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THE COMPUTATIONAL EFFICIENCY OF WALSH APPROXIMATION FOR TWO-DIMENSIONAL VOLTERRA INTEGRAL EQUATIONS

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In this research, we give details of a new numerical method for the approximate solution of a general two-dimensional Volterra integral equation, using the discontinuous wavelet packets e.g. Walsh functions. The double Walsh approximation we have adopted utilizes a simple robust numerical scheme for approximate solution of the equations. The two-dimensional operational matrix of integration for each subinterval $[\frac{i-1}{m}, \frac{i}{m}]$ is explicitly constructed, where m is a power of 2. Finally the reliability and efficiency of the proposed scheme are demonstrated by some numerical results.

Keywords: Walsh function; operational matrix; two-dimensional Walsh transformation; integral equations; numerical treatments.

AMS Subject Classification: 65R20, 34A25

1. Introduction

In recent years, the Walsh theory has been innovated and investigated to various fields of applied sciences e.g. signal processing, communication and pattern recognition [18], transform spectroscopy [10], heart rate, optimal linear systems [17], variational problems [6, 20], etc. The basic properties and notations of Walsh functions may be found in [18, 21, 9] and references therein. This paper is concerned with the development of efficient numerical method based on double Wlash transformation for the solution of two-dimensional Volterra integral equations of the second kind

$$u(x,y) = f(x,y) + \int_0^x \int_0^y k(x,y,s,t)u(s,t)dsdt, \qquad (x,y) \in D, \tag{1.1}$$

where the known functions $f(x,y)$ and $k(x,y,s,t)$ are defined, respectively, on $D := [0,1] \times [0,1]$ and $S := \{(x,y,s,t) : 0 \leq s \leq y \leq 1, \ \ 0 \leq t \leq x \leq 1\}$ and $u(x,y)$ is a solution to be determined. This equation may arise from certain hyperbolic differential equations (see [8], for an equivalent formulation of the Darboux problem). Actually, a few approximate methods for (1.1) are known. Beltyukov et al. in [1] proposed a class of explicit Runge-Kutta type methods of order 3. Bivariate cubic spline functions method was obtained by Singh in [23]. In [4] an exhaustive analysis of polynomial spline collocation and iterated methods was given by Brunner and Kauthen. The asymptotic error expansion of collocation and iterated collocation as well as Galerkin and iterated Galerkin solutions for two-dimensional linear and nonlinear Volterra integral equations were obtained by Guoqiang et al. in [14, 13]. Here, we restrict our attention to the approximate solution of the linear two-dimensional Volterra integral equations by the Walsh function spectral approach. There has been considerable interest in solving integral equations using techniques which involve Walsh functions. A summary of the historical developments of the Walsh approximation for integral equations may be found in [15, 3] where an excellent bibliography on Walsh functions and its applications is also given in [9]. One of the motivations for these developments is that these methods usually involve the use of fast Walsh Fourier transform, which is faster than the corresponding transforms such as the trigonometric fast Fourier transform. Also, Walsh functions appear to be easily incorporated into a wide variety of robust general purpose algorithms.

2. Preliminaries and Basic Idea

Let $f \in L^2[0,1)$, then $f(x)$ can be expanded as a series of Walsh functions $f(x) = \sum_{i=0}^{\infty} c_i W_i(x)$, where $c_i = \int_0^1 f(x)W_i(x)dx$. It is well known that its integral from 0 to x have Walsh series with coefficients of b_i, where $\int_0^x f(t)dt = \sum_{i=0}^{\infty} b_i W_i(x)$. If we truncate to $m = 2^n$ terms and use to obvious vector notation, then integration is performed by matrix multiplication $b = P_m^t c$, where

$$P_m^t = \begin{pmatrix} P_{\frac{m}{2}} & \frac{1}{2m} I_{\frac{m}{2}} \\ \\ \frac{-1}{2m} I_{\frac{m}{2}} & O_{\frac{m}{2}} \end{pmatrix}, \qquad P_2^t = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \\ \frac{-1}{4} & 0 \end{pmatrix},$$

and I_m and O_m are the unit matrix and zero matrix of order m, respectively. (For details see [6, 2].) Now, if $f(x,y)$ has a Walsh series with coefficients c_{ij} and its integral has a truncated Walsh series with coefficients of b_{ij}, such that

$$\int_0^x \int_0^y f(t,s)dtds = \int_0^x \int_0^y \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} W_i(t) W_j(s) dtds = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} b_{ij} W_i(x) W_j(y),$$

we will show that the integration is performed by system of equations $b = \mathbf{P}'_m c$. The solution process of this equation leads to a linear system in which the solution at the dyadic grid points is recovered by a method which requires as an input the initial value and average value of the function over the subintervals $[\frac{i-1}{m}, \frac{i}{m}]$ and $[\frac{j-1}{m}, \frac{j}{m}]$ and gives the approximate values of the solution at the grid points. In the next section, the two-dimensional operational matrix of integration is constructed.

2.1. *Operational matrix*

Before giving a more details of the Walsh approximation for two-dimensional integral equations, we obtain a general formula for integration of two-dimensional Walsh functions. The integration process of a step function is defined as: $\int\int W_0(t)W_0(s)dtds$. We will show that, the function f can be expressed by a Walsh series. However, the coefficients c_i of the Walsh series for the function $f(x)$ are given by $c_m = \frac{1}{m} W_m f_m$, and so for the function $f(x, y)$, the double Walsh coefficients are:

$$c_m = \frac{1}{m^2} W_m f_m W_m.$$

We may set $m = 2$, therefore:

$$xy = c_{00}W_0(x)W_0(y) + c_{01}W_0(x)W_1(y) + c_{10}W_1(x)W_0(y) + c_{11}W_1(x)W_1(y),$$

where the coefficients c_{ij} are calculated as follows:

$$\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{9}{16} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{1}{16} \end{pmatrix}.$$

Similarly, we can evaluate the Walsh series coefficients of the integration of functions $W_0(s)W_1(t), W_1(s)W_0(t)$ and $W_1(s)W_1(t)$. The operational matrix $\mathbf{P}'_2$ with some changes in entries for the integration process $\int_0^x \int_0^y W_i(t)W_j(s)dtds$, can be expressed in a block matrix form:

$$\mathbf{P}'_4 = \left(\begin{array}{cc|cc} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{16} \\ \frac{-1}{8} & 0 & \frac{-1}{16} & 0 \\ \hline \frac{-1}{8} & \frac{-1}{16} & 0 & 0 \\ \frac{1}{16} & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \frac{1}{8}P_2^t & \frac{1}{4}P_2^t \\ \hline \frac{-1}{4}P_2^t & O_2 \end{array}\right)_{4\times 4}.$$

It is straightforward to show that the general form for the operational matrix $\mathbf{P}'$ of order m^2 (which is positive integer power of 2) can be obtained as:

$$\mathbf{P}'_{m^2} = \left(\begin{array}{c|c|c|c} \frac{1}{2}P_m^t \quad \ddots & \frac{2}{m}\mathbf{P}_{\frac{m^2}{8}} & \frac{1}{m}\mathbf{P}_{\frac{m^2}{4}} & \frac{1}{2m}\mathbf{P}_{\frac{m^2}{2}} \\ \frac{-2}{m}\mathbf{P}_{\frac{m^2}{8}} & \mathbf{O}_{\frac{m^2}{8}} & & \\ \hline \multicolumn{2}{c|}{\frac{-1}{m}\mathbf{P}_{\frac{m^2}{4}}} & \mathbf{O}_{\frac{m^2}{4}} & \\ \hline \multicolumn{3}{c|}{\frac{-1}{2m}\mathbf{P}_{\frac{m^2}{2}}} & \mathbf{O}_{\frac{m^2}{2}} \end{array}\right)_{m^2\times m^2},$$

where $\mathbf{P}_{\frac{m^2}{2}}$ and $\mathbf{O}_{\frac{m^2}{2}}$ are the following matrices:

$$\mathbf{P}_{\frac{m^2}{2}} = \begin{pmatrix} P_m^t & O_m & \dots & O_m \\ O_m & P_m^t & \dots & O_m \\ \vdots & & & \vdots \\ O_m & O_m & \dots & P_m^t \end{pmatrix}_{\frac{m^2}{2} \times \frac{m^2}{2}}, \qquad \mathbf{O}_{\frac{m^2}{2}} = \begin{pmatrix} O_m & \dots & O_m \\ O_m & \dots & O_m \\ \vdots & & \vdots \\ O_m & \dots & O_m \end{pmatrix}_{\frac{m^2}{2} \times \frac{m^2}{2}}.$$

3. The Numerical Analysis of the Scheme

As a consequence of the previous section, here we derive formulas for numerical solvability of linear integral equation (1.1) based on double Walsh approximation and the two-dimensional operational matrix of integration. The first task is to replace all functions in (1.1) with their Walsh series as follows:

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{ij} W_i(x) W_j(y), \qquad c_{ij} = \int_0^1 \int_0^1 u(x,y) W_i(x) W_j(y) dxdy,$$

and

$$f(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c'_{ij} W_i(x) W_j(x), \qquad c'_{ij} = \int_0^1 \int_0^1 f(x,y) W_i(x) W_j(y) dxdy.$$

The kernel $k(x,y,s,t)$ is approximated by a fourth order Walsh series:

$$k(x,y,s,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} k_{ijmn} W_i(x) W_j(x) W_m(s) W_n(t),$$

where

$$k_{ijmn} = \int_0^1 \int_0^1 \int_0^1 \int_0^1 k(x,y,s,t) W_i(x) W_j(y) W_m(s) W_n(t) dxdydsdt.$$

Following Fine [9], the $m = 2^n$th partial sum of the Walsh series of a function f is a piecewise constant, equal to the L^1 mean of f on each subinterval $[\frac{i-1}{m}, \frac{i}{m}]$. In this case the coefficients c'_{ij} and k_{ijmn} of the Walsh series are:

$$c'_{ij} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \frac{1}{m^2} W_{ij} \widetilde{f}_{ij} W_{ij},$$

where $\widetilde{f}_{ij}$ is the average value of the function $f(x,y)$ in all subintervals and W_{ij} is the value of the ith Walsh function in the jth subinterval. Also, we get:

$$K_{m^2} = \frac{1}{m^4} W_{m^2} \widetilde{K}_{m^2} W_{m^2}.$$

Dividing the interval $[0,1)$ into m subintervals along the x-axis and the y-axis, respectively, we replace all functions by their Walsh series and integrate them in appropriate moments over the grids of the form $[\frac{i-1}{m}, \frac{i}{m})$ and $[\frac{j-1}{m}, \frac{j}{m})$. Finally, the following matrix equation is obtained:

$$W_m C_m W'_m = W_m C'_m W'_m + \sum_{l=0}^{m-1}\sum_{k=0}^{m-1}\sum_{g=0}^{m-1}\sum_{h=0}^{m-1}\sum_{p=0}^{m-1}\sum_{q=0}^{m-1} k_{ghpq} c_{lk}$$

$$\times \begin{bmatrix} W'_{i+1,1}a_{11}W_{j+1,1} & \cdots & W'_{i+1,1}a_{1m}W_{j+1,m} \\ W'_{i+1,2}a_{21}W_{j+1,1} & \cdots & W'_{i+1,2}a_{2m}W_{j+1,m} \\ \vdots & \vdots & \vdots \\ W'_{i+1,m}a_{m1}W_{j+1,1} & \cdots & W'_{i+1,m}a_{mm}W_{j+1,m} \end{bmatrix}. \qquad (3.1)$$

Note that in our innovate matrix, the $W'_{i+1,t}$ $(1 \leq t < m)$ is the $(i+1, t)$th entry of the Walsh matrix along the x-axis, $W_{i+.1,t}$ is the $(i+1, t)$th entry of the Walsh matrix along the y-axis and a_{ij} $(1 \leq i, j < m)$ is defined as follows:

$$W_m \mathbf{P}_1 W'_m = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}.$$

For computing the above matrix, we need to calculate:

$$\int_0^x \int_0^y W_l(s)W_k(t)W_p(s)W_q(t)dsdt.$$

Following [11, p. 1159], using the multiplication rule of two-dimensional Walsh series, we take $z = l \bigotimes p$ and $v = k \bigotimes q$, then we must compute the double integral $\int_0^x \int_0^y W_z(s)W_v(t)dsdt$. Note that one of the columns of operational matrix $\mathbf{P}'$ for $l, k, p, q = 0, \ldots, m-1$ is each of integration process. Changing each m^2 entries of column of operational matrix to the $m \times m$ matrix, the matrix $\mathbf{P}_1$ is obtained.

Finally, we obtain a linear system of m^2 equation with m^2 unknown coefficients, which gives the Walsh coefficients. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments, in the next section.

4. Numerical Experiments

Test problem 1. Consider the following two-dimensional Volterra integral equation:

$$u(x, y) = x^2 + y^2 - \frac{y^3}{3}x - \frac{x^3}{3}y + \int_0^x \int_0^y u(t, s)dtds, \qquad (x, y) \in [0, 1] \times [0, 1],$$

with the exact solution $u(x, y) = x^2 + y^2$. Let $u_N(x, y)$ be the approximate solution of the equation which is approximated by the truncated double Walsh series:

$$u_N(x, y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} W_i(x) W_j(y).$$

m	h	Max error
2	$\frac{1}{2}$	2.00×10^{-3}
4	$\frac{1}{4}$	1.10×10^{-5}
8	$\frac{1}{8}$	6.80×10^{-9}

We choose a uniform partition of the unit interval $[0,1)$ and set $h = \frac{1}{m}$ for $m = 2, 4, 8, ...$ The m^2 Walsh coefficients of approximate solution have been obtained using the linear system of equation (3.1). All computations were carried out in double precision arithmetic with Maple® software. The maximum errors of the scheme are given in Table 1.

However, better approximation is expected by choosing a larger values of m.

Test problem 2. (From [16]) Consider the first kind of Volterra integral equation:

$$f(x,y) = \int_0^x \int_0^y (\sin(x+t) + \sin(y+s) + 3)\, u(t,s) dt ds, \qquad (x,y) \in [0,1] \times [0,1],$$

and choose $f(x,y)$ such that the exact solution is $u(x,y) = \cos(x+y)$.

The L_∞ errors in some subintervals for $m = 4$ are of special interest and can be compared with those obtained by previous work [16]. The matrix form of above equation for $m = 4$ is as follows:

$$W_4 C_4 W_4' = W_4 C_4' W_4' + \sum_{l=0}^{3}\sum_{k=0}^{3}\sum_{i=0}^{3}\sum_{j=0}^{3} \sum_{p=0}^{3}\sum_{q=0}^{3} k_{ijpq} c_{lk}$$

$$\times \begin{bmatrix} W'_{i+1,1} a_{11} W_{j+1,1} & W'_{i+1,1} a_{12} W_{j+1,2} & W'_{i+1,1} a_{13} W_{j+1,3} & W'_{i+1,1} a_{14} W_{j+1,4} \\ W'_{i+1,2} a_{21} W_{j+1,1} & W'_{i+1,2} a_{22} W_{j+1,2} & W'_{i+1,2} a_{23} W_{j+1,3} & W'_{i+1,2} a_{24} W_{j+1,4} \\ W'_{i+1,3} a_{31} W_{j+1,1} & W'_{i+1,3} a_{32} W_{j+1,2} & W'_{i+1,3} a_{33} W_{j+1,3} & W'_{i+1,3} a_{34} W_{j+1,4} \\ W'_{i+1,4} a_{41} W_{j+1,1} & W'_{i+1,4} a_{42} W_{j+1,2} & W'_{i+1,4} a_{43} W_{j+1,3} & W'_{i+1,4} a_{44} W_{j+1,4} \end{bmatrix}.$$

We take $h = 0.25$, and $m_i = [\frac{i-1}{m}, \frac{i}{m}], n_j = [\frac{j-1}{m}, \frac{j}{m}]$, so, the results for different values of $i = j = 1, 2, 3, 4$ are obtained.

$i(=j)$	$h = 0.25$
1	9.21×10^{-4}
2	7.19×10^{-4}
3	3.02×10^{-3}
4	1.71×10^{-5}
$\|e(h)\|_\infty$	3.02×10^{-3}

For the Walsh approximation scheme, the errors on the subintervals are mainly less than the corresponding errors at the other grid points for the Euler and trapezoidal methods. (For details see [16].)

5. Conclusion

The intent of this note was to confirm the Walsh approximation for numerical solution of general two-dimensional Volterra integral equations. The reliability and efficiency of the scheme are demonstrated by some numerical experiments. Due to the nature of Walsh functions, the process of solution mainly depends on m, where for the larger values of m, we have some restrictions dealing with large matrices. However, following [19], using the generalization of the single-term Walsh series strategy we can address this problem in the proposed scheme that will be addressed in an upcoming paper.

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