Exercises 13.1

Finding Regions of Integration and Double Integrals

In Exercises 1–10, sketch the region of integration and evaluate the integral.

1. $\displaystyle\int_0^3 \int_0^2 (4 - y^2)\, dy\, dx$

2. $\displaystyle\int_0^3 \int_{-2}^0 (x^2 y - 2xy)\, dy\, dx$

3. $\displaystyle\int_{-1}^0 \int_{-1}^1 (x + y + 1)\, dx\, dy$

4. $\displaystyle\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y)\, dx\, dy$

5. $\displaystyle\int_0^{\pi} \int_0^{x} x \sin y\, dy\, dx$

6. $\displaystyle\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$

7. $\displaystyle\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx\, dy$

8. $\displaystyle\int_1^2 \int_y^{y^2} dx\, dy$

9. $\displaystyle\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

10. $\displaystyle\int_1^4 \int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}}\, dy\, dx$

In Exercises 11–16, integrate f over the given region.

11. $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$

12. $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

13. $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

14. $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

15. $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

16. $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 17–20 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

17. $\displaystyle\int_{-2}^0 \int_v^{-v} 2\, dp\, dv$ (the pv-plane)

18. $\displaystyle\int_0^1 \int_0^{\sqrt{1-s^2}} 8t\, dt\, ds$ (the st-plane)

19. $\displaystyle\int_{-\pi/3}^{\pi/3} \int_0^{\sec t} 3 \cos t\, du\, dt$ (the tu-plane)

20. $\displaystyle\int_0^3 \int_{-2}^{4-2u} \frac{4 - 2u}{v^2}\, dv\, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 21–30, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

21. $\displaystyle\int_0^1 \int_2^{4-2x} dy\, dx$

22. $\displaystyle\int_0^2 \int_{y-2}^0 dx\, dy$

23. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} dx\, dy$

24. $\displaystyle\int_0^1 \int_{1-x}^{1-x^2} dy\, dx$

25. $\displaystyle\int_0^1 \int_1^{e^x} dy\, dx$

26. $\displaystyle\int_0^{\ln 2} \int_{e^y}^2 dx\, dy$

27. $\displaystyle\int_0^{3/2} \int_0^{9-4x^2} 16x\, dy\, dx$

28. $\displaystyle\int_0^2 \int_0^{4-y^2} y\, dx\, dy$

29. $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$

30. $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x\, dy\, dx$

Evaluating Double Integrals

In Exercises 31–40, sketch the region of integration, determine the order of integration, and evaluate the integral.

31. $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$

32. $\displaystyle\int_0^2 \int_x^2 2y^2 \sin xy\, dy\, dx$

33. $\displaystyle\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy$

34. $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4 - y}\, dy\, dx$

35. $\displaystyle\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dx\, dy$

36. $\displaystyle\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dy\, dx$

37. $\displaystyle\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5)\, dx\, dy$

38. $\displaystyle\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4 + 1}$

39. $\displaystyle\iint_R (y - 2x^2)\, dA$ where R is the region inside the square $|x| + |y| = 1$

40. $\displaystyle\iint_R xy\, dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

41. Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

42. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

43. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

44. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

45. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

46. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

47. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

48. Find the volume of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

49. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

50. Find the volume of the solid that is bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm \sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

Integrals over Unbounded Regions

Evaluate the improper integrals in Exercises 51–54 as iterated integrals.

51. $\displaystyle\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\,dy\,dx$

52. $\displaystyle\int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y+1)\,dy\,dx$

53. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy$

54. $\displaystyle\int_0^\infty \int_0^\infty xe^{-(x+2y)}\,dx\,dy$

Approximating Double Integrals

In Exercises 55 and 56, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\,dA \approx \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

55. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

56. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided it lies within R)

Theory and Examples

57. Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \le 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

58. Integrate $f(x, y) = 1/[(x^2 - x)(y - 1)^{2/3}]$ over the infinite rectangle $2 \le x < \infty,\ 0 \le y \le 2$.

59. A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1 \int_0^y (x^2 + y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2 + y^2)\,dx\,dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

60. Evaluate the integral

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)\ dx.$$

(*Hint:* Write the integrand as an integral.)

61. What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\ dA?$$

Give reasons for your answer.

62. What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\ dA?$$

Give reasons for your answer.

63. Is it all right to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

64. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices (0, 1), (2, 0), and (1, 2)? Give reasons for your answer.

65. Prove that $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy = \lim_{b\to\infty} \int_{-b}^b \int_{-b}^b e^{-x^2-y^2}\,dx\,dy$

$$= 4\left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

66. Evaluate the improper integral $\displaystyle\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\,dy\,dx.$

Numerical Evaluation

Use a double-integral evaluator to estimate the values of the integrals in Exercises 67–70.

67. $\displaystyle\int_1^3 \int_1^x \frac{1}{xy}\,dy\,dx$

68. $\displaystyle\int_0^1 \int_0^1 e^{-(x^2+y^2)}\,dy\,dx$

69. $\displaystyle\int_0^1 \int_0^1 \tan^{-1} xy\,dy\,dx$

70. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1 - x^2 - y^2}\,dy\,dx$