key equations to construct the functional - the potential energy



1. The element displacements vs. nodal displacements via Interpolation function:

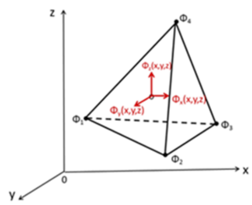
$$\{\Phi(x,y,z)\} = [N(x,y,z)]\ \{\Phi\}$$

2. Element strain vs. nodal displacements:

$$\{\varepsilon(x,y,z)\} = [D]\{\Phi(x,y,z)\}$$

in which $[D]$ in Equation (4.4)

Hence **$\{\varepsilon\}$ = $[D][N(x,y,z)]\{\Phi\}$= $[B(x,y,z)]\{\Phi\}$**
with $[B(x,y,z)] = [D][N(x,y,z)]$



Typical tetrahedron element for 3-D FE models

3. Element stresses vs. nodal displacements:

$$\{\sigma\} = [C]\{\varepsilon\}$$

in which the elasticity matrix [D] in Equation (4.7)
Hence **$\{\sigma\} = [D]\ [B(x,y,z)]\{\Phi\}$**

4. Strain energy with nodal displacements:

$$U = \frac{1}{2}\int_v \{\varepsilon\}^T \{\sigma\} dv$$

Hence

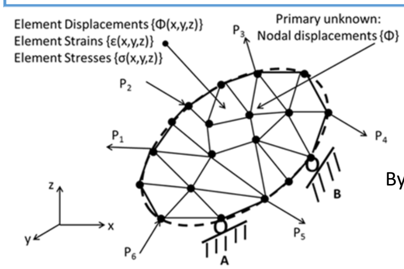
$$U = \frac{1}{2}\int_v \left(\left[B(x,y,z)\right]\{\varphi\}\right)^T [\mathrm{D}]\left(\left[B(x,y,z)\{\varphi\}\right]\right) dv$$

or

$$U = \frac{1}{2}\int_v \{\varphi\}^T \left[B(x,y,z)\right]^T [\mathrm{D}]\left[B(x,y,z)\right]\{\varphi\}\, dv$$

Derivation of Element Equations – cont'd

The functional for Variational process



We mentioned that the functional for deriving the element equations for discretized solid structure is the POTENTIAL ENERGY (P) as shown below:

$$P = U - W = \frac{1}{2}\int_v \{\varepsilon(x,y,z)\}^T \{\sigma(x,y,z)\} dv - \left(\int_v \{\phi(x,y,z)\}^T \{f\} dv + \int_s \{\phi(x,y,z)\}^T \{t\} ds\right)$$

By substituting the Strain energy expressed in Equation (4.16) into the above equation, we get:

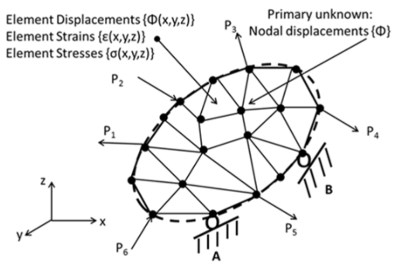
$$P(\varphi) = \frac{1}{2}\int_v \{\varphi\}^T \left[B(x,y,z)\right]^T [\mathrm{D}]\left[B(x,y,z)\right]\{\varphi\}\, dv - \int_v \{\varphi\}^T \left[N(x,y,z)\right]^T \{f\}\, dv - \int_s \{\varphi\}^T \left[N(x,y,z)\right]^T \{t\}\, ds$$

Due to the fact that nodal displacement {Φ} have "fixed value" but not a function og (x,y,z), so they can be factored Out of the integration with respect to (x,y,z). We thus have the following:

$$P(\varphi) = \{\varphi\}^T \left(\frac{1}{2}\int_v \{\varphi\}^T \left[B(x,y,z)\right]^T [\mathrm{D}]\left[B(x,y,z)\right] dv\right)\{\varphi\} - \{\varphi\}^T \left(\int_v \left[N(x,y,z)\right]^T \{f\}\, dv\right) - \{\varphi\}^T \left(\int_s \left[N(x,y,z)\right]^T \{t\}\, ds\right)$$

Derivation of Element Equations – cont'd

Element equation by Variational process



All elements in discretized solids subjected to external force require to satisfy the following condition that:

$$\frac{\partial P(\phi)}{\partial\{\phi\}} = \begin{Bmatrix} \frac{\partial P}{\partial \phi_1} \\ \frac{\partial P}{\partial \phi_2} \\ \bullet \\ \bullet \\ \bullet \end{Bmatrix} = 0$$

By substituting the potential energy P

$$\frac{\partial P(\varphi)}{\partial \varphi} = \frac{\partial}{\partial \varphi}\left(\begin{aligned} &\{\varphi\}^T \left(\frac{1}{2}\int_v \{\varphi\}^T [B(x,y,z)]^T [\mathrm{D}][B(x,y,z)]dv\right)\{\varphi\} \\ &-\{\varphi\}^T\left(\int_v [N(x,y,z)]^T\{f\}dv\right) - \{\varphi\}^T\left(\int_s [N(x,y,z)]^T\{t\}ds\right) \end{aligned}\right) = 0$$

Derivation of Element Equations – cont'd

Element equation by Variational process

The above variation results in:

$$\left(\int_v [B(x,y,z)]^T [\mathrm{D}][B(x,y,z)]dv\right)\{\varphi\} - \left(\int_v [N(x,y,z)]^T \{f\}dv\right) - \left(\int_s [N(x,y,z)]^T \{t\}ds\right) = 0$$

Upon moving the last two items to the right-hand side:

$$\left(\int_v [B(x,y,z)]^T [\mathrm{D}][B(x,y,z)]dv\right)\{\varphi\} = \left(\int_v [N(x,y,z)]^T \{f\}dv\right) + \left(\int_s [N(x,y,z)]^T \{t\}ds\right)$$

We may represent Equation by the following **element equation**:

$$[K_e]\{\Phi\} = \{q\}$$

where $[K_e] = \int_v [B(x,y,z)]^T [\mathrm{D}][B(x,y,z)]dv$ = Element stiffness matrix

$\{\phi\} = Nodal\ displacement\ copmponents$

$\{q\} = \int_v [N(x,y,z)]^T \{f\}dv + \int_s [N(x,y,z)]^T \{t\}ds = Nodal\ \ force\ matrix$

{f} = Body forces

{t} = Surface tractions

[N(x,y,z)] in Step 3, Chapter 3, [B(x,y,z)] in Equation (4.13), [C] in Equation (4.7)

Examples of FE Stress Analysis of Solid Structures

NOTE: In FE stress analysis of solid structures, it is customer to represent the **element displacements** by:

$$\{U(x,y,z)\} = \begin{Bmatrix} U_x(x,y,z) \\ U_y(x,y,z) \\ U_z(x,y,z) \end{Bmatrix}$$

and **nodal displacements** by: {u}.

The relationship between the element displacements and the nodal displacements are:

{U} = [N(x,y,z)] {u}

where [N(x,y,z)] = the Interpolation function. It is a row matrix for 1-D bar elements,
rectangular matrices for 2- or 3-D elements.

- **Interpolation function enables the determination of the primary quantities in the element with specified coordinate (x,y,z) with the same primary quantities of the associate nodes**