On the approximate solution of integro-differential equations arising in oscillating magnetic fields

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Abstract

In this work, we propose the Shannon wavelets approximation for the numerical solution of a class of integro-differential equations which describe the charged particle motion for certain configurations of oscillating magnetic fields. We will show that using the Galerkin method and the connection coefficients of the Shannon wavelets, the problem is transformed to an infinite algebraic system, which can be solved by fixing a finite scale of approximation. The error analysis of the method is also investigated. Finally, some numerical experiments are reported to illustrate the accuracy and applicability of the method.

key words: Charged particle motion, Oscillating magnetic field, Integro-differential equation, Shannon wavelet, Numerical treatment.

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1 Introduction

Integral and integro-differential equations of Volterra type arise in many modeling problems in physical fields such as optics, electromagnetics, electrodynamics, statistical physics, inverse scattering problems [1, 2, 3] and many other practical applications.

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In this paper we consider the following Volterra integro-differential equation from [4]:

$$\begin{cases} y^{(2)}(t) + a(t)y(t) = g(t) + b(t)\displaystyle\int_0^t \cos(\omega_p s)y(s)ds, \\ y(0) = \alpha, \\ y'(0) = \beta, \end{cases} \tag{1.1}$$

where $a(t)$, $b(t)$ and $g(t)$ are given periodic functions of time and $y(t)$ is an unknown function to be determined. The existence and the uniqueness results of this type of problems have been investigated by many authors. See e.g. [5, 6, 7]. For instance, using Theorem 4 in [5], we can obtain the existence and the uniqueness issues of the second kind integro-differential equation (1.1).

Throughout this paper, we assumed that the conditions of the given functions of the equation (1.1) are somehow that the existence and the uniqueness results of the solution of (1.1) are satisfied.

This equation, which describes the charged particle motion for certain configurations of oscillating magnetic fields, may be easily found in the charged particle dynamics for some field configurations [4].

For clarifying the model, suppose that the three mutually orthogonal magnetic field components are defined as:

$$\begin{cases} B_x = B_1 \sin \omega_p t, \\ B_y = 0, \\ B_z = B_0. \end{cases} \tag{1.2}$$

So, the nonrelativistic equations of motion for a particle of mass m and charge q in this field configuration are

$$m\frac{d^2x}{dt^2} = q(B_0\frac{dy}{dt}), \tag{1.3}$$

$$m\frac{d^2y}{dt^2} = q(B_1 \sin \omega_p t\frac{dz}{dt} - B_0\frac{dx}{dt}), \tag{1.4}$$

$$m\frac{d^2z}{dt^2} = q(-B_1 \sin \omega_p t\frac{dy}{dt}). \tag{1.5}$$

Integrating (1.3) and (1.5) and substituting the results into (1.4) yield

$$\begin{aligned} \tfrac{d^2y}{dt^2} &= -(\omega_c^2 + \omega_f^2 \sin^2 \omega_p t)y + \omega_f^2\omega_p(\sin \omega_p t)\textstyle\int_0^t (\cos \omega_p s)y(s)ds \\ &+\omega_f(\sin \omega_p t)z'(0) + \omega_c^2 y(0) + \omega_c x'(0), \end{aligned}$$

where $\omega_c = q\frac{B_0}{m}$ and $\omega_f = q\frac{B_1}{m}$, which is corresponding to model (1.1) with the following periodic functions:

$$
\begin{aligned}
a(t) &= \omega_c^2 + \omega_f^2 \sin^2(\omega_p t),\\
b(t) &= \omega_f^2 \omega_p \sin(\omega_p t),\\
g(t) &= \omega_f(\sin(\omega_p t))z'(0) + \omega_c^2 y(0) + \omega_c x'(0).
\end{aligned}
$$

The numerical solvability of equation (1.1) and other related equations has been pursued by several authors. Dehghan and Shakeri [4] in 2008, applied the Homotopy perturbation method for solving equation (1.1). Machado et al. in [8] solved (1.1) by using Adomian's method. In [9], Volterra integro-differential equation with periodic solution was considered, via the Mixed collocation method. Akyuz-Dasciogln et al. in [10, 11] presented numerical verification of solutions of systems of integro-differential equations using Chebyshev series.

In recent years, the Shannon approximation method is used for problems whose solutions may have singularities, or infinite domains, or boundary layers. This method has increasingly been recognized as a powerful tool for attacking problems in applied physics. In particular, it has become very popular in solving initial and boundary value problems of ordinary or partial differential equations as well as integral equations.

Here, we are interested in the numerical Shannon approximation of the integro-differential equation (1.1). Our discussion is based on the connection coefficients of the Shannon wavelets which were proposed by Cattani in [12]. Detailed description and analysis of this technique may be found in [12, 13, 14] and references therein.

The layout of the paper is as follows: In Section 2, we give basic definitions, assumptions and preliminaries of the Shannon wavelets. In Section 3, we clarify how the Shannon approximation including the Galerkin method transform the equation (1.1) to an explicit system of linear algebraic equations. The error analysis of the proposed method is given in Section 4. Finally, some numerical results are reported to clarify the efficiency of the method.

2 A survey and some properties of the Shannon wavelets

At first, we recall the definitions and the notations of the Shannon wavelets family from [14]. The starting point for the definition of the Shannon wavelets family is the *Sinc* or Shannon scaling function. The *Sinc* function is defined on the whole real line by:

$$
Sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0,\\ 1, & t = 0. \end{cases}
$$

The Shannon scaling functions, mother wavelets and their Fourier transforms can be defined as:

$$\begin{cases} \varphi_{j,k}(t) = 2^{j/2} Sinc(2^j t - k) = 2^{j/2} \frac{\sin \pi(2^j t-k)}{\pi(2^j t-k)}, & j,k \in Z, \\ \psi_{j,k}(t) = 2^{j/2} \frac{\sin \pi(2^j t-k-\frac{1}{2}) - \sin 2\pi(2^j t-k-\frac{1}{2})}{\pi(2^j t-k-\frac{1}{2})}, & j,k \in Z, \\ \widehat{\varphi}_{j,k}(\omega) = \frac{2^{-j/2}}{2\pi} e^{-i\omega k/2^j} \chi(\frac{\omega}{2^j} + 3\pi), & j,k \in Z, \\ \widehat{\psi}_{j,k}(\omega) = -\frac{2^{-j/2}}{2\pi} e^{-i\omega(k+1/2)/2^j} [\chi(\frac{\omega}{2^{j-1}}) + \chi(-\frac{\omega}{2^{j-1}})], & j,k \in Z, \end{cases}$$

where the characteristic function $\chi(\omega)$ is defined as:

$$\chi(\omega) = \begin{cases} 1, & 2\pi \leq \omega < 4\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Using the definition of the inner product of two functions and the Parseval equality, it can be easily shown that the following results hold [12]:

$$\begin{cases} < \psi_{j,k}, \psi_{m,h} >= \delta_{jm}\delta_{hk}, \\ < \varphi_{0,k}, \varphi_{0,h} >= \delta_{kh}, \\ < \varphi_{0,k}, \psi_{j,h} >= 0, & j \geq 0, \end{cases} \tag{2.1}$$

where δ_{jm} (δ_{hk}) denotes the Kronecker delta.

Orthogonality conditions (2.1) lead to the following theorem which is concerned with approximation of the function $y(t)$:

Theorem 1 (From [12]) *If $y(t) \in L_2(\mathbb{R})$, then the series*

$$\sum_{k=-\infty}^{\infty} \alpha_k \varphi_{0,k}(t) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \beta_{j,k} \psi_{j,k}(t) \tag{2.2}$$

converges to $y(t)$, with α_k and $\beta_{j,k}$ being given by:

$$\alpha_k =< y, \varphi_{0,k} >= \int_{-\infty}^{\infty} y(t)\varphi_{0,k}(t)dt, \tag{2.3}$$

$$\beta_{j,k} =< y, \psi_{j,k} >= \int_{-\infty}^{\infty} y(t)\psi_{j,k}(t)dt. \tag{2.4}$$

3 Method of the solution

As a consequence of the previous section, in this section we drive formulas for numerical solvability of integro-differential equation (1.1) based on the connection coefficients of the Shannon wavelets.

Using a finite truncated series of the above equation, we can define an approximation function of the exact solution $y(t)$ as follows:

$$y(t) \simeq \sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(t). \tag{3.1}$$

Also, expression (2.2) enables us to compute the derivatives of $y(t)$ in terms of the Shannon wavelets decomposition

$$\frac{d^l}{dt^l} y(t) \simeq \sum_{k=-M}^{M} \alpha_k \frac{d^l}{dt^l} \varphi_{0,k}(x) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \frac{d^l}{dt^l} \psi_{j,k}(x). \tag{3.2}$$

According to expression (3.2), the derivatives of $y(t)$ are known when the functions $\frac{d^l}{dt^l}\varphi_{0,k}(t)$ and $\frac{d^l}{dt^l}\psi_{j,k}(t)$ are given. Indeed, due to (3.2), we can compute the wavelet decomposition of the derivatives from the following relations which are given in [12]:

$$\begin{cases} \frac{d^l}{dt^l}\varphi_{0,k}(t) = \sum_{h=-M}^{M} \lambda_{kh}^{(l)} \varphi_{0,h}(t) + \sum_{j=0}^{N} \sum_{h=-M}^{M} \Lambda_{kh}^{(l)j} \psi_{j,h}(t), \\ \frac{d^l}{dt^l}\psi_{j,k}(t) = \sum_{h=-M}^{M} \xi_{kh}^{(l)j} \varphi_{0,h}(t) + \sum_{j=0}^{N} \sum_{h=-M}^{M} \gamma_{kh}^{(l)jj} \psi_{j,h}(t), \end{cases} \tag{3.3}$$

with

$$\begin{aligned} \Lambda_{kh}^{(l)j} &\equiv \left\langle \frac{d^l}{dt^l}\varphi_{o,k}, \psi_{j,h} \right\rangle, \quad \lambda_{kh}^{(l)} \equiv \left\langle \frac{d^l}{dt^l}\varphi_{o,k}, \varphi_{o,h} \right\rangle, \\ \gamma_{kh}^{(l)jj} &\equiv \left\langle \frac{d^l}{dt^l}\psi_{j,k}, \psi_{j,h} \right\rangle, \quad \xi_{kh}^{(l)j} \equiv \left\langle \frac{d^l}{dt^l}\psi_{j,k}, \varphi_{o,h} \right\rangle, \end{aligned} \tag{3.4}$$

which are known as the connection coefficients.

In this position, we recall an auxiliary lemma for computing these coefficients and their properties from [12]:

Lemma 1 *The connection coefficients $\Lambda_{kh}^{(l)j}$ and $\xi_{kh}^{(l)j}$ are zero. Also, the any order connection coefficients $\lambda_{kh}^{(l)}$ and $\gamma_{kh}^{(l)jj}$ are:*

$$\lambda_{kh}^{(l)} = \begin{cases} (-1)^{k-h} \frac{i^l}{2\pi} \sum_{s=1}^{l} \frac{l!\pi^s}{s![i(k-h)]^{l-s+1}} [(-1)^s - 1], & k \neq h, \\ \frac{i^l \pi^{l+1}}{2\pi(l+1)} [1 + (-1)^l], & k = h, \end{cases}$$

and

$$\gamma_{kh}^{(l)jj} = \begin{cases} \frac{i^l 2^{jl}}{2\pi} \sum_{s=1}^{l} (-1)^l \frac{l!\pi^s (2^s - 1)}{s![i(h-k)]^{l-s+1}} [(-1)^s - 1], & k \neq h, \\ \frac{i^l 2^{jl} \pi^{l+1}}{2\pi(l+1)} [(2^{l+1} - 1)(1 + (-1)^l)], & k = h, \end{cases}$$

3.1 Description of the proposed method

In order to obtain the numerical solution, we apply the previous results for constructing the Shannon approximate solution of the Volterra integro-differential equation (1.1). According to Lemma 1, the equation (3.2) can be rewritten as:

$$\frac{d^l}{dt^l}y(t) \simeq \sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda_{kh}^{(l)} \varphi_{0,h}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{(l)jj} \psi_{j,h}(t). \quad (3.5)$$

By using expression (3.5), we can approximate $\frac{d^2}{dt^2}y(t)$ as follows:

$$\frac{d^2}{dt^2}y(t) \simeq \sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda_{kh}^{(2)} \varphi_{0,h}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{(2)jj} \psi_{j,h}(t). \quad (3.6)$$

Now, by substituting this relation and expression (3.1) into (1.1), we get the following result:

$$\begin{aligned} g(t) \quad &\simeq \left[\textstyle\sum_{h=-M}^{M} \lambda_{kh}^{(2)} \varphi_{0,h}(t) + a(t)\varphi_{0,h}(t) - b(t)\int_0^t \cos(\omega_p s)\varphi_{0,h}(s)ds\right] \\ &\times \textstyle\sum_{k=-M}^{M} \alpha_k \\ &+ \left[\textstyle\sum_{h=-M}^{M} \gamma_{kh}^{(2)jj} \psi_{j,h}(t) + a(t)\psi_{j,h}(t) - b(t)\int_0^t \cos(\omega_p s)\psi_{j,h}(s)ds\right] \\ &\times \textstyle\sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k}. \end{aligned}$$

Here, we can define the residual vector as follows:

$$r_{(2M+1)(N+2)} = [\Gamma_{1k} + \Gamma_{2k}] \sum_{k=-M}^{M} \alpha_k + [\Delta_{1k} + \Delta_{2k}] \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} - g(t),$$

with

$$\begin{aligned} \Gamma_{1k} &= \textstyle\sum_{h=-M}^{M} \lambda_{kh}^{(2)} \varphi_{0,h}(t) + a(t)\varphi_{0,h}(t), \\ \Gamma_{2k} &= -b(t)\textstyle\int_0^t \cos(\omega_p s)\varphi_{0,h}(s)ds, \\ \Delta_{1k} &= \textstyle\sum_{h=-M}^{M} \gamma_{kh}^{(2)jj} \psi_{j,h}(t) + a(t)\psi_{j,h}(t), \\ \Delta_{2k} &= -b(t)\textstyle\int_0^t \cos(\omega_p s)\psi_{j,h}(s)ds. \end{aligned}$$

Also, the following relations can be obtained from the boundary conditions (1.1):

$$\textstyle\sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(0) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(0) = \alpha,$$

$$\textstyle\sum_{k=-M}^{M} \alpha_k \sum_{h=-M}^{M} \lambda'_{kh} \varphi_{0,h}(0) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \sum_{h=-M}^{M} \gamma_{kh}^{'jj} \psi_{j,h}(0) = \beta. \quad (3.7)$$

Let us introduce the notations Θ and ν, for $j = 0, ..., N$ and $k = -M, ..., M$, by the following relations

$$\begin{cases} \Theta_{k+M+1} := \varphi_{0,k}, \\ \Theta_{(2M+1)(j+1)+k+M+1} := \psi_{j,k}, \\ \nu_{k+M+1} := \alpha_k, \\ \nu_{(2M+1)(j+1)+k+M+1} := \beta_{j,k}. \end{cases} \tag{3.8}$$

With these notations, we have

$$\sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(t) = \sum_{i=1}^{(2M+1)(N+2)} \nu_i \Theta_i(t). \tag{3.9}$$

We can compute ν_i, for $i = 1, ..., (2M+1)(N+2)$, using the Galerkin method such that

$$< r_{(2M+1)(N+2)}, \Theta_n >= 0, \qquad n = 1, ..., (2M+1)(N+2) - 2. \tag{3.10}$$

This relation together with the boundary condition (3.7) gives a system of $(N+2)(2M+1)$ algebraic equations for $(N+2)(2M+1)$ unknown coefficients α_k and $\beta_{j,k}$. Determining these coefficients, an approximate solution will be obtained from (3.1) for the equation (1.1).

The following algorithm summarizes our proposed method:

Algorithm: The construction of the Shannon approximation for the Volterra integro-differential equation (1.1)

Step 1. Input:
N, M;
$\varphi_{0,k}(t), \psi_{j,k}(t), g(t), \alpha, \beta, \omega_p, a(t), b(t)$;

Step 2. Compute:
2.1. $\gamma_{kh}^{(1)}$, $\lambda_{kh}^{(1)jj}$, $\gamma_{kh}^{(2)}$, $\lambda_{kh}^{(2)jj}$, Γ_{1k}, Γ_{2k}, Δ_{1k}, Δ_{2k}; $\quad j = 0, ..., N$; $k, h = -M, ..., M$.

2.2. $r_{(2M+1)(N+2)}$.

Step 3. Set:

$$\begin{cases} \Theta_{k+M+1} \longleftarrow \varphi_{0,k}, & \Theta_{(2M+1)(j+1)+k+M+1} \longleftarrow \psi_{j,k}, \\ \nu_{k+M+1} \longleftarrow \alpha_k, & \nu_{(2M+1)(j+1)+k+M+1} \longleftarrow \beta_{j,k}, \end{cases}$$

$j = 0, ..., N$ and $k, h = -M, ..., M$.

Step 4. Compute α_k and $\beta_{j,k}$ from $< r_{(2M+1)(N+2)}, \Theta_{j'} >= 0$ and (3.7) for $j = 0, ..., N$ and $k, h = -M, ..., M$.

Step 5. Set:
$y(t) \simeq \sum_{k=-M}^{M} \alpha_k \varphi_{0,k}(t) + \sum_{j=0}^{N} \sum_{k=-M}^{M} \beta_{j,k} \psi_{j,k}(t)$.

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4 Error analysis

In this section, a convergence analysis of the numerical scheme for the Volterra integro-differential equation (1.1) will be provided. The following theorem expresses the convergence properties of the proposed method, which is as a consequence of Theorem 4.2 from our recent paper [14].

Theorem 2 *Assume that $\widetilde{y}(t)$ be the approximate solution of the equation (1.1). If $y^{(2)}(t) \in L_2(\mathbb{R})$, then the obtained approximation solution of the proposed method converges to the exact solution, where α_k and $\beta_{j,k}$ are given by (2.3) and (2.4).*

Proof. Note that

$$\begin{aligned}\widetilde{y}(t) &= \textstyle\sum_{k=-\infty}^{\infty}\langle y,\varphi_{0,k}\rangle\varphi_{0,k}(t) + \sum_{j=0}^{N-1}\sum_{k=-\infty}^{\infty}\langle y,\psi_{j,k}\rangle\psi_{j,k}(t)\\ &= \textstyle\sum_{j=-\infty}^{N-1}\sum_{k=-\infty}^{\infty}\langle y,\psi_{j,k}\rangle\psi_{j,k}(t).\end{aligned}$$

Actually, as stated in [14, pp. 2675] the following relation holds

$$\left\| D^{(l)}[\sum_{j=-\infty}^{N-1}\sum_{k=-\infty}^{\infty}\langle y,\psi_{j,k}\rangle\psi_{j,k}(t) - y(t)]\right\|_2 \to 0, \quad asN \to \infty,$$

or equivalently

$$\left\| \sum_{k=-\infty}^{\infty}\langle y,\varphi_{0,k}\rangle\varphi_{0,k}^{(l)}(t) + \sum_{j=-\infty}^{N-1}\sum_{k=-\infty}^{\infty}\langle y,\psi_{j,k}\rangle\psi_{j,k}^{(l)}(t) - y^{(l)}(t)\right\|_2 \to 0,$$

as $N \to \infty$.

Substituting the relations (2.3), (2.4), and (3.3) into the above equation for $l = 2$, we get

$$\lim_{N\to\infty}\left[\sum_{k=-\infty}^{\infty}\alpha_k\sum_{h=-\infty}^{\infty}\lambda_{kh}^{(2)}\varphi_{0,h}(t) + \sum_{j=0}^{N-1}\sum_{k=-\infty}^{\infty}\beta_{j,k}\sum_{h=-\infty}^{\infty}\gamma_{kh}^{(2)jj}\psi_{j,h}(t)\right] = y^{(2)}(t),$$

which proves the theorem. □

We emphasize that Theorem 4.2 in [14] gives an error upper bound for $|y(t) - \widetilde{y}_M(t)|$. Here, in order to compute an error estimation for the solution of Volterra integro-differential equation (1.1), we determine an upper bound for $|y^{(2)}(t) - \widetilde{y}_M^{(2)}(t)|$ as follows

Theorem 3 *Let $\widetilde{y}_M^{(2)}(t)$ be the second order derivative of the approximate solution of equation (1.1), then there exist constants C_1 and C_2 independent of N*

and M, such that

$$\left|y^{(2)}(t)-\widetilde{y}_M^{(2)}(t)\right| \leq |C_1(y(-M-1)+y(M+1))$$
$$-C_2\left[\frac{3\sqrt{3}}{\pi}[y(2^{-N-1}(-M-\tfrac{1}{2}))+y(2^{-N-1}(M+\tfrac{3}{2}))]\right]|,$$

where $C_1 = Max\{|\sum_k\sum_h \lambda_{kh}^{(2)}|\}$, $C_2 = Max\{|\sum_k\sum_h \gamma_{kh}^{(2)jj}|\}$ *and* M, N *are referred to the given values of* j *and* k.

Proof. The proof follows immediately from a consequence of Theorem 4.3 in our recent paper [14], and we refrain from going into details. □

5 Numerical experiments and some comments

In order to illustrate the performance of the Shannon approximation in solving integro-differential equations (1.1), we consider the following two cases from [4]:

$$\begin{cases} \omega_p &= 1, \qquad a(t) = -\sin(t), \qquad b(t) = \sin(t), \\ g(t) &= -\sin(t)(-\frac{3}{10}\cos(t)e^{-\frac{t}{3}} + \frac{9}{10}e^{\frac{-t}{3}}\sin(t) + \cos(t) + t\sin(t) - \frac{7}{10}) \\ & +\frac{1}{9}e^{-\frac{t}{3}} - \sin(t)(e^{-\frac{t}{3}} + t), \\ y(0) &= 1, \qquad y^{'}(0) = \frac{2}{3}\ , \end{cases} \tag{5.1}$$

with the exact solution $y(t) = e^{-\frac{t}{3}} + t$, and

$$\begin{cases} \omega_p &= 2, \qquad a(t) = \cos(t), \qquad b(t) = \sin(\frac{t}{2}), \\ g(t) &= \cos(t) - t\sin(t) + \cos(t)(t\sin(t) + \cos(t)) - \sin(\frac{t}{2})(\frac{2}{9}\sin(3t) \\ & -\frac{t}{6}\cos(3t) + \frac{t}{2}\cos(t)), \\ y(0) &= 1, \qquad y^{'}(0) = 0, \end{cases} \tag{5.2}$$

with the exact solution $y(t) = t\sin(t) + \cos(t)$.

N	M	Maximal Error
2	2	$2.16E-7$
4	2	$2.58E-8$
5	4	$3.37E-11$
8	4	$1.56E-12$
9	5	$5.16E-15$

Table 1. Numerical results for (5.1) at $t = 1$

N	M	Maximal Error
2	3	$3.02E-9$
4	3	$3.28E-10$
6	4	$1.83E-14$
8	4	$2.30E-15$
10	4	$2.40E-18$

Table 2. Numerical results for (5.2) at $t = 1$

To investigate the high accurate solution of the present method, the computational results of (5.1) and (5.2) have been reported in Tables 1 and 2, respectively. Graphs of the error functions for several values of M and N are also given in Figures 1-4.

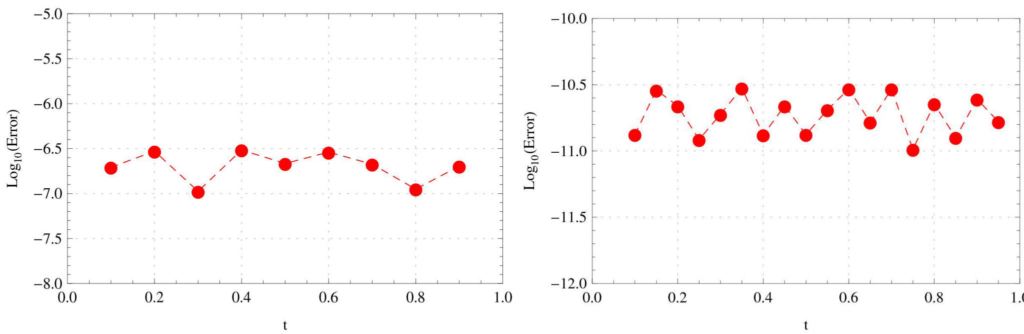

Fig. 1. *Graphs of the Shannon wavelets approximation errors with respect to* $0 \leq t \leq 1$ in the cases $(M = N = 2)$(left) and $(M = 4, N = 5)$ (right) for Example 1.

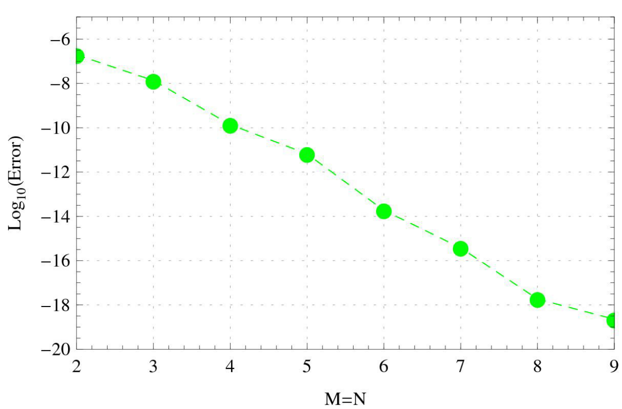

Fig. 2. *Error behaviours of the method with respect to different values of M and N in Example 1.*

Figures 1 and 3 represent the Shannon wavelets approximation errors in the cases $M = N = 2$ and $M = 4, N = 5$ of (5.1) and (5.2) with respect to t in the interval $0 \leq t \leq 1$. The "Maximal Error" refers to the maximal difference between the approximation and the exact solutions.

These problems were solved in [4] by a method based on He's homotopy perturbation. In this case, the best result has the error of order $O(10^{-13})$. The comparison between our results and the method of [4] indicates that for small values of N and M, both methods have produced nearly equivalent approximate solutions. However, the additional numerical experiments show that the good numerical results can be achieved with other values of N and M (e.g. with $N = 5$ and $M = 4$). Also, due to some restrictions of the method in [4], which requires good initial approximations, the comparative effect of our proposed Shannon wavelets approximations will become obvious.

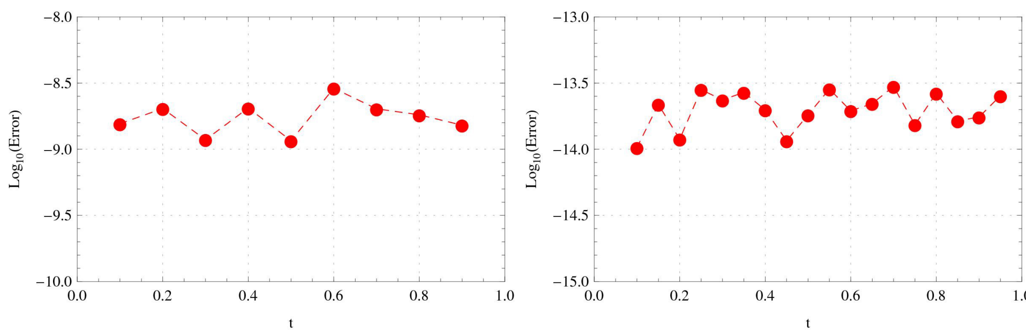

Fig. 3. *Graphs of the Shannon wavelets approximation errors with respect to* $0 \leq t \leq 1$ in the cases $(M = N = 2)$(left) and $(M = 4, N = 5)$ (right) for Example 2.

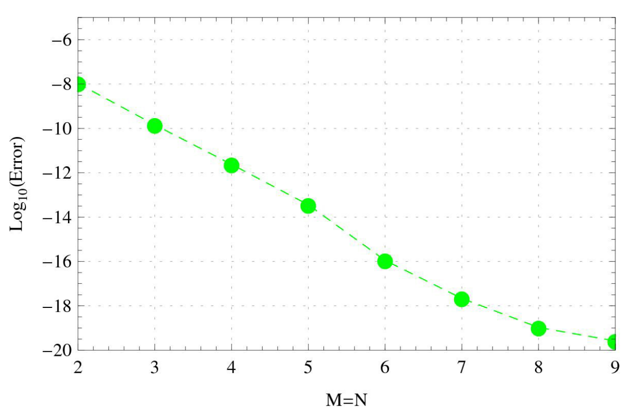

Fig. 4. *Error behaviours of the method with respect to different values of M and N in Example 2.*

Noting that for the numerical implementation in the case $N = M = 1$, the following results will be obtained:

$$\nu_1 = \alpha_{-1} = -0.552305, \qquad \nu_2 = \alpha_0 = 0.412517, \qquad \nu_3 = \alpha_1 = 0.992934,$$

$$\nu_4 = \beta_{0,-1} = -0.202466, \qquad \nu_5 = \beta_{0,0} = -0.16766, \qquad \nu_6 = \beta_{0,1} = -0.8964427,$$

$$\nu_7 = \beta_{1,-1} = 0.761246, \qquad \nu_8 = \beta_{1,0} = 0.125023, \qquad \nu_9 = \beta_{1,1} = 0.59325.$$

Substituting these values into expression (3.1), an approximate solution is obtained. The numerical results are given in Table 3.

t	Approximate solution	Exact solution
0	1.00040	1.00000
0.2	1.13551	1.13551
0.4	1.27518	1.27517
0.6	1.41875	1.41873
0.8	1.56595	1.56593
1	1.716530	1.71653

Table 3. Numerical results of (5.1) for the case $N = M = 1$

6 Conclusions

In this research, a computational method is applied to special class of Volterra integro-differential equations which describe the charged particle motion for certain configurations of oscillating magnetic fields. We use connection coefficients of the Shannon wavelets together with the Galerkin method to obtain numerical solutions for the problem. With the availability of this methodology, it will be possible to investigate the approximate solution of some applicable integro-differential equations. Also, the accuracy of the solution can be improved by selecting the large values of M and N.

References

[1] A. Yildirim, Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method, Comput. Math. Appl. 56 (2008) 3175-3180.

[2] G. Cheng Wu, E. W. M. Lee, Fractional variational iteration methods and its application, Phys. Lett. A 374 (2010) 2506-2509.

[3] K. Domke, L. Hacia, Integral equation in some thermal problems, Int. J. Math. Comput. Simulation. 2 (2007) 184-189.

[4] M. Dehghan, F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's homptopy perturbation method, Progress In Electromagnetics Research. 78 (2008) 361-376.

[5] A. A. Bojeldain, On the numerical solving of nonlinear Volterra integro-differential equations, Annales Univ. Sci. Budapest. Sect. Comp. 11 (1991) 105-125.

[6] R. P. Agarwal, Boundary Value Problems for Higher Order Differential Equations, World Scientific, Singapore, Philadelphia. (1986).

[7] V. Muresan, Existence, uniqueness and data dependence for the solutions of some integro-differential equations of mixed type in Banach space, J. Anal. Appl. 23 (2004) 205-216.

[8] J. M. Machado, M. Tsuchida, Solution for a class of integro-differential equations with time periodic coefficients, Appl. Math. E-Notes. 2 (2002) 66-71.

[9] H. Brunner, A. Makroglou and R. K. Miller, Mixed collocation methods for first and second order Volterra integro-differential equations with periodic solution, Appl. Numer. Math. 23 (1997) 381-402.

[10] A. Akyuz-Dascioglu, Chebyshev polynomial solutions of systems of linear integral equations, Appl. Math. Comput. 151 (2004) 221-232.

[11] A. Akyuz-Dascioglu, M. Sezer, Chebyshev polynomial solutions of systems of higher-order linear FredholmVolterra integro-differential equations, J. Franklin Inst. 342 (2005) 688-701.

[12] C. Cattani, Connection coefficients of Shannon wavelets, Math. Model. Anal. 11 (2006) 117-132.

[13] C. Cattani, Shannon wavelets for the solution of integro-differential equations, Math. Probl. Eng. (2010) 1-22.

[14] K. Maleknejad, M. Attary, An efficient numerical approximation for the linear class of Fredholm integro-differential equations based on Cattani's method, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 2672-2679.