Chapter 3

1 Banach Spaces

It is assumed the reader is familiar with the concept of an abstract vector space over the real or complex numbers. Banach spaces are vector spaces with additional structure. Most of this course is concerned only with finite-dimensional spaces. However, much of the theory is really no harder in the general case, and the infinite-dimensional case is essential in more advanced work.

1. Definition. A norm on a real (complex) vector space E is a mapping from E into the real numbers, $\|\cdot\| : E \to \mathbb{R}$ by $e \mapsto \|e\|$, such that

(N1) $\|e\| \geqslant 0$ for all $e \in E$ and $\|e\| = 0$ iff $e = 0$ (positive definiteness)

(N2) $\|\lambda e\| = |\lambda| \, \|e\|$ for all $e \in E$ and $\lambda \in \mathbb{R}$ (homogeneity)

(N3) $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$ for all $e_1, e_2 \in E$ (triangle inequality)

The pair $(E, \|\cdot\|)$ is called a normed space. If there is no danger of confusion, we sometimes just say "E is a normed space". To distinguish different norms, different notations are sometimes used, e.g. $\|\cdot\|_E$, $\|\cdot\|_1$, $|||\cdot|||$, etc. for the norm.

Remark: The triangle inequality (N3) has the following important consequence:

$$\big|\,\|e_1\| - \|e_2\|\,\big| \le \|e_1 - e_2\| \quad \text{for all } e_1, e_2 \in E$$

which is proved in the following way:

$$\|e_2\| = \|e_1 + (e_2 - e_1)\| \le \|e_1\| + \|e_1 - e_2\|$$

$$\|e_1\| = \|e_2 + (e_1 - e_2)\| \le \|e_2\| + \|e_1 - e_2\|$$

so that both $\|e_2\| - \|e_1\|$ and $\|e_1\| - \|e_2\|$ are smaller than or equal to $\|e_1' - e_2\|$.

If (N1) in 1 is replaced by

(N1)$'$ $\|e\| \geq 0$ for all $e \in E$ and $e = 0$ implies $\|e\| = 0$,

the mapping $\|\cdot\| : E \to \mathbb{R}$ is called a seminorm.

For example, $\mathbb{R}^n$ with the standard norm

$$\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}, \qquad x = (x_1, \dots, x_n)$$

is a normed space. Actually, the standard norm on $\mathbb{R}^n$ comes from a more special structure define as follows.

<u>2 Definition.</u> An inner product on a real vector space E is a mapping

$$\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R} \;;\; (e_1, e_2) \mapsto \langle e_1, e_2 \rangle$$

such that

(I1) $\langle e, e_1 + e_2 \rangle = \langle e, e_1 \rangle + \langle e, e_2 \rangle$;

(I2) $\langle e, \alpha e_1 \rangle = \alpha \langle e, e_1 \rangle$;

(I3) $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle$;

(I4) $\langle e, e \rangle \geq 0$ and $\langle e, e \rangle = 0$ iff $e = 0$.

The standard inner product on $\mathbb{R}^n$ is

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i$$

and (I1) – (I4) are readily checked.

For complex vector spaces the difinition is modified slightly as follows.

<u>2' Definition</u>. A complex inner product (or a Hermitian inner product) on a complex vector space E is a mapping

$$\langle\cdot,\cdot\rangle : E\times E \longrightarrow \mathbb{C}$$

such that

(CI1) $\langle e, e_1+e_2\rangle = \langle e,e_1\rangle + \langle e,e_2\rangle$;

(CI2) $\langle \alpha e, e_1\rangle = \alpha\langle e,e_1\rangle$;

(CI3) $\langle e_1,e_2\rangle = \overline{\langle e_2,e_1\rangle}$ (so $\langle e,e\rangle$ is real);

(CI4) $\langle e,e\rangle \geq 0$ and $\langle e,e\rangle = 0$ iff $e=0$.

These properties are to hold for all $e, e_1, e_2 \in E$ and $\alpha\in\mathbb{C}$; $^{-}$ denotes complex conjugation.

Note that (CI2) and (CI3) imply that

$$\langle e_1, \alpha e_2\rangle = \bar{\alpha}\langle e_1,e_2\rangle$$

Properties (CI1) – (CI3) are also known in the literature under the name <u>sesquilinearity</u>.

The standard inner product on $\mathbb{C}^n = \mathbb{C}\times\cdots\times\mathbb{C}$ is

$$\langle z,w\rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

and (CI1) – (CI4) are readily checked. Also $\mathbb{C}^n$ is a normed sp. with

$$\|z\|^2 = \sum_{i=1}^{n} |z_i|^2$$

In $\mathbb{R}^n$ or $\mathbb{C}^n$, property (N3) is a little harder to check directly. However, as we shall show in Theorem .4, (N3) follows from (I1) – (I4) or (CI1) – (CI4).

In a (real or complex) inner product space E, two vectors $e_1, e_2 \in E$ are called orthogonal denoted $e_1 \perp e_2$, if $\langle e_1, e_2 \rangle = 0$. For any set $A \subset E$,

$$A^\perp = \{ e \in E \mid \langle e, x \rangle = 0 \text{ for all } x \in A \}$$

is called the orthogonal complement of A. Two sets $A, B \subset E$ are called orthogonal, denoted $A \perp B$, if $\langle A, B \rangle = 0$; i.e., $e_1 \perp e_2$ for all $e_1 \in A$ and $e_2 \in B$.

<u>3 Theorem.</u> <u>(Cauchy-Schwarz Inequality)</u>. In a (real or complex) inner product space we have

$$|\langle e_1, e_2 \rangle| \leq \langle e_1, e_1 \rangle^{1/2} \langle e_2, e_2 \rangle^{1/2}$$

Equality holds iff e_1, e_2 are linearly dependent.

<u>Proof.</u> It suffices to prove the complex case. If $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned} 0 \leq \langle \alpha e_1 + \beta e_2, \alpha e_1 + \beta e_2 \rangle &= \langle \alpha e_1, \alpha e_1 \rangle + \langle \alpha e_1, \beta e_2 \rangle + \langle \beta e_2, \alpha e_1 \rangle + \langle \beta e_2, \beta e_2 \rangle \\ &= \alpha\bar{\alpha} \langle e_1, e_1 \rangle + \alpha\bar{\beta} \langle e_1, e_2 \rangle + \bar{\alpha}\beta \langle e_2, e_1 \rangle + \beta\bar{\beta} \langle e_2, e_2 \rangle \\ &= |\alpha|^2 \langle e_1, e_1 \rangle + \alpha\bar{\beta} \langle e_1, e_2 \rangle + \overline{\alpha\bar{\beta} \langle e_1, e_2 \rangle} + |\beta|^2 \langle e_2, e_2 \rangle \\ &= |\alpha|^2 \langle e_1, e_1 \rangle + 2 \,\mathrm{Re}\, (\alpha\bar{\beta} \langle e_1, e_2 \rangle) + |\beta|^2 \langle e_2, e_2 \rangle \end{aligned}$$

If $\alpha = \langle e_2, e_2\rangle$, $\beta = -\langle e_1, e_2\rangle$. Then

$$\alpha\bar\beta\langle e_1, e_2\rangle = \langle e_2, e_2\rangle\left(\overline{-\langle e_1, e_2\rangle}\right)\langle e_1, e_2\rangle$$

$$= -\langle e_2, e_2\rangle\,|\langle e_1, e_2\rangle|^2$$

$$2\mathrm{Re}(\alpha\bar\beta\langle e_1, e_2\rangle) = -2\langle e_2, e_2\rangle\,|\langle e_1, e_2\rangle|^2$$

So

$$0 \le \langle e_2, e_2\rangle^2\langle e_1, e_1\rangle - 2\langle e_2, e_2\rangle|\langle e_1, e_2\rangle|^2 + |\langle e_1, e_2\rangle|^2\langle e_2, e_2\rangle$$

or

$$\langle e_2, e_2\rangle\,|\langle e_1, e_2\rangle|^2 \le \langle e_2, e_2\rangle^2\langle e_1, e_1\rangle$$

$$|\langle e_1, e_2\rangle|^2 \le \langle e_1, e_1\rangle\,\langle e_2, e_2\rangle.$$

So

$$|\langle e_1, e_2\rangle| \le \langle e_1, e_1\rangle^{1/2}\,\langle e_2, e_2\rangle^{1/2}$$

If $e_2 = 0$ equality results in the statement of the theorem and there is nothing to prove. If $e_2 \neq 0$, the term $\langle e_2, e_2\rangle$ in the preceding inequality can be cancelled since $\langle e_2, e_2\rangle \neq 0$ by (CI4). Taking square roots yields the statement of the Theorem. Finally, equality results iff

$$\langle e_2, e_2\rangle e_1 - \langle e_1, e_2\rangle e_2 = 0$$

□

<u>4. Theorem.</u> Let $(E, \langle\cdot,\cdot\rangle)$ be a (real or complex) inner product space and set

$$\|e\| = (\langle e, e\rangle)^{1/2}$$

Then $(E, \|\cdot\|)$ is a normed space.

<u>Proof.</u> (N1) and (N2) are trivial verifications. As for (N3), the Cauchy–Schwartz inequality and the obvious inequality

$\mathrm{Re}(\langle e_1, e_2\rangle) \leq |\langle e_1, e_2\rangle|$ imply

$$\begin{aligned}\|e_1+e_2\|^2 &= \|e_1\|^2 + 2\,\mathrm{Re}(\langle e_1,e_2\rangle) + \|e_2\|^2\\ &\leq \|e_1\|^2 + 2\,|\langle e_1,e_2\rangle| + \|e_2\|^2\\ &\leq \|e_1\|^2 + 2\|e_1\|\,\|e_2\| + \|e_2\|^2\\ &= (\|e_1\| + \|e_2\|)^2\end{aligned}$$

Some other useful facts about inner products are given next.

5. <u>Theorem</u>. Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space and $\|\cdot\|$ the corresponding norm. Then

(i) (Polarization)

$$4\langle e_1,e_2\rangle = \begin{cases} \|e_1+e_2\|^2 - \|e_1-e_2\|^2 + i\|e_1+ie_2\|^2 - i\|e_1-ie_2\|^2 & \text{if } E \text{ is complex}\\[2ex] \|e_1+e_2\|^2 - \|e_1-e_2\|^2 & \text{if } E \text{ is real}\end{cases}$$

(ii) (Parallelogram law)

$$2\|e_1\|^2 + 2\|e_2\|^2 = \|e_1+e_2\|^2 + \|e_1-e_2\|^2$$

<u>Proof.</u> (i)

$$\begin{aligned}&\|e_1+e_2\|^2 - \|e_1-e_2\|^2 + i\|e_1+ie_2\|^2 - i\|e_1-ie_2\|^2\\ &\quad= \|e_1\|^2 + 2\,\mathrm{Re}(\langle e_1,e_2\rangle) + \|e_2\|^2 - \|e_1\|^2 + 2\,\mathrm{Re}(\langle e_1,e_2\rangle) - \|e_2\|^2\\ &\quad\quad + i\|e_1\|^2 + 2i\,\mathrm{Re}(\langle e_1,ie_2\rangle) + i\|e_2\|^2 - i\|e_1\|^2 + 2i\,\mathrm{Re}(\langle e_1,e_2\rangle) - i\|e_2\|^2\\ &\quad= 4\,\mathrm{Re}(\langle e_1,e_2\rangle) + 4i\,\mathrm{Re}(-i\langle e_1,e_2\rangle)\\ &\quad= 4\,\mathrm{Re}(\langle e_1,e_2\rangle) + 4i\,\mathrm{Im}(\langle e_1,e_2\rangle)\\ &\quad= 4\langle e_1,e_2\rangle\end{aligned}$$

The real case is proved in a similar way.

(ii') $$\|e_1+e_2\|^2+\|e_1-e_2\|^2 = \|e_1\|^2+2\,\mathrm{Re}\,(\langle e_1,e_2\rangle)+\|e_2\|^2$$
$$+\|e_1\|^2-2\,\mathrm{Re}\,(\langle e_1,e_2\rangle)+\|e_2\|^2$$
$$=2\,\|e_1\|^2+2\,\|e_2\|^2$$

<u>Remark</u>. Not all norms come from an inner product. For example in $\mathbb{R}^n$,

$$|||x||| = \sum_{i=1}^{n} |x_i|$$

is a norm, but there is no inner product with this as norm, since this norm fails to satisfy the parallelogram law

<u>6. Theorem</u>. Let $(E, \|\cdot\|)$ be a normed (seminormed) space and let $d(e_1,e_2) = \|e_1-e_2\|$. Then (E,d) is a metric (pseudometric space).

<u>Proof</u>. The only nonobvious verification is the triangle inequality. By (N3) we have

$$\begin{aligned} d(e_1,e_2) &= \|e_1-e_2\| \\ &= \|(e_1-e_2)+(e_3-e_3)\| \\ &= \|(e_1-e_3)+(e_3-e_2)\| \\ &\le \|e_1-e_3\|+\|e_2-e_3\| \\ &= d(e_1,e_3)+d(e_3,e_2) \end{aligned}$$

<u>Note</u>: Thus we have the following hierarchy of generality.

more general $\rightarrow$

inner product space $\subset$ normed spaces $\subset$ metric spaces $\subset$ topological spaces

$\leftarrow$ more special

Since inner product and normed spaces are metric spaces, we can use the concepts from metric spaces. In a normed space, (N1) and (N2) imply that the maps

$$(e_1, e_2) \mapsto e_1 + e_2 \quad , \quad (\alpha, e) \mapsto \alpha e$$

of $E \times E \to E$ and $\mathbb{C} \times E \to E$, respectively, are continuous. Hence for $e_0 \in E$, $\alpha_0 \in \mathbb{C}$, $\alpha_0 \neq 0$ fixed, the mappings

$$e \mapsto e_0 + e \quad , \quad e \mapsto \alpha_0 e$$

are homeomorphism. Thus U is a nhbd. of the origin iff $e + U = \{e + x \mid x \in U\}$ is a nhbd. of $e \in E$.

In other words all the nhbds. of $e \in E$ are sets that contain translates of disks centered at the origin. This constitutes a complete description of the topology of a normed vector space $(E, \|\cdot\|)$.

Finally, note that the inequality $\big| \|e_1\| - \|e_2\| \big| \leq \|e_1 - e_2\|$ implies that the norm is uniformly continuous on E. In inner product spaces, the Cauchy-Schwartz inequality implies the continuity of the inner product as a function of two variables.

7. Definition Let $(E, \|\cdot\|)$ be a normed space. If the corresponding metric d is complete, we say $(E, \|\cdot\|)$ is a Banach Space. If $(E, \langle \cdot, \cdot \rangle)$ is an inner product space whose corresponding metric is complete, we say $(E, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

For example, it is proved in advanced calculus that $\mathbb{R}^n$, is complete. Thus, $\mathbb{R}^n$ with the standard norm is a Banach space and with the standard inner product is a Hilbert space.

Not only is the standard norm on $\mathbb{R}^n$ complete, but so is the nonstandard one

$$|||x||| = \sum_{i=1}^{n} |x_i| .$$

However, it is equivalent to the standard one in the following sense.

8. Definition. Two norms on a vector space E are equivalent if they induce the same topology on E.

9. Theorem. Two norms $\|\cdot\|$ and $|||\cdot|||$ on E are equivalent iff there is a constant M such that

$$M^{-1} |||e||| \leq \|e\| \leq M\, |||e|||$$

for all $e \in E$.

Proof. Let

$$\overline{D_r^1}(x) = \{ y \in E \mid \|y-x\| \leq r \}$$
$$\overline{D_r^2}(x) = \{ y \in E \mid |||y-x||| \leq r \}$$

denote the two closed disks of radius r centered at $x \in E$ in the two metrics defined by the norms $\|\cdot\|$ and $|||\cdot|||$, respectively. Since neighborhoods of an arbitrary point are translates of neighborhoods of the origin, the two topologies are the same iff for every $R > 0$, there exists $M_1, M_2 > 0$ such that

$$\overline{D^2_{M_1}}(0) \subset \overline{D^1_R}(0) \subset \overline{D^2_{M_2}}(0)$$

The first inclusion say that if $|||x||| \le M_1$, then $\|x\| \le R$, i.e., if $|||x||| \le 1$ then $\|x\| \le R/M_1$. Thus, if $e \ne 0$, then

$$\left\| \frac{e}{|||e|||} \right\| = \frac{\|e\|}{|||e|||} \le \frac{R}{M_1}$$

That is, $\|e\| \le (R/M_1)\,|||e|||$ for all $e \in E$. Similarly, the second inclusion is equivalent to $(M_2/R)\,|||e||| \le \|e\|$ for all $e \in E$. Thus the two topologies are the same if there exist constants $N_1 > 0$, $N_2 > 0$ such that

$$N_1 |||e||| \le \|e\| \le N_2 |||e|||$$

for all $e \in E$. Taking $M = \max\{N_2, 1/N_1\}$ gives the statement of the theorem.

Remark: If E and F are normed spaces, the map

$$\|\cdot\| : E \times F \longrightarrow \mathbb{R}$$

defined by $\|(e,e')\| = \|e\|_E + \|e'\|_F$ is a norm on $E \times F$ inducing the product topology. Equivalent norms on $E \times F$ are

$$(e,e') \mapsto \max\{\|e\|_E, \|e'\|_F\} \quad \text{and} \quad (e,e') \mapsto (\|e\|_E^2 + \|e'\|_F^2)^{1/2}$$

The normed vector space $E \times F$ is usually denoted by $E \oplus F$ and called the direct sum of E and F. Note that $E \oplus F$ is a Banach space iff both E and F are. These statements are readily checked.

In finite dimensions some special things occur as follows.

10. Theorem. Let E be a finite-dimensional real or com-

plex vector space. Then

(i) There is a norm on E;

(ii) all norms on E are equivalent;

(iii) all norms on E are complete.

Proof. Let $e_1, \dots, e_n$ denote a basis of E, where n is the dimension of E.

(i) A norm on E is given for example by

$$\|e\| = \sum_{i=1}^{n} |a^i| \quad \text{where} \quad e = \sum_{i=1}^{n} a^i e_i$$

Note that this norm coincides with the $\mathbb{R}^n$ or $\mathbb{C}^n$ – norm

$$|||(a^1, \dots, a^n)||| = \sum_{i=1}^{n} |a^i|$$

(ii) Let $\|.\|'$ be any other norm on E. If

$$e = \sum_{i=1}^{n} a^i e_i \quad \text{and} \quad f = \sum_{i=1}^{n} b^i e_i$$

the inequality

$$\big|\|e\|' - \|f\|'\big| \le \|e - f\|' \le \sum_{i=1}^{n} |a^i - b^i| \, \|e_i\|'$$
$$\le \max_{1 \le i \le n}\{\|e_i\|'\} \, |||(a^1, \dots, a^n) - (b^1, \dots, b^n)|||$$

shows that the map

$$(x^1, \dots, x^n) \in \mathbb{C}^n \longmapsto \Big\|\sum_{i=1}^{n} x^i e_i\Big\|' \in [0, \infty)$$

is continuous w.r.t. the $|||.|||$ – norm on $\mathbb{C}^n$. Since the set $S = \{x \in \mathbb{C}^n \mid |||x||| = 1\}$ is closed and bounded, it is compact. The restriction of this map to S is a continuous, strictly positive function, so it attains its minimum M_1 and maximum M_2 on S; that is,

$$0 < M_1 \le \Big\|\sum_{i=1}^{n} x^i e_i\Big\|' \le M_2$$

for all $(x^1, \dots, x^n) \in \mathbb{C}^n$ such that $|||(x^1, \dots, x^n)||| = 1$. Thus

$$M_1 |||(x^1,\dots,x^n)||| \le \|\sum_{i=1}^n x^i e_i\|' \le M_2 |||(x^1,\dots,x^n)|||,$$

or

$$M_1 \|e\| \le \|e\|' \le M_2 \|e\|$$

for

$$e = \sum_{i=1}^n x^i e_i.$$

Taking $M = \max\{M_2, 1/M_1\}$, Theorem .9 shows that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

(iii) It is enough to observe that

$$(x^1,\dots,x^n) \in \mathbb{C}^n \longmapsto \sum_{i=1}^n x^i e_i \in E$$

is a norm-preserving map (i.e., an isometry) between $(\mathbb{C}^n, |||\cdot|||)$ and $(E, \|\cdot\|)$.

The unit spheres for the three common norms on $\mathbb{R}^2$ are shown in Fig. .1

The foregoing proof shows that compactness of the unit sphere in a finite-dimensional space is crucial. This fact is exploited in the following Theorem.

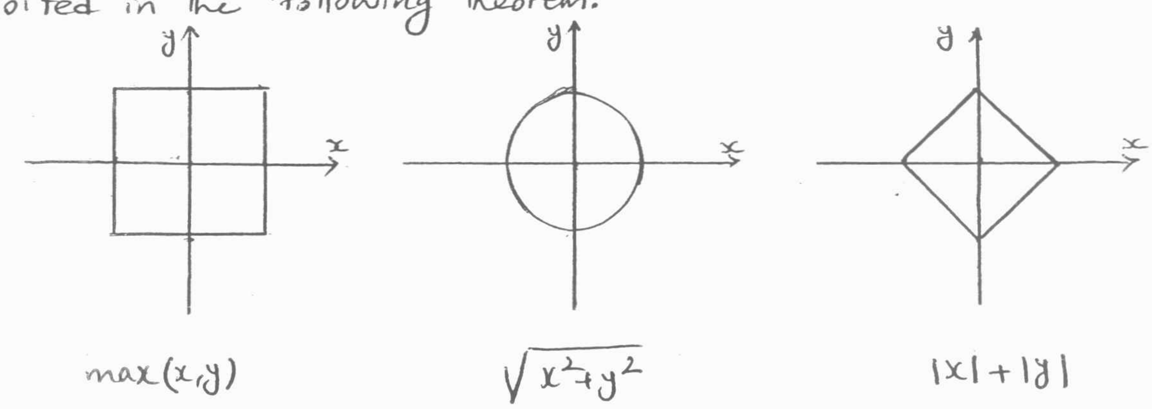

Fig. 1

A characterization of finite-dimensional spaces.

11. Theorem. A normed vector space is finite dimensional iff it is locally compact. (Lacal compactness is equivalent to the fact that the closed unit disk is compact.)

Proof. If E is finite dimensional, the previous proof of (iii) shows that E is locally compact.

Conversely, assume the closed unit disk $\overline{D}_1(0) \subset E$ is cpct. and let $\{D_{1/2}(x_i) \mid i=1,\dots,n\}$ be a finite cover with open disks of radii $1/2$. Let $F = \text{span}\{x_1,\dots,x_n\}$. Then F is finite dimensional, hence homeomorphic to $\mathbb{C}^k$ (or $\mathbb{R}^k$) for some $k \leq n$, and thus complete. Being a complete subspace of the metric space $(E, \|\cdot\|)$, it is closed. We shall prove $F=E$.

Suppose the contrary, that is, there exists $v \in E, v \notin F$. Since $F = cl(F)$, the number $d = \inf\{\|v-e\| \mid e \in F\} > 0$. Let $r > 0$ be such that $\overline{D}_r(v) \cap F \neq \emptyset$. The set $\overline{D}_r(v) \cap F$ is closed and bounded in the finite-dimensional space F, and thus is cpct. Since

$$\inf\{\|v-e\| \mid e \in F\} = \inf\{\|v-e\| \mid e \in \overline{D}_r(v) \subset F\}$$

and the continuous fn.

$$e \in \overline{D}_r(v) \cap F \longmapsto \|v-e\| \in (0,\infty)$$

attains its minimum, there exists a point $e_0 \in \overline{D}_r(v) \cap F$ s.t. $d = \|v - e_0\|$. But then there exists x_i such that

$$\left\| \frac{v-e_0}{\|v-e_0\|} - x_i \right\| < \frac{1}{2}$$

so that

$$\| v-e_0 - (\| v-e_0 \|) x_i \| < \| v-e_0 \| = \frac{d}{2}$$

since $e_0 + \| v-e_0 \| x_i \in F$, it follows that

$$\| v-e_0 - (\| v-e_0 \|) x_i \| \geq d$$

which is a contradiction.

12. Eamples.

A. Let X be a set and F a normed vector space. Let

$$B(X,F) = \{ f : X \to F \mid \sup_{x \in X} \| f(x) \| < \infty \}$$

Then $B(X,F)$ is easily seen to be a normed vector space with respect to the so-called sup-norm, $\| f \|_\infty = \sup_{x \in X} \| f(x) \|$.

We prove that if F is complete, then $B(X,F)$ is a Banach space. Let $\{f_n\}$ be a Cauchy sequence in $B(X,F)$, i.e.,

$$\| f_n - f_m \|_\infty < \varepsilon \qquad \text{for } m,n \geq N(\varepsilon)$$

Since for each $x \in X$, $\| f(x) \| \leq \| f \|_\infty$, it follows that $\{f_n(x)\}$ is a Cauchy sequence in F, whose limit we denote by $f(x)$. In the inequality $\| f_n(x) - f_m(x) \| < \varepsilon$ for all $n,m \geq N(\varepsilon)$, let $m \to \infty$ and get $\| f_n(x) - f(x) \| < \varepsilon$ for $n \geq N(\varepsilon)$, i.e., $\| f_n - f \|_\infty < \varepsilon$ for $n \geq N(\varepsilon)$. This shows that $f_n - f \in B(X,F)$, i.e., that $f \in B(X,F)$, and that $\| f_n - f \|_\infty \to 0$ as $n \to \infty$. As a particular case, we get the Banach space c_b of bounded real sequences $\{a_n\}$ with norm $\| \{a_n\} \|_\infty = \sup_n |a_n|$.

B. If X is a topological space the space

$$CB(X,F) = \{ f : X \to F \mid f \text{ is cont.}, f \in B(X,F) \}$$

is closed in $B(X,F)$. Thus if F is Banach, so is $CB(X,F)$.

In particular $C(X,F)=\{f: X\to F \mid f \text{ is continuous}\}$, for X cpct. and F Banach, is a Banach space.

In particular, $C([0,1],\mathbb{R})=\{f:[0,1]\to\mathbb{R} \mid f \text{ is continuous}\}$ is a Banach space with norm

$$\|f\|_\infty = \sup\{|f(x)| \mid x\in[0,1]\}$$

As in the case of vector spaces, quotient spaces of normed vector spaces play a fundamental role.

13. Theorem. Let E be a normed vector space, F a closed subspace, E/F the quotient vector space and $\pi: E\to E/F$ the canonical projection defined by $\pi(e)=[e]=e+F\in E/F$. Then

(i) The mapping $\|\cdot\|: E/F\to\mathbb{R}$,

$$\|[e]\| = \inf\{\|e+v\| \mid v\in F\}$$

defines a norm on E/F.

(ii) π is continuous and in fact the topology on E/F defined by the norm coincides with the quotient topology. In particular π is open.

(iii) If E is a Banach space, so is E/F.

Proof. (i) Clearly $\|[e]\|\geqslant 0$ for all $[e]\in E/F$ and

$$\|[0]\| = \inf\{\|v\| \mid v\in F\} = 0$$

If $\|[e]\|=0$, then there exists a sequence $\{v_n\}\subset F$ such that

$$\lim_{n\to\infty}\|e+v_n\| = 0$$

Thus $\lim_{n\to\infty} v_n = -e$ and since F is closed $e \in F$; i.e. $[e]=0$. Thus (N1) is verified and the necessity of having F closed becomes apparent. (N2) and (N3) are straightforward verifications.

(ii) Since $\|[e]\| \leq \|e\|$, it is obvious that $\lim_{n\to\infty} e_n = e$ implies $\lim_{n\to\infty} \pi(e_n) = \lim_{n\to\infty} [e_n] = [e]$ and hence π is continuous.

Translation by a fixed vector is a homeomorphism. Thus to show that the topology of E/F is the quotient topology, it suffices to show that if $[0] \in U$ and $\pi^{-1}(U)$ is a neighborhood of zero in E, then U is a nhbd. of $[0]$ in E/F. Since $\pi^{-1}(U)$ is a nhbd. of zero in E, there exists a disk $D_r(0)$ s.t. $D_r(0) \subset \pi^{-1}(U)$. But then $\pi(D_r(0)) \subset U$ and

$$\pi(D_r(0)) = \{[e] \mid e \in D_r(0)\} = \{[e] \mid \|[e]\| < r\}$$

so that U is a nhbd. of $[0]$ in E/F.

(iii) Let $\{[e_n]\}$ be a Cauchy sequence in E/F. We may assume without loss of generality that $\|[e_n]-[e_{n+1}]\| \leq 1/2^n$. Inductively we can find $e_n' \in [e_n]$ such that $\|e_n' - e_{n+1}'\| < \frac{1}{2^n}$. Thus $\{e_n'\}$ is Cauchy in E so it converges to, say, $e \in E$. Continuity of π implies that $\lim_{n\to\infty} [e_n] = [e]$.

Note: The codimension of F in E is defined to be the dimension of E/F. We say F is of finite codimension if E/F is finite dimensional.

Split Subspaces

14 Definition: The closed subspace F of the Banach sp. E is said to be split, or complemented, if there is a closed subspace $G \subset E$ such that $E = F \oplus G$.

Remark: Definition 14 implicitly asks that the topology of E coincide with the product topology of $F \oplus G$. We shall show, in next section, that this topological condition can be dropped; i.e., F is split iff E is the algebraic direct sum of F and the closed subspace G.

We note that if $E = F \oplus G$ then G is isomorphic to E/F. However, F need not split for E/F to be a Banach space, as we proved in 13.

In finite-dimensional spaces, any subspace is closed and splits; however, in infinite dimensions this is false. In next section, we give some general criteria useful in nonlinear analysis for a subspace to be split. But the simplest situation occurs in Hilbert spaces.

. 15 Theorem. If E is a Hilbert space and F a closed subspace then $E = F \oplus F^{\perp}$. Thus every closed subspace of a Hilbert space splits.

The proof of this theorem is done in three steps, the first two being important results in their own right.

.16 Lemma. (Existence and uniqueness of minimal norm elements in closed convex sets). If C is a closed, convex set (i.e., $x, y \in C$ and $0 \leq t \leq 1$ implies $tx+(1-t)y \in C$) in E, then there exists a unique $e_0 \in C$ such that $\|e_0\| = \inf\{\|e\| \mid e \in C\}$.

Proof. Let $\sqrt{d} = \inf\{\|e\| \mid e \in C\}$. Then there exists a sequence $\{e_n\}$ satisfying $d \leq \|e_n\|^2 < d + \frac{1}{n}$; hence $\|e_n\|^2 \to d$. Since $(e_n+e_m)/2 \in C$, C being convex, it follows that

$$\|(e_n+e_m)/2\|^2 \geq d.$$

By the parallelogram law

$$\begin{aligned}\left\|\frac{(e_n-e_m)}{2}\right\|^2 &= 2\left\|\frac{e_n}{2}\right\|^2 + 2\left\|\frac{e_m}{2}\right\|^2 - \left\|\frac{(e_n+e_m)}{2}\right\|^2 \\ &< \frac{d}{2} + \frac{1}{2n} + \frac{d}{2} + \frac{1}{2m} - d \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\end{aligned}$$

That is, $\{e_n\}$ is a Cauchy sequence in E. Let $\lim_{n\to\infty} e_n = e_0$. Continuity of the norm implies that

$$\sqrt{d} = \lim_{n\to\infty} \|e_n\| = \|e_0\|$$

and so the existence of an element of minimum norm in C is proved.

Finally, if e_0' is such that $\|e_0\| = \|e_0'\| = \sqrt{d}$, the parallelogram law implies

$$\begin{aligned}\left\|\frac{e_0-e_0'}{2}\right\|^2 &= 2\left\|\frac{e_0}{2}\right\|^2 + 2\left\|\frac{e_0'}{2}\right\|^2 - \left\|\frac{e_0+e_0'}{2}\right\|^2 \\ &\leq \frac{d}{2} + \frac{d}{2} - d = 0\end{aligned}$$

i.e. $e_0 = e_0'$

17. Lemma. Let $F \subset E$, $F \neq E$ be a closed subspace of E. Then there exists a nonzero element $e_0 \in E$ s.t. $e_0 \perp F$.

Proof. Let $e \in E$, $e \notin F$. The set

$$e - F = \{e - v \mid v \in F\}$$

is convex and closed, so by the previous lemma it contains a unique element $e_0 = e - v_0 \in e - F$ of minimum norm. Since F is closed and $e \notin F$, it follows that $e_0 \neq 0$. We shall prove that $e_0 \perp F$.

Since e_0 is of minimal norm in $e - F$, for any $v \in F$ and $\lambda \in \mathbb{C}$ $(\mathbb{R})$, we have

$$\|e_0\| = \|e - v_0\| \leq \|e - v_0 + \lambda v\| = \|e_0 + \lambda v\|$$

i.e.,

$$2 \operatorname{Re}(\lambda \langle v, e_0 \rangle) + |\lambda|^2 \|v\|^2 \geqslant 0$$

If $\lambda = a \langle e_0, v \rangle$, $a \in \mathbb{R}$, $a \neq 0$, this becomes

$$a |\langle v, e_0 \rangle|^2 (2 + a \|v\|^2) \geqslant 0$$

for all $v \in F$, $a \in \mathbb{R}$, $a \neq 0$. This forces $\langle v, e_0 \rangle = 0$ for all $v \in F$, since if $-2/\|v\|^2 < a < 0$, the preceding expression is negative.

Proof of Theorem 15. First of all, it is easy to see that $F^\perp$ is closed (Exercise.). We now show that $F \oplus F^\perp$ is a closed subspace. If

$$\{e_n + e_n'\} \subset F \oplus F^\perp, \ \{e_n\} \subset F, \ \{e_n'\} \subset F^\perp$$

The relation

$$\|(e_n+e_n')-(e_m+e_m')\|^2=\|e_n-e_n'\|^2+\|e_m-e_m'\|^2$$

shows that $\{e_n+e_n'\}$ is Cauchy iff both $\{e_n\}\subset F$ and $\{e_n'\}\subset F^\perp$ are Cauchy. Thus if $\{e_n+e_n'\}$ converges, then there exist $e\in F$, $e'\in F^\perp$ such that $\lim_{n\to\infty} e_n=e$, $\lim_{n\to\infty} e_n'=e'$. Thus

$$\lim_{n\to\infty}(e_n+e_n')=e+e'\in F\oplus F^\perp$$

If $F\oplus F^\perp\neq E$, then by the previous lemma there exists $e_0\in E$, $e_0\notin F\oplus F^\perp$, $e_0\neq 0$, $e_0\perp(F\oplus F^\perp)$. Hence $e_0\in F^\perp$ and $e_0\in F$ so that

$$\langle e_0,e_0\rangle=\|e_0\|^2=0$$

i.e., $e_0=0$, a contradiction.

Linear and Multilinear Mappings

This section deals with various aspects of linear and multilinear maps between Banach spaces. We begin with a study of continuity and go on to study spaces of continuous linear and multilinear maps and some related fundamental theorems of linear analysis.

18. Theorem. Let $A : E \to F$ be a linear map of normed spaces. Then A is continuous iff there is a constant $M > 0$ s.t.

$$\|Ae\|_F \leq M \|e\|_E \quad \text{for all } e \in E.$$

Proof. Continuity of A at $e_0 \in E$ means that for any $r > 0$, there exists $\rho > 0$ such that

$$A(e_0 + \bar{D}_\rho(0_E)) \subset Ae_0 + \bar{D}_r(0_F)$$

(0_E denote the zero element in E and $\bar{D}_1(0_E)$ denotes the closed disk of radius 1 centered at the origin in E). Since A is linear, this is equivalent to:

$$\text{If } \|e\| \leq \rho \quad \text{then } \|Ae\| \leq r$$

If $M = \rho r$, continuity of A is thus equivalent to the following:

$$\|e\|_E \leq 1 \quad \text{implies} \quad \|Ae\|_F \leq M$$

which in turn is the same as: there exists $M > 0$ such that $\|Ae\|_F \leq M \|e\|_E$, which is seen by taking $e/\|e\|_E$ in the preceding implication.

Because of this theorem one says that a continuous linear map is bounded.

19 . Theorem. If E is finite dimensional and $A: E \to F$ is linear, then A is continuous.

proof. Let $e_1, \ldots, e_n$ be a basis for E. Letting

$$M_1 = \max\{\|Ae_1\|, \ldots, \|Ae_n\|\}$$

and $e = a^1 e_1 + \cdots + a^n e_n$, we see that

$$\begin{aligned} \|Ae\| &= \|a^1 Ae_1 + \cdots + a^n Ae_n\| \\ &\leq |a^1|\,\|Ae_1\| + \cdots + |a^n|\,\|Ae_n\| \\ &\leq M_1(|a^1| + \cdots + |a^n|) \end{aligned}$$

since E is finite dimensional, all norms on it are equivalent. Since $|||e||| = \sum_{i=1}^{n} |a^i|$ is a norm, it follows that $|||e||| \leq C\|e\|$ for a constant C. Let $M = M_1 C$ and use Thm. (18)

20 . Definition. If E and F are normed spaces and $A: E \to F$ is a continuous linear map, let the operator norm of A be defined by

$$\|A\| = \sup\left\{ \frac{\|Ae\|}{\|e\|} \;\middle|\; e \in E,\ e \neq 0 \right\}$$

(which is finite by Thm. (18). Let $L(E,F)$ denote the space of all continuous linear maps of E to F. If $F = \mathbb{C}$ (resp. $\mathbb{R}$), then $L(E, \mathbb{C})$ (resp. $L(E, \mathbb{R})$) is denoted by E^* and is called the complex (resp. real) dual space of E. (It will always be clear from the context whether $L(E,F)$ or E^* means the real or complex linear maps or dual space; in most of the work later in this course it will mean the real case).

A straightforward verification gives the following equivalent definitions of $\|A\|$:

$$\begin{aligned}\|A\| &= \inf\{M>0 \mid \|Ae\| \le M\|e\| \text{ for all } e\in E\}\\ &= \sup\{\|Ae\| \mid \|e\|\le 1\}\\ &= \sup\{\|Ae\| \mid \|e\| = 1\}\end{aligned}$$

In particular, $\|Ae\| \le \|A\|\,\|e\|$.

If $A\in L(E,F)$ and $B\in L(F,G)$, where E, F and G are normed spaces, then

$$\|(B\circ A)(e)\| = \|B(A(e))\| \le \|B\|\,\|Ae\| \le \|B\|\,\|A\|\,\|e\|$$

which shows that

$$\|B\circ A\| \le \|B\|\,\|A\|$$

Equality does not hold in general. A simple example is with $E=F=G=\mathbb{R}^2$, $A(x,y)=(x,0)$, and $B(x,y)=(0,y)$ so that $B\circ A=0$ and $\|A\| = \|B\| = 1$

21. Theorem. $L(E,F)$ with the norm just defined is a normed space. It is a Banach space if F is.

proof. Clearly $\|A\|\geqslant 0$ and $|0|=0$. If $\|A\|=0$ then for any $e\in E$, $\|Ae\| \le \|A\|\,\|e\| = 0$, so that $A=0$ and thus (N1) is verified. (N2) and (N3) are also straightforward to check.

Now let F be a Banach space and $\{A_n\}\subset L(E,F)$

be a Cauchy sequence. Since

$$\|A_n e - A_m e\| \leq \|A_n - A_m\| \, \|e\| \quad \text{for all } e \in E$$

the sequence $\{A_n e\}$ is Cauchy in F and hence is convergent. Let $Ae = \lim_{n\to\infty} A_n e$. This defines a map $A: E \to F$, which is evidently linear. It remains to be shown that A is continuous and $\|A_n - A\| \to 0$.

If $\varepsilon > 0$ is given, there exists a natural number $N(\varepsilon)$ such that for all $m, n \geq N(\varepsilon)$ we have $\|A_n - A_m\| < \varepsilon$. If $\|e\| \leq 1$, this implies $\|A_n e - A_m e\| < \varepsilon$ and now letting $m \to \infty$, it follows that $\|A_n e - Ae\| \leq \varepsilon$ for all e with $\|e\| \leq 1$. Thus $A_n - A \in L(E,F)$, hence $A \in L(E,F)$ and $\|A_n - A\| \leq \varepsilon$ for all $n \geq N(\varepsilon)$, i.e., $\|A_n - A\| \to 0$.

Note: If a sequence $\{A_n\}$ converges to A in $L(E,F)$ in the sense that $\|A_n - A\| \to 0$, i.e., if $A_n \to A$ in the norm topology, we say $A_n \to A$ in norm, this phrase is necessary since other topologies on $L(E,F)$ are possible. For example, we say that $A_n \to A$ strongly, if $A_n e \to Ae$ for each $e \in E$. Since

$$\|A_n e - Ae\| \leq \|A_n - A\| \, \|e\|$$

norm convergence implies strong convergence.

The converse is false as the following example shows. Let $E = \ell^2(\mathbb{R}) = \{\{a_n\} \mid \sum_{n=1}^{\infty} a_n^2 < \infty\}$ with inner product

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_{n=1}^{\infty} a_n b_n$$

Let

$$e_n = (\underbrace{0,\dots,0}_{(n-1)\text{ times}}, 1, 0, \dots) \in E \quad , \quad F = \mathbb{R}$$

and $A_n = \langle e_n, \cdot \rangle \in L(E,F)$, The sequence $\{A_n\}$ is not Cauchy in norm since $\|A_n - A_m\| = \sqrt{2}$, but if $e = \{a_m\}$, $A_n(e) = \langle e_n, e \rangle = a_n \to 0$ i.e., $A_n \to 0$ strongly. If both E and F are finite dimensional strong convergence implies norm convergence. (To see this; choose a basis $e_1, \dots, e_n$ of E and note that strong convergence is equivalent to $A_k e_i \to A e_i$ as $k \to \infty$ for $i = 1, \dots, n$. Hence $\max_i \|Ae_i\| = |||A|||$ is a norm yielding strong convergence. But all norms are equivalent in finite dimensions).

Dual Spaces

Recall from elementary linear algebra that the dual-space of $\mathbb{C}^n$ ($\mathbb{R}^n$) may be identified with itself. For general Banach spaces this is no longer true. An important result-asserts that this fact still holds for Hilbert space.

22 . Riesz Representation Theorem. Let E be a real (resp. complex) Hilbert space. The map $e \mapsto \langle \cdot, e \rangle$ is a linear (resp. antilinear : A map $A: E \to F$ between complex vector spaces is called antilinear if $A(e+e') = Ae + Ae'$, and $A(\alpha e) = \bar{\alpha} Ae$.) norm-preserving isomorphism of E with E^*; for short, $E \approx E^*$.

Proof: Let $f_e = \langle \cdot, e \rangle$. Then $\|f_e\| = \|e\|$ and thus $f_e \in E^*$. The map $A: E \to E^*$, $Ae = f_e$ is clearly linear (resp. antilinear), norm preserving, and thus injective. It remains to prove surjectivity.

Let $f \in E^*$ and $\ker f = \{e \in E \mid f(e) = 0\}$. $\ker f$ is a closed subspace in E. If $\ker f = E$, then $f = 0$ and $f = A(0)$ and there is nothing to prove. If $\ker f \neq E$, then by Lemma (17) there exists $e \neq 0$, such that $e \perp \ker f$. Then we claim that

$$f = A(f(e)e/\|e\|^2)$$

Indeed, any $v \in E$ can be written as

$$v = v - \frac{f(v)}{f(e)}e + \frac{f(v)}{f(e)}e \quad \text{and} \quad v - \frac{f(v)}{f(e)}e \in \ker f.$$

Remarks. In a real Hilbert space E every continuous linear function $\ell : E \longrightarrow \mathbb{R}$ can be written

$$\ell(e) = \langle e, e_0 \rangle$$

for some $e_0 \in E$ and $\|\ell\| = \|e_0\|$.

In a general Banach space E we do not have such a concrete realization of E^*. However one should not always attempt to identify E and E^*, even in finite dimensions. In fact, distinguishing these spaces is fundamental in tensor analysis.

We have a canonical map $i : E \to E^{**}$ defined by

$$i(e)(\ell) = \ell(e)$$

(Pause and look again at this strange but natural formula: $i(e) \in E^{**} = (E^*)^*$, so $i(e)$ is applied to the element $\ell \in E^*$).

It is easy to check that i is norm preserving. One calls E reflexive when i is onto. Hilbert spaces are reflexive, as is seen by using Theorem (22).

Next we shall discuss integration of vector valued functions. We shall require the following.

23. Lemma (Linear Extension theorem). Let E, F and G be normed vector spaces where

(i) $F \subset E$

(ii) G is a Banach space and

(iii) $T \in L(F,G)$.

Then the closure $cl(F)$ of F is a normed vector subspace of E and T can be unique extended to a map $\bar{T} \in L(cl(F), G)$. Moreover $\|T\| = \|\bar{T}\|$.

Proof. The fact that $cl(F)$ is a linear subspace of E is easily checked. Note that if $\bar{T}$ exists it is unique by continuity. Let us prove the existence of $\bar{T}$. If $e \in cl(F)$, we can write $e = \lim_{n\to\infty} e_n$, where $e_n \in F$, so that

$$\|Te_n - Te_m\| \le \|T\| \, \|e_n - e_m\|$$

which shows that the sequence $\{Te_n\}$ is Cauchy in the Banach space G. Let $\bar{T}e = \lim_{n\to\infty} Te_n$. This limit is independent of the sequence $\{e_n\}$, for if $e = \lim_{n\to\infty} e_n'$, then

$$\|Te_n - Te_n'\| \le \|T\| \left(\|e_n - e\| + \|e - e_n'\|\right)$$

which proves that $\lim_{n\to\infty}(Te_n) = \lim_{n\to\infty}(Te_n')$. It is simple to check the linearity of $\bar{T}$. Since $Te = \bar{T}e$ for $e \in F$ (because $e = \lim_{n\to\infty} e$), $\bar{T}$ is an extension of T. Finally

$$\|\bar{T}e\| = \|\lim_{n\to\infty}(Te_n)\| = \lim_{n\to\infty}\|Te_n\| \le \|T\| \lim_{n\to\infty}\|e_n\| = \|T\| \, \|e\|$$

shows that $\bar{T} \in L(cl(F), G)$ and $\|\bar{T}\| \leq \|T\|$. The inequality $\|T\| \leq \|\bar{T}\|$ is obvious since $\bar{T}$ extends T.

Remark. As an application of this lemma we define a Banach space valued integral that will be of use later on.

Fix the closed interval $[a,b] \subset \mathbb{R}$ and the Banach space E. A map $f:[a,b] \longrightarrow E$, is called a step fn., if there exists a partition $a = t_0 < t_1 < t_2 < \dots < t_n = b$ such that f is constant on each interval $[t_i, t_{i+1})$. Using the standard notion of a refinement of a partition it is clear that the sum of two step fns, and the scalar multiples of step fns, are also step fns. Thus the set $S([a,b],E)$ of step functions is a vector subspace of $B([a,b],E)$, the Banach space of all bounded functions. The integral of a step fn. f is defined by

$$\int_a^b f = \sum_{i=0}^{n} (t_{i+1} - t_i) f(t_i)$$

It is easily verified that this definition is independent of the partition. Also note that

$$\left\| \int_a^b f \right\| \leq \int_a^b \|f\| \leq (b-a) \|f\|_\infty$$

where $\|f\|_\infty = \sup_{a \leq t \leq b} |f(t)|$, that is,

$$\int_a^b : S([a,b],E) \longrightarrow E$$

is continuous and linear. By the linear extension theorem, it extends to a continuous linear map

$$\int_a^b \in L(cl(S([a,b],E)), E).$$

24. Definition. $\int_a^b$ is called the Cauchy-Bochner integral.

Note. Note that
$$\left\| \int_a^b f \right\| \leq \int_a^b \|f\| \leq (b-a)\,\|f\|_\infty .$$
the usual properties of the integral such as
$$\int_a^b f = \int_a^c f + \int_c^b f \qquad \text{and} \qquad \int_a^b f = -\int_b^a f$$
are easily verified since they clearly hold for step functions.

The space $cl(\mathcal{S}([a,b],E))$ contains enough interesting functions for our purposes, namely
$$C_0([a,b],E) \subset cl(\mathcal{S}([a,b],E)) \subset B([a,b],E)$$
The first inclusion is proved in the following way. Since $[a,b]$ is compact, each $f \in C_0([a,b],E)$ is uniformly continuous. For $\varepsilon > 0$, let $\delta > 0$ be given by uniform continuity of f for $\varepsilon/2$. Then take a partition $a = t_0 < \dots < t_n = b$ such that $|t_{i+1} - t_i| < \delta$ and define a step fn. g by $g|_{[t_i, t_{i+1}]} = f(t_i)$. Then the ε-disk $D_\varepsilon(f)$ in $B([a,b],E)$ contains g.

Finally, note that if E and F are Banach spaces, $A \in L(E,F)$ and $f \in cl(\mathcal{S}([a,b],E))$, we have $A \circ f \in cl(\mathcal{S}([a,b],F))$ since $\|A\circ f_n - A \circ f\| \leq \|A\| \, \|f_n - f\|_\infty$ for f_n. step functions in E. Moreover $\int_a^b A\circ f = A\left(\int_a^b f\right)$ since this relation is obtained as the limit of the same relation for step fns.

Note. we note that the Riemann integral exists also for functions outside of $cl(\mathcal{S}([a,b],\mathbb{R}))$.

Next we turn to multilinear mappings. If $E_1, \dots, E_k$ and F are linear spaces, a map

$$A: E_1 \times \dots \times E_k \longrightarrow F$$

is called k-multilinear if $A(e_1, \dots, e_k)$ is linear in each argument separately. Linearity in the first argument means that

$$A(\lambda e_1 + \mu \tilde{e}_1, e_2, \dots, e_k) = \lambda A(e_1, e_2, \dots, e_k) + \mu A(\tilde{e}_1, e_2, \dots, e_k)$$

We shall study multilinear mappings in detail in our study of tensors, numerical Analysis, Partial differential equations and etc. we shall require a few facts about them for that purpose.

25. Definition. The space of continuous k-multilinear maps of $E_1, \dots, E_k$ to F is denoted $L(E_1, \dots, E_k; F)$. If $E_i = E$, $1 \le i \le k$, This space is denoted $L^k(E, F)$.

Note. As in (18), a k-multilinear map A is continuous iff There is an $M > 0$ such that

$$\| A(e_1, \dots, e_k) \| \le M \|e_1\| \dots \|e_k\|$$

for all $e_i \in E_i$, $1 \le i \le k$. We set

$$\|A\| = \sup \left\{ \frac{\|A(e_1, \dots, e_k)\|}{\|e_1\| \dots \|e_k\|} \;\middle|\; e_1, \dots, e_k \neq 0 \right\}$$

which makes $L(E_1, \dots, E_k; F)$ into a normed space that is complete if F is.

Again $\|A\|$ can also be defined as

$$\|A\| = \inf \{ M > 0 \mid \|A(e_1, \dots, e_n)\| \le M \|e_1\| \dots \|e_n\| \}$$
$$= \sup \{ \|A(e_1, \dots, e_n)\| \mid \|e_1\| \le 1, \dots, \|e_n\| \le 1 \}$$

$$= \sup\{\|A(e_1,\dots,e_n)\| \mid \|e_1\| = \dots = \|e_n\| = 1\}$$

26. Theorem. There is a (natural) norm-preserving isomorphism

$$L(E_1, L(E_2,\dots,E_k;F)) \approx L(E_1,\dots,E_k;F)$$

Proof. For $A \in L(E_1, L(E_2,\dots,E_k;F))$ we define $\tilde{A} \in L(E_1,\dots,E_k;F)$ by

$$\tilde{A}(e_1,\dots,e_k) = A(e_1)(e_2,\dots,e_k)$$

The association $A \mapsto \tilde{A}$ is clearly linear and $\|\tilde{A}\| = \|A\|$.

Remarks. (i) In a similar way, we can identify $L(\mathbb{R},F)$ (or $L(\mathbb{C},F)$ if F is complex) with F: to $A \in L(\mathbb{R},F)$ we associate $A(1) \in F$; again $\|A\| = \|A(1)\|$.

(ii) As a special case .9 note that

$$L(E,E^*) \approx L^2(E,\mathbb{R}) \quad (\text{or } L^2(E,\mathbb{C}), \text{ if } E \text{ is complex})$$

This isomorphism will be useful when we consider second derivatives.

(iii) We shall need a few facts about the permutation group on k elements. The information we cite is obtainable from virtually any elementary algebra book. The permutation group on k elements, denoted S_k, consists of all bijections

$$\sigma: \{1,\dots,k\} \to \{1,\dots,k\}$$

together with the structure of a group under composition. Clearly

S_k has order $k!$. Letting $(\mathbb{R},\times)$ denote $\mathbb{R}\setminus\{0\}$ with the multiplicative group structure, we have a homomorphism sign: $S_k \to (\mathbb{R},\times)$. That is, for $\sigma,\tau\in S_k$, $\operatorname{sign}(\sigma\circ\tau)=(\operatorname{sign}\sigma)(\operatorname{sign}\tau)$. The image of sign is the subgroup $\{-1,1\}$, while its kernel consists of the subgroup of even permutations. Thus, a permutation σ is even when $\operatorname{sign}\sigma=+1$ and is odd when $\operatorname{sign}\sigma=-1$.

A transposition is a permutation that swaps two elements of $\{1,\dots,k\}$, leaving the remainder fixed. An even (odd)-permutation can be written as the product of an even (odd) number of transpositions.

(iv) The group S_k acts on $L^k(E,F)$; i.e., each $\sigma\in S_k$ defines a map $\sigma: L^k(E,F)\to L^k(E,F)$ by

$$(\sigma A)(e_1,\dots,e_k)=A(e_{\sigma(1)},\dots,e_{\sigma(k)})$$

(respectively $\sigma A=(\operatorname{sign}\sigma)A$).

27. Definition. Let E and F be normed vector spaces. Let $L^k_s(E,F)$ and $L^k_a(E,F)$ denote the subspaces of symmetric and antisymmetric elements of $L^k(E,F)$. Write $S^0(E,F)=L^0(E,F)=F$ and $S^k(E,F)=\{p:E\to F \mid p(e)=A(e,\dots,e),\ A\in L^k(E,F)\}$. We call $S^k(E,F)$ the space of homogeneous polynomials of degree k from E to F.

Remark. Note that $L^k_s(E,F)$ and $L^k_a(E,F)$ are closed in $L^k(E,F)$; thus if F is a Banach space, so are $L^k_s(E,F)$ and

$L^k_a(E,F)$. The antisymmetric maps $L^k_a(E,F)$ will be studied in detail in Integration on Manifolds.

Homogeneous Polynomials

28. Theorem. (i) $S^k(E,F)$ is a normed vector space with respect to the following norm:

$$\begin{aligned}\|f\| &= \inf\{M>0 \mid \|f(e)\| \le M\|e\|^k\} \\ &= \sup\{\|f(e)\| \mid \|e\| \le 1\} \\ &= \sup\{\|f(e)\| \mid \|e\| = 1\}\end{aligned}$$

It is complete if F is.

(ii) If $f \in S^k(E,F)$ and $g \in S^\ell(F,G)$, then $g\circ f \in S^{k\ell}(E,G)$ and $\|g\circ f\| \le \|g\|\,\|f\|$.

(iii) (Polarization). The mapping $\hat{\ }: L^k(E,F) \longrightarrow S^k(E,F)$ defined by $\hat{A}(e) = A(e,\dots,e)$ restricted to $L^k_s(E,F)$ has an inverse $\check{\ }: S^k(E,F) \longrightarrow L^k_s(E,F)$ given by

$$\check{f}(e_1,\dots,e_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\Big|_{t=0} f(t_1e_1+\dots+t_ke_k)$$

(note that $f(t_1e_1+\dots+t_ke_k)$ is polynomial in $t_1,\dots,t_k$ so there is no problem in understanding what the derivatives on the right hand side mean).

(iv) For $A \in L^k(E,F)$, $\|\hat{A}\| \le \|A\| \le (k^k/k!)\,\|\hat{A}\|$, which implies that the maps $\hat{\ }$, $\check{\ }$ are continuous.

proof. (i) and (ii) are proved exactly as for $L(E,F) = S^1(E,F)$.

(iii) For $A \in L^k_s(E,F)$ we have

$$\hat{A}(t_1e_1+\cdots+t_ke_k)$$
$$= \sum_{a_1+\cdots+a_j=k} \frac{k!}{a_1!\cdots a_j!} t_1^{a_1}\cdots t_j^{a_j} A(\underbrace{e_1,\dots,e_1}_{a_1},\dots,\underbrace{e_j,\dots,e_j}_{a_j})$$

and

$$\frac{\partial^j}{\partial t_1\cdots\partial t_j}\Big|_{t=0} t_1^{a_1}\cdots t_j^{a_j} = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{if } k\neq j\end{cases}$$

It follows that

$$A(e_1,\dots,e_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\hat{A}(t_1e_1+\cdots+t_ke_k)$$

and for $j\neq k$,

$$\frac{\partial^j}{\partial t_1\cdots\partial t_j}\hat{A}(t_1e_1+\cdots+t_ke_k) = 0$$

This means that

$$(\hat{A})^{\vee} = A \quad \text{for any } A\in L_s^k(E,F)$$

Conversely, if $f\in S^k(E,F)$, then

$$(\hat{\check{f}})(e) = \check{f}(e,\dots,e) = \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\Big|_{t=0} f(t_1e+\cdots+t_ke)$$
$$= \frac{1}{k!}\frac{\partial^k}{\partial t_1\cdots\partial t_k}\Big|_{t=0}(t_1+\cdots+t_k)^k f(e) = f(e)$$

iv) $\|\hat{A}(e)\| = \|A(e,\dots,e)\| \leq \|A\|\,\|e\|^k$, so $\|\hat{A}\|\leq\|A\|$.

To prove the other inequality, note that if $A\in L_s^k(E,F)$, then

$$A(e_1,\dots,e_k) = \frac{1}{k!\,2^k}\sum \epsilon_1\cdots\epsilon_k\,\hat{A}(\epsilon_1e_1+\cdots+\epsilon_ke_k)$$

where the sum is taken over all the 2^k possibilities $\epsilon_1=\pm1,\dots,\epsilon_k=\pm1$. Put $\|e_1\|=\cdots=\|e_k\|=1$ and get

$$\|\hat{A}(\epsilon_1e_1+\cdots+\epsilon_ke_k)\| \leq \|\hat{A}\|\,\|\epsilon_1e_1+\cdots+\epsilon_ke_k\|^k$$
$$\leq \|\hat{A}\|\,(|\epsilon_1|\,\|e_1\|+\cdots+|\epsilon_k|\,\|e_k\|)^k$$
$$= \|\hat{A}\|\,k^k$$

whence

$$\|A(e_1,\dots,e_k)\| \leq \frac{k^k}{k!}\|\hat{A}\|, \text{ i.e. } \|A\|\leq\frac{k^k}{k!}\|\hat{A}\|.$$

Remark: Let $E = \mathbb{R}^n$, $F = \mathbb{R}$, and $e_1, \dots, e_n$ be the standard basis in $\mathbb{R}^n$. For $f \in S^k(\mathbb{R}^n, \mathbb{R})$, set

$$c_{a_1 \cdots a_n} = \check{f}(\underbrace{e_1, \dots, e_1}_{a_1}, \dots, \underbrace{e_n, \dots, e_n}_{a_n})$$

If $e = t_1 e_1 + \dots + t_n e_n$, the proof of (iii) shows that

$$f(e) = \check{f}(e, \dots, e) = \sum_{a_1 + \dots + a_n = k} c_{a_1 \cdots a_n} t_1^{a_1} \cdots t_n^{a_n}$$

i.e., f is a homegeneous polynomial of degree k in $t_1, \dots, t_n$ in the usual algebraic sense.

The constant $k^k/k!$ in (iv) is the best possible, as the following example shows. Write elements of $\mathbb{R}^k$ as $x = (x^1, \dots, x^k)$ and introduce the norm $|||(x^1, \dots, x^k)||| = |x^1| + \dots + |x^k|$. Define $A \in L_s^k(\mathbb{R}^k, \mathbb{R})$ by

$$A(x_1, \dots, x_k) = \frac{1}{k!} \sum x^1_{i_1} \cdots x^k_{i_k},$$

where $x_i = (x_i^1, \dots, x_i^k) \in \mathbb{R}^k$ and the sum is taken over all permutations of $\{1, \dots, k\}$. It is easily verified that

$$\|\hat{A}\| = 1/k^k \qquad \text{i.e.,} \quad \|A\| = (k^k/k!)\,\|\hat{A}\|$$

Thus, except for $k=1$, the isomorphism $\hat{}$ is not norm preserving (This is a source of annoyance in the theory of formal-power series and infinite-dimensional holomorphic mappings.

The Three Pillars of Linear Analysis

The three fundamental theorems of linear analysis are the Hahn–Banach theorem, the open mapping theorem and the uniform boundedness principle. We begin with the first pillar.

29 . **Hahn–Banach Theorem.** Let E be a real or complex vector space, $\|\cdot\|: E \to \mathbb{R}$ a seminorm and $F \subset E$ a subspace. If $f \in F^*$ satisfies $|f(e)| \le \|e\|$ for all $e \in F$, then there exists a linear map $\tilde{f}: E \to \mathbb{R}$ (or $\mathbb{C}$) such that $\tilde{f}|F = f$ and $|\tilde{f}(e)| \le \|e\|$ for all $e \in E$.

Proof: Real Case. First we show that $f \in F^*$ can be extended with the given property to $F \oplus \operatorname{span}\{e_0\}$, for a given $e_0 \notin F$. For $e_1, e_2 \in F$ we have

$$f(e_1) + f(e_2) = f(e_1 + e_2) \le \|e_1 + e_2\| \le \|e_1 + e_0\| + \|e_2 - e_0\|$$

so that

$$f(e_2) - \|e_2 - e_0\| \le \|e_1 + e_0\| - f(e_1)$$

and hence

$$\sup\{f(e_2) - \|e_2 - e_0\| \mid e_2 \in F\} \le \inf\{\|e_1 + e_0\| - f(e_1) \mid e_1 \in F\}$$

Let $a \in \mathbb{R}$ be any number between the sup and inf in the preceding expression and define $\tilde{f}: F \oplus \operatorname{span}\{e_0\} \to \mathbb{R}$ by

$$\tilde{f}(e + te_0) = f(e) + ta$$

It is clear that $\tilde{f}$ is linear and $\tilde{f}|F = f$. To show that

$$|\tilde{f}(e + te_0)| \le \|e + te_0\|$$

note that by the definition of a,

$$f(e_2) - \|e_2 - e_0\| \le a \le \|e_1 + e_0\| - f(e_1)$$

so that multiplying the second inequality by $t \geq 0$ and the first by $t < 0$ we get the desired result.

Second, one verifies that the set $\mathcal{S} = \{(G, g) \mid F \subset G \subset E$, G is a subspace of E, $g \in G^*$, $g|F = f$ and $|g(e)| \le \|e\|$ for all $e \in G\}$

is inductively ordered with respect to the ordering

$$(G_1, g_1) \leq (G_2, g_2) \text{ iff } G_1 \subset G_2 \text{ , } g_2|G_1 = g_1$$

Thus by Zorn's lemma there exists a maximal element (F_0, f_0) of $\mathcal{S}$.

Third, using the first step and the maximality of (F_0, f_0) one concludes that $F_0 = E$.

Complex Case. Let $f = \mathrm{Re} f + i \,\mathrm{Im} f$ and note that complex linearity implies $(\mathrm{Im} f)(e) = -(\mathrm{Re} f)(ie)$ for all $e \in F$. By the real-case, $\mathrm{Re} f$ extends to a real linear continuous map $(\mathrm{Re} f)^{\sim} : E \to \mathbb{R}$ such that $|(\mathrm{Re} f)^{\sim}(e)| \leq \|e\|$ for all $e \in E$. Define $\tilde{f} : E \to \mathbb{C}$ by $\tilde{f}(e) = (\mathrm{Re} f)^{\sim}(e) - i(\mathrm{Re} f)^{\sim}(ie)$ and note that f is complex linear and $\tilde{f}|F = f$.

To show that $|\tilde{f}(e)| \leq \|e\|$ for all $e \in E$, write

$$\tilde{f}(e) = |\tilde{f}(e)| \exp(i\theta)$$

so complex linearty of $\tilde{f}$ implies $\tilde{f}(e.\exp(-i\theta)) \in \mathbb{R}$ and hence

$$|\tilde{f}(e)| = \tilde{f}(e.\exp(-i\theta)) = (\mathrm{Re} f)^{\sim}(e.\exp(-i\theta))$$
$$\leq \|e.\exp(-i\theta)\| = \|e\|.$$

30, Corollary. Let $(E, \|\cdot\|)$ be a normed space, $F \subset E$ a subspace and $f \in F^*$ (the topological dual). Then there exists $\tilde{f} \in E^*$ such that $\tilde{f}|_F = f$ and $\|\tilde{f}\| = \|f\|$.

Proof. We can assume $f \neq 0$. Then $|||e||| = \|f\| \|e\|$ is a norm on E and $|f(e)| \leq \|f\| \|e\| = |||e|||$ for all $e \in F$. Applying the preceding Theorem we get a linear map $\tilde{f} : E \to \mathbb{R}$ (or $\mathbb{C}$) such that $\tilde{f}|F = f$ and $|\tilde{f}(e)| \leq |||e|||$ for all $e \in E$. This says that $\|\tilde{f}\| \leq \|f\|$ and since $\tilde{f}$ extends f, it follows that $\|f\| \leq \|\tilde{f}\|$; i.e. $\|\tilde{f}\| = \|f\|$ and $\tilde{f} \in E^*$.

31. Corollary. Let E be a normed vector space and $e \neq 0$ then there exists $f \in E^*$ such that $f(e) \neq 0$. In other words if $f(e) = 0$ for all $f \in E^*$, then $e = 0$; i.e. E^* separate points of E.

Proof. Apply Corollary (30) to the linear function

$$\{\alpha e \mid \alpha \in \mathbb{C}\} \longrightarrow \mathbb{C}, \quad \alpha e \mapsto \alpha.$$

The second pillar is as follows.

32. Open Mapping Theorem of Banach and Schauder. Let E and F be Banach spaces and suppose $A \in L(E,F)$ is onto. Then A is an open mapping.

Proof. To show that A is an open mapping, it suffices to prove that $A(\mathrm{cl}(D_1(0)))$ contains a disk centered at zero in F. Let $r > 0$. Since $E = \bigcup_{n=1}^{\infty} D_{nr}(0)$, it follows that $F = \bigcup_{n=1}^{\infty} (A(D_{nr}(0)))$ and hence $\bigcup_{n=1}^{\infty} \mathrm{cl}(A(D_{nr}(0))) = F$. Completeness of F implies that at least one of the $\mathrm{cl}(A(D_{nr}(0)))$ has nonempty interior by the Baire category theorem (B-17). The mapping $e \in E \mapsto ne \in E$ being a homeomorphism, this says that $\mathrm{cl}(A(D_r(0)))$ contains some open set $V \subset F$. We shall prove that in fact the origin of F is in $\mathrm{int}(\mathrm{cl}(A(D_r(0))))$. Continuity of $(e_1, e_2) \in E \times E \longmapsto e_1 - e_2 \in E$ assures

The existence of an open set $U \subset E$ such that

$$U - U = \{e_1 - e_2 \mid e_1, e_2 \in U\} \subset D_r(0)$$

Thus $cl(A(D_1(0))) \supset cl(A(U) - A(U)) \supset cl(A(U)) - cl(A(U)) \supset V - V$. But $V - V = \bigcup_{e \in V}(V - e)$ is open and clearly contains $0 \in F$. It follows that there exists a disk $D_\delta(0) \subset F$ such that $D_\delta(0) \subset cl(A(D_r(0)))$.

Now let $\varepsilon_n = 1/2^{n+1}$, $n = 0, 1, 2, \ldots$, so that $1 = \sum_{n=0}^{\infty} \varepsilon_n$. By the foregoing result there exists an $\eta_n > 0$ such that $D_{\eta_n}(0) \subset cl(A(D_{\varepsilon_n}(0)))$. Clearly $\eta_n \to 0$. We shall prove that

$$D_{\eta_0}(0) \subset A(cl(D_1(0)))$$

For $v \in D_{\eta_0}(0) \subset cl(A(D_{\varepsilon_0}(0)))$ there exists $e_0 \in D_{\varepsilon_0}(0)$ such that $\|v - Ae_0\| < \eta_1$ and thus $v - Ae_0 \in cl(A(D_{\varepsilon_1}(0)))$, so there exists $e_1 \in D_{\varepsilon_1}(0)$ such that $\|v - Ae_0 - Ae_1\| < \eta_2$, etc. Inductively one constructs a sequence $e_n \in D_{\eta_n}(0)$ such that

$$\|v - Ae_0 - \cdots - Ae_n\| < \eta_{n+1}$$

The series $\sum_{n=0}^{\infty} e_n$ is convergent because

$$\Big\|\sum_{i=n+1}^{m} e_i\Big\| \le \sum_{i=n+1}^{m} 1/2^{i+1} \quad \text{and} \quad \sum_{n=0}^{\infty} 1/2^{n+1} = 1$$

and E is complete. Let

$$e = \sum_{n=0}^{\infty} e_n \in E$$

Thus

$$Ae = \sum_{n=0}^{\infty} Ae_n = v, \text{ and } \|e\| \le \sum_{n=0}^{\infty} \|e_n\| \le \sum_{n=0}^{\infty} 1/2^{n+1} = 1;$$

i.e., $v \in D_{\eta_0}(0)$ implies $v = Ae$, $\|e\| \le 1$; that is, $D_{\eta_0}(0) \subset A(cl(D_1(0)))$.

An immediate consequence is the following, 33 . Banach's Isomorphism Theorem. A continuous linear isomorphism of Banach spaces is a homeomorphism.

Thus, if F, G are closed subspaces of the Banach space E and E is the algebraic direct sum of F and G, then the mapping $(e, e') \in F \times G \mapsto e + e' \in E$ is a continuous isomorphism, and hence a homeomorphism; i.e., $E = F \oplus G$; this proves the comment at the beginning of split subspaces.

34 . Closed Graph Theorem. Let E, F be Banach sps. and $A : E \to F$ a linear map. Then A is continuous iff its graph $\Gamma_A = \{ (e, Ae) \in E \times F \mid e \in E \}$ is a closed subspace of $E \oplus F$.

proof. It is readily verified that Γ_A is a linear subspace of $E \oplus F$. If $A \in L(E,F)$, then Γ_A is closed. Conversely, if Γ_A is closed, then it is a Banach subspace of $E \oplus F$ and since the mapping $(e, Ae) \in \Gamma_A \mapsto e \in E$ is a continuous isomorphism, its inverse $e \in E \mapsto (e, Ae) \in \Gamma_A$ is also continuous by Theorem (33). Since $(e, Ae) \in \Gamma_A \mapsto Ae \in F$ is clearly continuous, so is the composition $e \mapsto (e, Ae) \mapsto Ae$.

Note. The closed graph theorem is often used in the following way. To show that a linear map $A : E \to F$ is continuous for E, F Banach spaces, it suffices to show that if $e_n \to 0$ and $Ae_n \to e'_n$, then $e' = 0$.

35. Corollary. Let E be a Banach space and F a closed subspace of E. Then F is split iff there exists $P \in L(E,E)$ such that $P \circ P = P$ and $F = \{e \in E \mid Pe = e\}$.

proof. If such a P exists, then clearly $\ker P$ is a closed subspace of E that is an algebraic complement of F; any $e \in E$ is of the form $e = e - Pe + Pe$ with $e - Pe \in \ker P$ and $Pe \in F$.

Conversely, if $E = F \oplus G$, define $P : G \to E$ by $P(e) = e_1$, where $e = e_1 + e_2$, $e_1 \in F$, $e_2 \in G$. P is clearly linear, $P^2 = P$ and $F = \{e \in E \mid Pe = e\}$, so all there is to show is that P is continuous. Let $e_n = e_{1n} + e_{2n} \to 0$ and $P(e_n) = e_{1n} \to e'$; i.e., $-e_{2n} \to e'$ and since F and G are closed this implies that $e' \in F \cap G = \{0\}$. By the closed graph theorem, $P \in L(E,E)$.

36. Fundamental Isomorphism Theorem. Let $A \in L(E,F)$ be surjective where E and F are Banach spaces. Then $E/\ker A$ and F are isomorphic Banach spaces.

proof. The map $[e] \mapsto Ae$ is bijective and continuous (its norm is $\leq \|A\|$), so it is a homeomorphism.

Remark and Definition: A sequence of maps

$$\cdots \longrightarrow E_{i-1} \xrightarrow{A_i} E_i \xrightarrow{A_{i+1}} E_{i+1} \longrightarrow \cdots$$

of Banach spaces is said to be split exact if for all i, $\ker A_{i+1} = \operatorname{range} A_i$ and both $\ker A_i$ and $\operatorname{range} A_i$ split.

With this terminology, .19 can be reformulated in the following way: If $0 \to G \to E \to F \to 0$ is a split exact sequence of Banach spaces, then E/G is Banach space isomorphic to F (thus $E \cong G \oplus F$).

Now we are ready for the third pillar.

37. Uniform Boundedness Principle of Banach and Steinhaus. Let E and F be normed vector spaces, E complete, and $\{A_i\}_{i\in I} \subset L(E,F)$. If for each $e \in E$ the set $\{A_i e\}_{i \in I}$ is bounded in F, then $\{\|A_i\|\}_{i\in I}$ is a bounded set of real-numbers.

Proof. Let $\varphi(e) = \sup\{\|A_i e\| \mid i \in I\}$ and note that

$$S_n = \{e \in E \mid \varphi(e) \le n\} = \bigcap_{i \in I} \{e \in E \mid \|A_i e\| \le n\}$$

is closed and $\bigcup_{n=1}^{\infty} S_n = E$. Since E is a complete metric space, the Baire category theorem (B-17) tells us that some S_n has nonempty interior; i.e., there exist $r > 0$, $e_0 \in E$ such that $\varphi(e) \le M$, for all $e \in cl(D_r(e_0))$, where $M > 0$ is some constant.

For each $i \in I$, and $\|e\| = 1$, we have

$$\|A_i(re + e_0)\| \le \varphi(re + e_0) \le M$$

so that

$$\|A_i e\| = \frac{1}{r}\|A_i(re + e_0 - e_0)\| \le \frac{1}{r}\|A_i(re+e_0)\| + \frac{1}{r}\|A_i e_0\|$$
$$\le (M + \varphi(e_0))/r$$

i.e. $\|A_i\| \le (M + \varphi(e_0))/r$ for all $i \in I$.

38. . Corollary. If $\{A_n\} \subset L(E,F)$ is a strongly convergent sequence; i.e., $\lim_{n\to\infty} A_n e = Ae$ exists for every $e \in E$, then $A \in L(E,F)$.

Proof. A is clearly a linear map. Since $\{A_n e\}$ is convergent, it is a bounded set for each $e \in E$, so that by .20 , $\{\|A_n\|\}$ is bounded by, say $M > 0$. But then

$$\begin{aligned} \|Ae\| &= \lim_{n\to\infty} \|A_n e\| \\ &\le (\limsup_{n\to\infty} \|A_n\|)\, \|e\| \\ &\le M\, \|e\| \end{aligned}$$

i.e., $A \in L(E,F)$.