

If $z = 0$, then solving Eqs. (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points $(1, 0, 0)$ and $(0, 1, 0)$. This makes sense when you look at Fig. 12.66.

If $x = y$, then Eqs. (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm \frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2} \right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2} \right).$$

But here we need to be careful. While P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1 .

The points on the ellipse closest to the origin are $(1, 0, 0)$ and $(0, 1, 0)$. The point on the ellipse farthest from the origin is P_2 . \square

Exercises 12.9

Two Independent Variables with One Constraint

- Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.
- Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.
- Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ (Fig. 12.58).
- Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.
- Find the points on the curve $xy^2 = 54$ nearest the origin.
- Find the points on the curve $x^2y = 2$ nearest the origin.
- Use the method of Lagrange multipliers to find
 - the minimum value of $x + y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$;
 - the maximum value of xy , subject to the constraint $x + y = 16$.

Comment on the geometry of each solution.

- Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy -plane that are nearest to and farthest from the origin.
- Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is $16\pi \text{ cm}^3$.
- Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a . What is the largest surface area?

- Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.
- Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What is the largest perimeter?
- Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.
- Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.
- The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?
- Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

- Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.
- Find the point on the sphere $x^2 + y^2 + z^2 = 4$ which is farthest from the point $(1, -1, 1)$.

19. Find the minimum distance from the surface $x^2 + y^2 - z^2 = 1$ to the origin.
20. Find the point on the surface $z = xy + 1$ nearest the origin.
21. Find the points on the surface $z^2 = xy + 4$ closest to the origin.
22. Find the point(s) on the surface $xyz = 1$ closest to the origin.
23. Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.
25. Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.
26. Find the largest product the positive numbers x , y , and z can have if $x + y + z^2 = 16$.
27. Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.
28. Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.
29. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.
31. *An example from economics.* In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

32. You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Lagrange Multipliers with Two Constraints

33. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.
34. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.
35. Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.
36. Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.
37. Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.
38. a) Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.
b) Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w .
39. Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.
40. Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

41. *The condition $\nabla f = \lambda \nabla g$ is not sufficient.* While $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the condition $g(x, y) = 0$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.
42. *A least squares plane.* The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k) :

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 1, 1), \quad (1, 0, -1).$$

Find the values of A , B , and C that minimize the sum

$$\sum_{k=1}^4 (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

43. a) Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc -coordinate system is $(r^2/3)^3$.
 b) Using part (a), show that for nonnegative numbers a, b , and c ,

$$(abc)^{1/3} \leq \frac{a+b+c}{3}.$$

That is, the *geometric mean* of three numbers is less than or equal to the *arithmetic mean*.

44. Let a_1, a_2, \dots, a_n be n positive numbers. Find the maximum of $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.

CAS Explorations and Projects

In Exercises 45–50, use a CAS to perform the following steps implementing the method of Lagrange multipliers for finding constrained extrema:

- a) Form the function $h = f - \lambda_1 g_1 - \lambda_2 g_2$, where f is the function to optimize subject to the constraints $g_1 = 0$ and $g_2 = 0$.
 b) Determine all the first partial derivatives of h , including the partials with respect to λ_1 and λ_2 , and set them equal to 0.
 c) Solve the system of equations found in (b) for all the unknowns, including λ_1 and λ_2 .
 d) Evaluate f at each of the solution points found in (c) and select the extreme value subject to the constraints asked for in the exercise.
45. Minimize $f(x, y, z) = xy + yz$ subject to the constraints $x^2 + y^2 - 2 = 0$ and $x^2 + z^2 - 2 = 0$.
 46. Minimize $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 - 1 = 0$ and $x - z = 0$.
 47. Maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + 4z - 5 = 0$ and $4x^2 + 4y^2 - z^2 = 0$.
 48. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$.
 49. Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the constraints $2x - y + z - w - 1 = 0$ and $x + y - z + w - 1 = 0$.
 50. Determine the distance from the line $y = x + 1$ to the parabola $y^2 = x$. (Hint: Let (x, y) be a point on the line and (w, z) a point on the parabola. You want to minimize $(x - w)^2 + (y - z)^2$.)

12.10

Taylor's Formula

This section uses Taylor's formula (Section 8.10) to derive the second derivative test for local extreme values (Section 12.8) and the error formula for linearizations of functions of two independent variables (Section 12.4, Eq. 5). The use of Taylor's formula in these derivations leads to an extension of the formula that provides polynomial approximations of all orders for functions of two independent variables.

The Derivation of the Second Derivative Test

Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$ (Fig. 12.67). Let h and k be increments small enough to put the point $S(a + h, b + k)$ and the line segment joining it to P inside R . We parametrize the segment PS as

$$x = a + th, \quad y = b + tk, \quad 0 \leq t \leq 1.$$

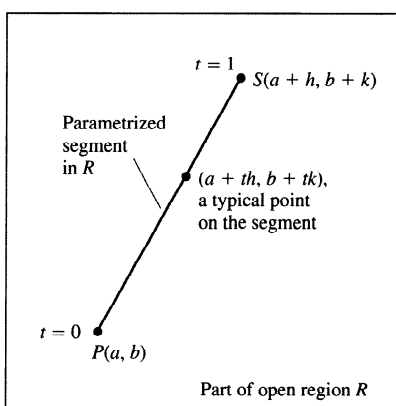
If $F(t) = f(a + th, b + tk)$, the Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable (they have continuous partial derivatives), F' is a differentiable function of t and

$$\begin{aligned} F'' &= \frac{\partial F'}{\partial x} \frac{dx}{dt} + \frac{\partial F'}{\partial y} \frac{dy}{dt} = \frac{\partial}{\partial x} (hf_x + kf_y) \cdot h + \frac{\partial}{\partial y} (hf_x + kf_y) \cdot k \\ &= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}. \end{aligned}$$

Since F and F' are continuous on $[0, 1]$ and F' is differentiable on $(0, 1)$, we can



12.67 We begin the derivation of the second derivative test at $P(a, b)$ by parametrizing a typical line segment from P to a point S nearby.