Chapter 4

L_p - Spaces

Our attention is now turned from the study of general normed sps. to fn. sps. Many of the classical spaces in analysis consist of measurable fns, and most of the important norms on such spaces are defined by integrals.

The theory of integration enables us to study the remarkable properties of these spaces. Here the classical L_p-spaces will be considered. As we shall see, they are special examples of Banach lattices.

Throughout this section $(X, \mathcal{S}, \mu)$ will be a fixed measure space, and unless otherwise specified, all properties of fns. will refer to this measure space. It is important to keep in mind that if f is a measurable fn. then $|f|^p$ is also measurable for each $p>0$.

Definition 1. Let $0<p<\infty$. Then the collection of all measurable functions f for which $|f|^p$ is integrable will be denoted by $L_p(\mu)$.

If clarity requires the measure space X to be indicated, then $L_p(\mu)$ will be denoted by $L_p(X)$.

It is easy to see that $L_p(\mu)$ is a vector space. Indeed, if $f \in L_p(\mu)$, then clearly, $\alpha f \in L_p(\mu)$ holds for all $\alpha \in \mathbb{R}$. On the

other hand, the inequality among the real numbers
$$|a+b|^p \le 2^p(|a|^p+|b|^p)$$
shows that $L_p(\mu)$ is closed under addition. Moreover, if $f\in L_p(\mu)$, then the inequalities $0\le f^+\le |f|$ and $0\le f^-\le |f|$ imply that f^+, f^- and $|f|$ belong to $L_p(\mu)$. In other words, $L_p(\mu)$ is a vector lattice.

For each $f\in L_p(\mu)$ let
$$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p}$$
The number $\|f\|_p$ is called the L_p-norm of f. Obviously, $\|f\|_p \geq 0$ and $\|\alpha f\|_p = |\alpha|\cdot\|f\|_p$ hold for all $f\in L_p(\mu)$ and $\alpha\in\mathbb{R}$.

To obtain additional properties of the L_p-norms, we need a inequality.

Lemma 2. If $0<t<1$, then
$$a^t b^{1-t} \le ta+(1-t)b$$
holds for every pair of nonnegative real numbers a and b.

pf. The inequality is trivial if either a or b equals zero. Hence, assume $a>0$ and $b>0$. Consider the fn. $f(x)=(1-t)+tx-x^t$ for $x>0$. Then $f'(x)=t(1-x^{t-1})$ and so $x=1$ is the only critical point of f. It follows that f attains its minimum at $x=1$. Thus, $f(1)=0\le(1-t)+tx-x^t$ holds for all $x>0$. Now set $x=\frac{a}{b}$ to obtain the desired inequality:
$$0\le(1-t)+\frac{ta}{b}-\frac{a^t}{b^t}=\frac{(1-t)b^t+tab^{t-1}-a^t}{b^t}$$
$$a^tb^{1-t}\le(1-t)b+ta$$

An important inequality between L_p-norms, known as Holder's inequality, is stated next.

Theorem 3. (Hölder's Inequality). Let $1<p<\infty$ and $1<q<\infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. If $f\in L_p(\mu)$ and $g\in L_q(\mu)$, then $fg\in L_1(\mu)$ and

$$\int |fg|\,d\mu \le \left(\int |f|^p d\mu\right)^{1/p}\left(\int |g|^q d\mu\right)^{1/q} = \|f\|_p\cdot\|g\|_q.$$

Proof. If $f=0$ a.e. or $g=0$ a.e. holds, then the inequality is trivial. So, assume $f\ne 0$ a.e. and $g\ne 0$ a.e.. Then $\|f\|_p>0$ and $\|g\|_q>0$. Now apply Lemma (2) with $t=\frac{1}{p}$, and $a=(|f(x)|/\|f\|_p)^p$ and $b=(|g(x)|/\|g\|_q)^q$ to obtain

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q}\le\frac{1}{p}\frac{|f(x)|^p}{(\|f\|_p)^p}+\frac{1}{q}\frac{|g(x)|^q}{(\|g\|_q)^q}$$

Hence $fg\in L_1(\mu)$ and by integrating, we get

$$\frac{\int|fg|\,d\mu}{\|f\|_p\|g\|_q}\le\frac{1}{p}+\frac{1}{q}=1.$$

That is $\int|fg|\,d\mu\le\|f\|_p\|g\|_q$, as required.

For the special case $p=q=2$, Holder's inequality is known as the Cauchy-Schwarz inequality. The triangle inequality of the function $\|\cdot\|_p$ is referred to as the Minkowski inequality. The details follow.

Theorem 4. (Minkowski Inequality). Let $1\le p<\infty$. Then for every pair $f,g\in L_p(\mu)$ the following inequality holds:

$$\|f+g\|_p\le\|f\|_p+\|g\|_p.$$

Proof. For $p=1$ the inequality is clearly true. Thus, assume $1<p<\infty$. Let $1<q<\infty$ be such that $\frac{1}{p}+\frac{1}{q}=1$. We already know that if f and g belong to $L_p(\mu)$, then $f+g$ likewise belongs to $L_p(\mu)$. Next observe that since $(p-1)q=p$, it follows that $|f+g|^{p-1}\in L_q(\mu)$. Thus, by theorem (3) both $|f|\cdot|f+g|^{p-1}$ and $|g|\cdot|f+g|^{p-1}$ belong to $L_1(\mu)$ and

$$\int|f|\,|f+g|^{p-1}d\mu\leq \|f\|_p\left(\int|f+g|^{(p-1)q}d\mu\right)^{1/q}=\|f\|_p\cdot\left(\|f+g\|_p\right)^{p/q}$$
$$\int|g|\,|f+g|^{p-1}d\mu\leq \|g\|_p\left(\|f+g\|_p\right)^{p/q}.$$

Therefore,

$$\begin{aligned}(\|f+g\|_p)^p&=\int|f+g|^p d\mu\\ &\leq\int|f|\,|f+g|^{p-1}d\mu+\int|g|\,|f+g|^{p-1}d\mu\\ &\leq\|f\|_p\left(\|f+g\|_p\right)^{p/q}+\|g\|_p\left(\|f+g\|_p\right)^{p/q}\\ &=\left(\|f\|_p+\|g\|_p\right)\left(\|f+g\|_p\right)^{p/q}.\end{aligned}$$

from which is easily follows that

$$\begin{aligned}\|f+g\|_p&=\left(\|f+g\|_p\right)^{p-\frac{p}{q}}\\ &\leq\|f\|_p+\|g\|_p.\end{aligned}$$

The proof of the theorem is now complete.

Summarizing the above discussion:

If $1\leq p<\infty$, then

a. $\|f\|_p \geq 0$

b. $\|\alpha f\|_p = |\alpha| \cdot \|f\|_p$ and

c. $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

hold for all $f, g \in L_p(\mu)$ and $\alpha \in \mathbb{R}$.

Obviously, by Theorem (1.76), $\|f\|_p = 0$ iff $f=0$ a.e. holds. Thus, unfortunately, the fn. $\|.\|_p$ on $L_p(\mu)$ fails to satisfy the norm requirement that $\|f\|_p = 0$ imply $f=0$. To avoid this difficulty, it is a custom to call two fns. of $L_p(\mu)$ equivalent if they are equal almost everywhere. Clearly, this introduces an equivalence relation on $L_p(\mu)$, and $\|.\|_p$ becomes a norm on the equivalence classes. In other words, $L_p(\mu)$, for $1 \leq p < \infty$, is a normed sp. if we do not distinguish between fns. that are equal almost everywhere. Thus, $L_p(\mu)$ in reality consists of equivalence classes of fns; but this should not pose a problem. In actual practice, the equivalence classes are relegated to the background, and the elements of $L_p(\mu)$ are thought of as fns. (where two fns. are considered identical if they are equal a.e.). Another advantage of the identification of fns. that are equal a.e. is the following: A fn. of $L_p(\mu)$ can assume infinite values or even be left undefined on a null set [since by assigning finite values to these points, the fn. becomes equivalent to a real valued fn. of $L_p(\mu)$].

Also, it should be clear that if g is a measurable fn.

and $f \in L_p(\mu)$ satisfies $|g| \le |f|$ a.e., then $g \in L_p(\mu)$ and $\|g\|_p \le \|f\|_p$ holds. In other words, $\|\cdot\|_p$ is a lattice norm. Therefore, for $1 \le p < \infty$ each $L_p(\mu)$ is a normed vector lattice and, in fact, a Banach lattice, as the next result shows.

Theorem 5. (Riesz - Fischer). If $1 \le p < \infty$, then $L_p(\mu)$ is a Banach lattice.

pf. Let $\{f_n\}$ be a Cauchy sequence. By passing to a subseq. if necessary, we can assume without loss of generality that $\|f_{n+1} - f_n\|_p < 2^{-n}$ holds for each n. We have to show the existence of some $f \in L_p(\mu)$ s.t. $\lim \|f - f_n\|_p = 0$

Set $g_1 = 0$ and $g_n = |f_1| + |f_2 - f_1| + \cdots + |f_n - f_{n-1}|$ for $n \geq 2$. Then $0 \le g_n \uparrow$ and

$$\begin{aligned} \int (g_n)^p \, d\mu &= (\|g_n\|_p)^p \\ &\le \Big(\|f_1\|_p + \sum_{i=2}^{\infty} \|f_i - f_{i-1}\|_p\Big)^p \\ &\le (\|f_1\|_p + 1)^p \end{aligned}$$

holds for all n. By Levi's Theorem (Theorem 2.52), there exists some $g \in L_p(\mu)$ such that $0 \le g_n \uparrow g$ a.e.

Because

$$|f_{n+k} - f_n| = \Big|\sum_{i=n+1}^{n+k} (f_i - f_{i-1})\Big| \le \sum_{i=n+1}^{n+k} |f_i - f_{i-1}| = g_{n+k} - g_n$$

it follows that $\{f_n\}$ converges pointwise (a.e.) to some fn. f.

Since

$$|f_n| = \Big|f_1 + \sum_{i=2}^{n} (f_i - f_{i-1})\Big| \le g_n \le g \quad \text{a.e.}$$

it follows that $|f| \le g$ a.e. holds, and hence $f \in L_p(\mu)$. But

Then $|f-f_n| \leq 2g$ a.e. and $\lim |f_n - f|^p = 0$, coupled with the Lebesgue dominated convergence theorem, imply that

$$\lim \|f - f_n\|_p = 0$$

and the proof is finished.

Remark 6. A glance at the preceding proof reveals that if $\{f_n\}$ is a seq. of some $L_p(\mu)$, with $1 \leq p < \infty$, such that $\lim \|f - f_n\|_p = 0$, then there exists a subsequence $\{f_{k_n}\}$ of $\{f_n\}$ and some $g \in L_p(\mu)$ such that $f_{k_n} \to f$ a.e. and $|f_{k_n}| \leq g$ a.e. for each n.

In general, it is not true that $\lim \|f - f_n\|_p = 0$ implies $f_n \to f$ a.e. For instance, the seq. $\{f_n\}$ of Example 8.6 satisfies $\lim \|f_n\|_p = 0$ (for each $1 \leq p < \infty$), but $\{f_n(x)\}$ does not converge for any $x \in [0,1]$.

It is easy to construct an example of a seq. of an L_p-space that converges pointwise to some fn. of the space, but fails to converge in the L_p-norm. For instance, consider $\mathbb{R}$ with the Lebesgue measure and $f_n = \chi_{(n,n+1)}$ for each n. Then $f_n(x) \to 0$ holds for each $x \in \mathbb{R}$ and $f_n \in L_p(\mathbb{R})$ for all n and $1 \leq p < \infty$. On the other hand, $\|f_n\|_p = 1$ holds for each n and $1 \leq p < \infty$, and so $\{f_n\}$ does not converge to zero with respect to any L_p-norm.

The next useful result gives a condition for pointwise convergence to imply norm convergence in L_p-spaces.

Theorem 7. Let $1 \le p < \infty$ and $f \in L_p(\mu)$ and let (f_n) be sequence of $L_p(\mu)$ such that $f_n \longrightarrow f$ a.e. If $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$ then $\lim \|f_n - f\|_p = 0$.

Proof. Start by observing that $(a+b)^p \le 2^{p-1}(a^p + b^p)$ holds for each pair of nonnegative real numbers a and b. Indeed for $p=1$ the inequality is trivial. On the other hand, if $1<p<\infty$ then the convexity of the function $g(x) = x^p$ $(x>0)$ implies

$$\left(\frac{a+b}{2}\right)^p \le \frac{1}{2}(a^p + b^p),$$

and hence, $(a+b)^p \le 2^{p-1}(a^p+b^p)$ holds. In particular, for each pair of real numbers a and b we have

$$|a-b|^p \le 2^{p-1}(|a|^p + |b|^p).$$

Thus,

$$0 \le 2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p \quad \text{a.e.}$$

and by applying Fatou's Lemma and using our hypothesis

$$\lim_{n\to\infty} \int |f_n|^p d\mu = \int |f|^p d\mu$$

we get

$$2^p \int |f|^p d\mu = \int \lim [2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p] d\mu$$

$$\le \liminf \int [2^{p-1}(|f_n|^p + |f|^p) - |f_n - f|^p] d\mu$$

$$= 2^{p-1}\int |f|^p d\mu + 2^{p-1} \lim \int |f_n|^p d\mu + \liminf\left(-\int |f_n - f|^p d\mu\right)$$

$$= 2^p \int |f|^p d\mu - \limsup \int |f_n - f|^p d\mu.$$

Now because $\int |f|^p d\mu < \infty$, the last inequality yields

$$\limsup \int |f_n - f|^p d\mu \leq 0$$

Hence,

$$\limsup \int |f_n - f|^p d\mu = \liminf \int |f_n - f|^p d\mu = 0$$

so that

$$\lim \int |f_n - f|^p d\mu = 0$$

Therefore, $\lim \|f_n - f\|_p = 0$ holds, as required.

A real number M is said to be an essential bound for a fn. f whenever $|f(x)| \leq M$ holds for almost all x. A fn. is called essentially bounded if it has an essential bound. Therefore, a function is essentially bounded if it is bounded except possibly on a set of measure zero. The essential supremum of a fn. f is defined by

$$\|f\|_\infty = \inf\{M : |f(x)| \leq M \text{ holds for almost all } x\}.$$

If f does not have any essential bound, then it is understood that $\|f\|_\infty = \infty$. Observe that $|f(x)| \leq \|f\|_\infty$ holds for almost all x.

The following properties are easily verified, and they are left as exercises.

1. If $f = g$ a.e., then $\|f\|_\infty = \|g\|_\infty$.
2. $\|f\|_\infty \geq 0$ for each fn. f, and $\|f\|_\infty = 0$ iff $f = 0$ a.e.
3. $\|\alpha f\|_\infty = |\alpha| \cdot \|f\|_\infty$ for all $\alpha \in \mathbb{R}$.
4. $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
5. If $|f| \leq |g|$, then $\|f\|_\infty \leq \|g\|_\infty$.

Def. 8. The collection of all essentially bdd. measurable fns. is denoted by $L_\infty(\mu)$.

Note 9. Here again, two fns. are considered identical if they are equal almost everywhere. It should be obvious that with the usual algebraic and lattice operations $L_\infty(\mu)$ is a vector lattice. Moreover, according to the above listed properties, $L_\infty(\mu)$ equipped with $\|\cdot\|_\infty$ is a normed vector lattice that is actually a Banach lattice.

Theorem 10. $L_\infty(\mu)$ is a Banach lattice.

pf. Let $\{f_n\}$ be a Cauchy seq. of $L_\infty(\mu)$. We have to show that there exists some $f \in L_\infty(\mu)$ such that $\lim \|f - f_n\|_\infty = 0$.

Since for each pair of indices m and n we have

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \quad \text{for almost all } x$$

it follows that there exists a set A of measure zero such that $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ holds for all m and n and all $x \notin A$. But then $\lim f_n(x) = f(x)$ exists in $\mathbb{R}$ for all $x \notin A$, and moreover, f is measurable and essentially bounded. That is, $f \in L_\infty(\mu)$.

Now let $\varepsilon > 0$. Choose k s.t. $\|f_n - f_m\|_\infty < \varepsilon$ for all $n, m > k$. Since $|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \le \varepsilon$ holds for all $x \notin A$ and $n > k$, it follows that $\|f - f_n\|_\infty \le \varepsilon$ for each $n > k$. This shows that $\lim \|f - f_n\|_\infty = 0$, and the proof of the theorem is complete.

It is easy to verify that every step function belongs to every L_p-space. Moreover, the collection of all step functions is a vector sublattice of every L_p-space. In addition, this sublattice is norm dense in $L_1(\mu)$. the next result tells us that actually the vector lattice of setep functions is norm dense in every $L_p(\mu)$ with $1 \le p < \infty$.

Theorem 11. For every $1 \le p < \infty$, the collection of all step functions is norm dense in $L_p(\mu)$.

Proof. Let $0 \le f \in L_p(\mu)$. then, there exists a sequence (φ_n) of simple functions such that $0 \le \varphi_n \uparrow f$ a.e. clearly, each φ_n is a step function and $(f-\varphi_n)^p \downarrow 0$ a.e. holds. By the Lebesgue dominated convergence theorem we get

$$\|f-\varphi_n\|_p = \left(\int |f-\varphi_n|^p d\mu\right)^{1/p} \downarrow 0.$$

Since every function of $L_p(\mu)$ can be written as a difference of two positive functions of $L_p(\mu)$, it follows from the above that the step functions are norm dense in $L_p(\mu)$.

In case the measure is a regular Borel measure, the continuous functions with cpct. support are also norm dense in each $L_p(\mu)$ for $1 \le p < \infty$. The details are included in the next Theorem.

Theorem 12. Let μ be a regular Borel measure on a Hausdorff locally compact topological space X. Then the collection of all continuous fns. with cpct. support is norm dense in $L_p(\mu)$ for every $1 \le p < \infty$.

pf. Clearly, every continuous fn. with cpct. support belongs to each L_p-space. Now let $1 \le p < \infty$, $f \in L_p(\mu)$, and $\varepsilon > 0$. We have to show that there exists some cont. fn. g with cpct. support s.t. $\|f-g\|_p < \varepsilon$. By Theorem (11) it is enough to assume that $f = \chi_A$, where A is a measurable set s.t. $\mu^*(A) < \infty$.

As in the proof of Theorem (2.76), there exists a continuous fn. $g : X \longrightarrow [0,1]$ with cpct. support s.t.

$$\int |\chi_A - g| \, d\mu < 2^{-p} \varepsilon^p$$

(Note that $|\chi_A - g| \le 2$ holds.) But then

$$\begin{aligned} \|\chi_A - g\|_p &= \left(\int |\chi_A - g|^p \, d\mu \right)^{1/p} \\ &= \left(\int |\chi_A - g| \cdot |\chi_A - g|^{p-1} \, d\mu \right)^{1/p} \\ &\le 2 \left(\int |\chi_A - g| \, d\mu \right)^{1/p} \\ &< 2 \cdot 2^{-1} \varepsilon = \varepsilon \end{aligned}$$

and the proof is finished.

Note 13. Consider $\mathbb{R}$ equipped with the measure μ that assigns to every subset of $\mathbb{R}$ the value zero, that is, $\mu = 0$. Then μ is a regular Borel measure, and obviously, any two fns. on $\mathbb{R}$ can be identified with the zero fn., a situation that is not very useful.

Therefore, it is desirable to deal with regular Borel measures for which distinct continuous fns. are not equivalent. To do

This, we need to know where the measure is "concentrated" in the space.

Theorem 14. Let μ be a regular Borel measure on a Hausdorff locally cpct. topological sp. X. Then there exists a unique closed subset E of X with the following two properties:

1. $\mu(E^c)=0$; and
2. if V is an open set s.t. $E\cap V\neq\emptyset$, then $\mu(E\cap V)>0$.

pf. Let $O=\cup\{V: V \text{ is open and } \mu(V)=0\}$. Clearly, O is an open set, and we claim that $\mu(O)=0$.

To see this, let K be a cpct. subset of O. From the definition of O it follows that there exist open sets $V_1,\dots,V_n$, all of measure zero, s.t. $K\subseteq\bigcup_{i=1}^{n}V_i$. Hence

$$\mu(K)=0$$

Our claim now follows from

$$\mu(O)=\sup\{\mu(K): K \text{ cpct. and } K\subseteq O\}$$

Now set $E=O^c$. Then E is a closed set and

$$\mu(E^c)=\mu(O)=0$$

On the other hand, if V is an open set s.t. $E\cap V\neq\emptyset$, then $\mu(E\cap V)>0$ must hold. Otherwise, if $\mu(E\cap V)=0$ holds, then

$$\mu(V)=\mu(E\cap V)+\mu(E^c\cap V)=0$$

also holds, implying $V\subseteq O=E^c$, contrary to $E\cap V\neq\emptyset$.

For the uniqueness of E assume that another closed set F satisfies (1) and (2). From (1) it follows at once that

$F^c \subseteq O$, and so $E = O^c \subseteq F$ holds. On the other hand, since $\mu(O \cap F) = 0$, it follows from (2) that $O \cap F = \emptyset$. Hence $F \subseteq O^c = E$, so that $F = E$, and the proof is finished.

Definition and Remarks 3.15. The unique set E determined by Theorem (14) is called the support of μ and is denoted by $\operatorname{Supp} \mu$. That is, $\operatorname{Supp} \mu = E$. If we think of the measure space as a set over which some material has been distributed, then $\operatorname{Supp} \mu$ represents the parts of the set at which the mass has been placed.

How "large" can the support of a regular Borel measure? If $X = \mathbb{R}^n$, then for example, the support of the zero measure is the empty set. On the other hand, the Lebesgue measure λ satisfies $\operatorname{Supp} \lambda = \mathbb{R}^n$.

Let μ be a regular Borel measure on a Hausdorff locally cpct. topological sp. X with $\operatorname{Supp} \lambda = X$. Then two continuous real valued fns. f and g on X satisfy $f = g$ a.e. iff $f(x) = g(x)$ holds for all $x \in X$. This follows immediately by observing that if $f(a) \neq g(a)$ holds for some $a \in X$, then $f(x) \neq g(x)$ holds for all x in some nonempty open set V. Because $\mu(V) > 0$, it is impossible for $f = g$ a.e. to hold. In particular, it follows that $\|f\|_p = (\int |f|^p d\mu)^{1/p}$ defines a lattice norm on $C_c(X)$, the vector lattice of all continuous real valued fns. on X with cpct. support. In general, $C_c(X)$ equipped with an L_p-norm is not a Banach lattice. However, by Theorem (12) the following result should be immediate.

Theorem 16. Let μ be a regular Borel measure on a Hausdorff locally cpct. topological space X with $\mathrm{Supp}\,\mu = X$. Then for each $1 \le p < \infty$, the completion of $C_c(X)$ with the L_p-norm is the Banach lattice $L_p(\mu)$.

Remark and Example 17. In general, the L_p-spaces are not "comparable". As an example, let $X = (0, \infty)$ with the Lebesgue measure. Then the fn. $f(x) = x^{-1/2}$ if $0 < x \le 1$ and $f(x) = 0$ if $x > 1$ belongs to $L_1(\mu)$, but it does not belong to $L_2(\mu)$. On the other hand, the fn. $g(x) = 0$ if $0 < x < 1$ and $g(x) = x^{-1}$ if $x \geq 1$ belongs to $L_2(\mu)$, but not to $L_1(\mu)$.

Two comparison results of the L_p-spaces are presented next. The first one is for the case that (X, S, μ) is a finite measure space.

Theorem 18. Let (X, S, μ) be a finite measure space, and let $1 \le p < q \le \infty$. Then $L_q(\mu) \subseteq L_p(\mu)$ holds.

pf. Clearly, in this case $L_\infty(\mu) \subseteq L_p(\mu)$ holds for each $1 \le p < \infty$. Thus, assume $1 \le p < q < \infty$.

Set $r = q/p > 1$, and then choose $s > 1$ s.t. $\frac{1}{r} + \frac{1}{s} = 1$. If

$f \in L_q(\mu)$, Then clearly, $|f|^p \in L_r(\mu)$. Since the constant fn. 1 — belongs to $L_s(\mu)$, it follows from Theorem (3) That

$$|f|^p = |f|^p \cdot 1 \in L_1(\mu)$$

That is, $f \in L_p(\mu)$, and the proof of the Theorem is complete.

Remark 19. It should be observed that if $L_q(\mu) \subseteq L_p(\mu)$ holds, Then $L_q(\mu)$ is an ideal of the vector lattice $L_p(\mu)$.

Some important examples of L_p-spaces are provided by considering the counting measure on $\mathbb{N}$. In this case the fns. on $\mathbb{N}$ are denoted as sequences, and integration is replaced by summation. These L_p-spaces are called the "little L_p-sps." and they are denoted by ℓ_p. In other words, if $1 \le p < \infty$, Then ℓ_p consists of all sequences $x = \{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, where $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$. Similarly, ℓ_∞ is the vector space of all bdd. sequences with the sup norm.

The ℓ_p-spaces, unlike the general L_p-spaces, are always comparable. Note the contrast between the next Theorem and the preceding one.

Theorem 20. If $1 \le p < q \le \infty$, Then $\ell_p \subseteq \ell_q$ holds. Moreover the inclusion is proper.

pf. Observe That if $x = \{x_n\}$ belongs to some ℓ_p-space with $1 \le p < \infty$, Then $\{x_n\}$ must be a bdd. sequence (actually, convergent to zero), and hence, $x \in \ell_\infty$. That is, $\ell_p \subseteq \ell_\infty$ holds for all $1 \leq p < \infty$.

Thus, as ume $1 < q < \infty$. Let $x = x_n \in \ell_p$. Since
$$\sum_{n=1}^{\infty} |x_n|^p < \infty,$$
There exists some k such that $|x_n| < 1$ for all $n \geq k$. Thus, $|x_n|^q \leq |x_n|^p$ holds for all $n \geq k$, and this shows that
$$\sum_{n=1}^{\infty} |x_n|^q < \infty.$$
Therefore, $x \in \ell_q$ and hence, $\ell_p \subseteq \ell_q$.

For the last part note that of $x_n = n^{-1/p}$ for all n, then $x = \{x_n\} \in \ell_q$ but $x \notin \ell_p$.

Two numbers p and q in $[1, \infty]$ are called conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$. We adhere to the convention $\frac{1}{\infty} = 0$, so that 1 and ∞ are conjugate exponents.

Let p and q be two conjugate exponents. If $g \in L_q(\mu)$ then it follows from theorem (3) that $fg \in L_1(\mu)$ for each $f \in L_p(\mu)$. Therefore, for each fixed $g \in L_q(\mu)$ a function F_g can be defined on $L_p(\mu)$ by
$$F_g(f) = \int fg\, d\mu,$$
for all $f \in L_p(\mu)$. Clearly, F_g is a linear functional and, in fact, as the next result shows, a bounded linear functional.

Theorem 21. Let $1 < p \leq \infty$, let q be its conjugate exponent, and let $g \in L_q(\mu)$. Then the linear functional defined by
$$F_g(f) = \int fg\, d\mu,$$

for $f \in L_p(\mu)$ is a bounded linear functional on $L_p(\mu)$ such that $\|F_g\| = \|g\|_q$ holds.

Proof. First we consider the case $p=\infty$ and $q=1$. Because $|F_g(f)| \leq \|g\|_1 \|f\|_\infty$ for each $f \in L_\infty(\mu)$, it follows that F_g is a bounded linear functional and that $\|F_g\| \leq \|g\|_1$ holds. On the other hand, let $f = \operatorname{sgn} g$, where $\operatorname{sgn} g(x) = 1$ if $g(x) \geqslant 0$ and $\operatorname{sgn} g(x) = -1$ if $g(x) < 0$. Then f belongs to $L_\infty(\mu)$ and satisfies $\|f\|_\infty = 1$ and

$$F_g(f) = \int |g| d\mu = \|g\|_1 .$$

Therefore, $\|F_g\| = \|g\|_1$.

Now we consider $1 < p < \infty$. By Hölder's inequality

$$|F_g(f)| = \left|\int fg\, d\mu\right| \leq \|g\|_q \ \|f\|_p .$$

holds for all $f \in L_p(\mu)$. Hence, F_g is a bounded linear functional, and $\|F_g\| \leq \|g\|_q$ holds. Now let $f = |g|^{q-1} \operatorname{sgn} g$. Clearly, f is a measurable function and $|f|^p = |g|^{p(q-1)} = |g|^q$ holds, so that $f \in L_p(\mu)$. Since $fg = |g|^q$, it follows that

$$\begin{aligned} F_g(f) &= \int fg\, d\mu \\ &= \int |g|^q d\mu \\ &= \left(\int |g|^q d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q} \\ &= \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q} \\ &= \|f\|_p \cdot \|g\|_q . \end{aligned}$$

That is, $\|F_g\| \geqslant \|g\|_q$. Thus $\|F_g\| = \|g\|_q$ holds, and the proof of the theorem is complete.

Remark 22. The preceding Theorem shows that for $1<p\le\infty$ a linear isometry $g \mapsto F_g$ can be defined from $L_q(\mu)$ into $L_p^*(\mu)$, the norm dual of $L_p(\mu)$. Observe that this isometry is also lattice preserving. Indeed, if $0\le f\in L_p(\mu)$, then by Theorem

$$
\begin{aligned}
(F_g)^+(f) &= \sup\{F_g(h) : 0\le h\le f\}\\
&= \sup\{\textstyle\int hg\,d\mu : 0\le h\le f\}\\
&= \textstyle\int f g^+\,d\mu = F_{g^+}(f)
\end{aligned}
$$

Therefore $(F_g)^+ = F_{g^+}$ holds, which implies that $g \mapsto F_g$ is also a lattice isometry.

Question: Is every bdd. linear functional on $L_p(\mu)$ representable, as in Theorem (21), by a fn. of $L_q(\mu)$?

The answer is yes if $1<p<\infty$. This is a classical result of F. Riesz. A proof of this theorem, as well as some of its applications, is deferred until Harmonic Analysis. Therefore $L_p^*(\mu)$ and $L_q(\mu)$ can be considered (under the above isomorphism) as identical Banach lattices. This is often expressed by saying that for $1<p<\infty$ the norm dual of $L_p(\mu)$ is $L_q(\mu)$; in symbols, $L_p^*(\mu) = L_q(\mu)$.

When $p=\infty$ the lattice isometry $g\mapsto F_g$ from $L_1(\mu)$ to $L_\infty^*(\mu)$ is rarely onto. The following example will clarify the situation.

Example 23. Let (X, S, μ) be a measure space such that there exists a disjoint seq. of measurable sets $\{E_n\}$ with $\mu^*(E_n) > 0$ for each n and $X = \bigcup_{n=1}^{\infty} E_n$. Let L be the collection of all real valued fns. f defined on X that are constant on each E_n, assuming on each E_n the value $f(E_n)$ and for which $\lim f(E_n)$ exists in $\mathbb{R}$. Clearly, L is a vector sublattice of $L_\infty(\mu)$.

Now define a linear functional F on L by

$$F(f) = \lim f(E_n) \quad \text{for each } f \in L$$

It is clear that $|F(f)| \le \|f\|_\infty$ holds for all $f \in L$, and so F is a continuous linear functional. By Theorem ( ), F can be extended to $L_\infty(\mu)$ with preservation of its original norm. Denote this extension by F again.

We claim that F cannot be represented by a fn. of $L_1(\mu)$. To see this, assume by way of contradiction that there exists some $g \in L_1(\mu)$ satisfying $F(f) = \int f g \, d\mu$ for all f in $L_\infty(\mu)$. Set $G_n = (\bigcup_{i=1}^{n} E_i)^c$, and $f_n = \chi_{G_n}$. Then $\{f_n\}$ is a sequence of L, and $F(f_n) = 1$ holds for each n. On the other hand, because $|f_n g| \le |g|$ and $f_n g \to 0$, it follows from the Lebesgue dominated convergence theorem that

$$F(f_n) = \int f_n g \, d\mu \longrightarrow 0$$

which is impossible.

Therefore, the lattice isometry $g \mapsto F_g$ from $L_1(\mu)$ to $L_\infty^*(\mu)$ is not onto.

Note 24. If the measure space is σ-finite, then the norm dual of $L_1(\mu)$ coincides with $L_\infty(\mu)$. (Exercise)

The representation theorem for the bounded linear functionals on the l_p-spaces can be proved directly.

Theorem 25. Let $1 \le p < \infty$, and let f be a continuous linear functional on l_p. Then there exists a unique $y = \{y_n\} \in l_q$ [where $\frac{1}{p} + \frac{1}{q} = 1$] s.t.

$$f(x) = \sum_{n=1}^{\infty} x_n y_n$$

holds for every $x = \{x_n\} \in l_p$.

pf. For each n, let e_n be the sequence having the valued one at the nth coordinate and zero at every other. Clearly if $x = \{x_n\} \in l_p$, then

$$\lim \| x - \sum_{i=1}^{n} x_i e_i \|_p = 0$$

Thus, $f(x) = \sum_{n=1}^{\infty} x_n f(e_n)$ holds. Let $y_n = f(e_n)$ for each n. To complete the proof, we have to show that $y = \{y_n\} \in l_q$. If $p = 1$, then $|y_n| = |f(e_n)| \le \|f\|$, so that $y \in l_\infty$.

Now for $1 < p < \infty$, set $a_n = y_n \cdot |y_n|^{q-2}$ if $y_n \ne 0$ and $a_n = 0$ if $y_n = 0$. Then

$$|a_n|^p = |y_n|^q = a_n y_n \quad \forall n$$

Moreover,

$$\sum_{i=1}^{n} |y_i|^q = \sum_{i=1}^{n} a_i y_i$$
$$= \sum_{i=1}^{n} a_i f(e_i)$$

$$
\begin{aligned}
&= f\Big(\sum_{i=1}^{n} a_i e_i\Big) \\
&\le \|f\| \cdot \Big\|\sum_{i=1}^{n} a_i e_i\Big\|_p \\
&= \|f\| \Big(\sum_{i=1}^{n} |a_i|^p\Big)^{1/p} \\
&= \|f\| \Big(\sum_{i=1}^{n} |y_i|^q\Big)^{1/p}
\end{aligned}
$$

Thus,

$$\Big(\sum_{i=1}^{n} |y_i|^q\Big)^{1/q} \le \|f\| < \infty \qquad \forall n$$

This implies that $y = \{y_n\}$ belongs to ℓ_q and $f(x) = \sum_{n=1}^{\infty} x_n y_n$ as required.

Preliminary Discussion. For a given normed space, it is often useful to have a characterization of its cpct. subsets. The Ascoli–Arzelá Theorem provided such a criterion for the cpct. subsets of $C(X)$ spaces. Next we shall characterize the cpct. substs of the Banach spaces $L_p([0,1])$. To do this, we need some preliminary discussion.

Every fn. $f \in L_p([0,1])$ will be considered defined on all of $\mathbb{R}$ by $f(t) = 0$ if $t \notin [0,1]$. Also, for simplicity, we shall write $\int_a^b f(x)dx$ instead of $\int_{[a,b]} f d\lambda$. If $1 \le p \le \infty$, then for $f \in L_p([0,1])$ and $h > 0$ define

$$f_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} f(x)\,dx$$

for each $t \in [0,1]$. Note that the integral exists, since by Theorem (18) we have $L_p([0,1]) \subseteq L_1([0,1])$.

Each f_h is a continuous fn.. Indeed, if $\lim t_n = t$, then

$$g_n = f\chi_{(t_n-h,\, t_n+h)} \longrightarrow f\chi_{(t-h,\, t+h)}$$

Hence, in view of $|g_n| \le |f|$, the Lebesgue dominated convergence theorem implies that

$$\lim f_h(t_n) = \frac{1}{2h} \lim \int_{t_n-h}^{t_n+h} f(x)\,dx$$
$$= \frac{1}{2h}\int_{t-h}^{t+h} f(x)\,dx$$
$$= f_h(t)$$

In particular, note that since $C[0,1] \subseteq L_p([0,1])$, it follows that $f_h \in L_p([0,1])$ for each $h>0$.

Lemma 26. Let $1 \le p < \infty$ and let $f \in L_p([0,1])$. Then for each $h>0$ the fn. f_h satisfies

a) $|f_h(t)| \le (2h)^{-1/p} \|f\|_p$ for all $t \in [0,1]$ and

b) $\|f_h\|_p \le \|f\|_p$.

pf. If $p>1$, then choose $1<q<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$ and apply Hölder's inequality to get

$$|f_h(t)|^p = \frac{1}{(2h)^p}\left|\int_{t-h}^{t+h} 1.\, f(x)\,dx\right|^p$$
$$\le \frac{1}{(2h)^p}\left(\int_{t-h}^{t+h} 1\,dx\right)^{p/q} \cdot \int_{t-h}^{t+h}|f(x)|^p dx$$
$$= \frac{1}{2h}\int_{t-h}^{t+h}|f(x)|^p dx$$

Thus

$$|f_h(t)|^p \le \frac{1}{2h}\int_{t-h}^{t+h}|f(x)|^p dx \qquad (1)$$

holds for each $1<p<\infty$ and $t\in[0,1]$. Also, (1) is obviously true for $p=1$ and thus, statement (a) follows immediately.

On the other hand, it follows from (1) that

$$\int_0^1 |f_h(t)|^p dt \le \frac{1}{2h}\int_0^1 \left[\int_{t-h}^{t+h}|f(x)|^p dx\right]dt$$
$$= \frac{1}{2h}\int_0^1\left[\int_{-h}^{h}|f(t+y)|^p dy\right]dt \qquad (2)$$

(Here we have used the substitution $x=t+y$) Since $f(t+y)$ is a Lebesgue measurable fn. on $\mathbb{R}^2$, it follows from Tonelli's Theorem (Theorem (2.93)) that

$$\int_0^1\int_{-h}^{h}|f(t+y)|^p\,dy\,dt=\int_{-h}^{h}\int_0^1|f(t+y)|^p\,dt\,dy$$
$$\leq 2h\int_0^1|f(x)|^p\,dx$$

Thus, (2) implies

$$\int_0^1|f_h(t)|^p\,dt\leq\int_0^1|f(x)|^p\,dx$$

so that $\|f_h\|_p\leq\|f\|_p$ holds.

A.N. Kolmogorov characterized the cpct. subsets of $L_p([0,1])$ when $1<p<\infty$. Later, A. Tulajkov proved that the same criterion holds true for the cpct. subsets of $L_1([0,1])$. This result is presented next.

Theorem 27. (Kolmogorov–Tuljakov). Let $1\leq p<\infty$, and let A be a closed and bdd. subset of $L_p([0,1])$. Then the following statements are equivalent:

1. The set A is compact (for the L_p-norm)
2. For each $\epsilon>0$ there exists some $\delta>0$ s.t. $\|f-f_h\|_p<\epsilon$ holds for all $f\in A$ and $0<h<\delta$.

pf. (1) $\Rightarrow$ (2). Let $\epsilon>0$. Since (by Theorem (12)) $C[0,1]$ is dense (for the L_p-norm) in $L_p([0,1])$ and A is compact, it is easy to see that there exist continuous fns. $f_1,\ldots,f_n$ s.t. $A\subseteq\bigcup_{i=1}^{n}B(f_i,\epsilon)$.

By the uniform continuity of each f_i, there exists some $\delta>0$ s.t. $|f_i(t)-f_i(x)|<\varepsilon$ holds for each $1\le i\le n$ whenever $t,x\in[0,1]$ satisfy $|x-t|<\delta$. In particular, if $0<h<\delta$, then

$$|f_i(t)-(f_i)_h(t)|=\frac{1}{2h}\left|\int_{t-h}^{t+h}[f_i(t)-f_i(x)]\,dx\right|\le\varepsilon$$

holds. Thus, $\|f_i-(f_i)_h\|_p\le\varepsilon$.

Now if $f\in A$, then choose $1\le i\le n$ with $f\in B(f_i,\varepsilon)$. By Lemma (26) we have

$$\|f_h-(f_i)_h\|_p\le\|f-f_i\|_p<\varepsilon$$

Therefore

$$\|f-f_h\|_p\le\|f-f_i\|_p+\|f_i-(f_i)_h\|_p+\|(f_i)_h-f_h\|_p<3\varepsilon$$

holds for all $f\in A$ and $0<h<\delta$.

$(2)\Rightarrow(1)$. According to Theorem( ), it is enough to show that A is totally bounded (for the L_p-norm).

To this end, let $\varepsilon>0$. Fix some $h>0$ s.t $\|f-f_h\|_p<\varepsilon$ holds for all $f\in A$. Next choose $M>0$ with $\|f\|_p\le M$ for all $f\in A$. Then by Lemma (26) it follows that

$$|f_h(t)|\le M(2h)^{-1/p}=K$$

holds for all $t\in[0,1]$ and $f\in A$. Set $A_h=\{f_{hh}: f\in A\}$, where

$$f_{hh}(t)=\frac{1}{2h}\int_{t-h}^{t+h}f_h(x)\,dx$$

Clearly, $|f_{hh}(t)|\le K$ holds for all $t\in[0,1]$ and $f\in A$, and hence, A_h is a uniformly bounded set. Next we claim that the set of continuous functions A_h is equicontinuous.

To see this, note that if $f\in A$ and $t<s$, then

$$
\begin{aligned}
|f_{hh}(s)-f_{hh}(t)| &= \frac{1}{2h}\left|\int_{s-h}^{s+h} f_h(x)\,dx - \int_{t-h}^{t+h} f_h(x)\,dx\right| \\
&= \frac{1}{2h}\left|\int_{t+h}^{s+h} f_h(x)\,dx - \int_{t-h}^{s-h} f_h(x)\,dx\right| \\
&\le \frac{1}{2h}\left[\int_{t+h}^{s+h} |f_h(x)|\,dx + \int_{t-h}^{s-h} |f_h(x)|\,dx\right] \\
&\le \frac{1}{2h}\left[2k(s-t)\right] = \frac{k}{h}(s-t)
\end{aligned}
$$

holds, and this shows that A_h is an equicontinuous set.

Now by the Ascoli–Arzelá Theorem, A_h is a totally bdd. subset of $C[0,1]$ (for the sup norm). Choose fns. $f_1, f_2, \dots, f_n \in A$ such that for each $f \in A$ there exists some $1 \le i \le n$ with

$$\|f_{hh} - (f_i)_{hh}\|_\infty < \varepsilon$$

In particular, note that

$$
\begin{aligned}
\|f - f_i\|_p &\le \|f - f_h\|_p + \|f_h - f_{hh}\|_p + \|f_{hh} - f_i\|_p \\
&< 2\varepsilon + \|f_{hh} - f_i\|_p \\
&< 2\varepsilon + \|f_{hh} - (f_i)_{hh}\|_p + \|(f_i)_{hh} - (f_i)_h\|_p + \|(f_i)_h - f_i\|_p \\
&< 5\varepsilon
\end{aligned}
$$

Thus, A is totally bounded (for the L_p-norm), and the proof of the Theorem is finished.