3

Strong and Weak Forms for One-Dimensional Problems

In this chapter, the strong and weak forms for several one-dimensional physical problems are developed. The strong form consists of the governing equations and the boundary conditions for a physical system. The governing equations are usually partial differential equations, but in the one-dimensional case they become ordinary differential equations. The weak form is an integral form of these equations, which is needed to formulate the finite element method.

In some numerical methods for solving partial differential equations, the partial differential equations can be discretized directly (i.e. written as linear algebraic equations suitable for computer solution). For example, in the finite difference method, one can directly write the discrete linear algebraic equations from the partial differential equations. However, this is not possible in the finite element method.

A roadmap for the development of the finite element method is shown in Figure 3.1. As can be seen from the roadmap, there are three distinct ingredients that are combined to arrive at the discrete equations (also called the system equations; for stress analysis they are called stiffness equations), which are then solved by a computer. These ingredients are

1. the strong form, which consists of the governing equations for the model and the boundary conditions (these are also needed for any other method);
2. the weak form;
3. the approximation functions.

The approximation functions are combined with the weak form to obtain the discrete finite element equations.

Thus, the path from for the governing differential equations is substantially more involved than that for finite difference methods. In the finite difference method, there is no need for a weak form; the strong form is directly converted to a set of discrete equations. The need for a weak form makes the finite element method more challenging intellectually. A number of subtle points, such as the difference between various boundary conditions, must be learned for intelligent use of the method. In return for this added complexity, however, finite element methods can much more readily deal with the complicated shapes that need to be analyzed in engineering design.

To demonstrate the basic steps in formulating the strong and weak forms, we will consider axially loaded elastic bars and heat conduction problems in one dimension. The strong forms for these problems will be developed along with the boundary conditions. Then we will develop weak forms for these problems and show that they are equivalent to the strong forms. We will also examine various degrees of continuity, or smoothness, which will play an important role in developing finite element methods.

A First Course in Finite Elements J. Fish and T. Belytschko

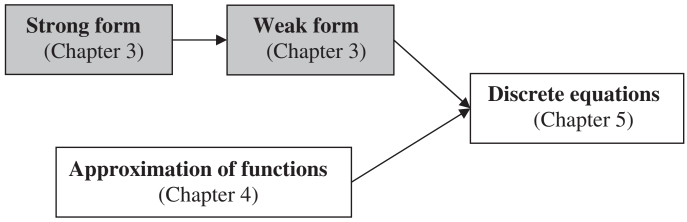

Figure 3.1 Roadmap for the development of the finite element method.

The weak form is the most intellectually challenging part in the development of finite elements, so a student may encounter some difficulties in understanding this concept; it is probably different from anything else that he has seen before in engineering analysis. However, an understanding of these procedures and the implications of solving a weak form are crucial to understanding the character of finite element solutions. Furthermore, the procedures are actually quite simple and repetitive, so once it is understood for one strong form, the procedures can readily be applied to other strong forms.

3.1 THE STRONG FORM IN ONE-DIMENSIONAL PROBLEMS

3.1.1 The Strong Form for an Axially Loaded Elastic Bar

Consider the static response of an elastic bar of variable cross section such as shown in Figure 3.2. This is an example of a problem in *linear stress analysis* or linear elasticity, where we seek to find the stress distribution $\sigma(x)$ in the bar. The stress will results from the deformation of the body, which is characterized by the displacements of points in the body, $u(x)$. The displacement results in a strain denoted by $\varepsilon(x)$; strain is a dimensionless variable. As shown in Figure 3.2, the bar is subjected to a body force or distributed loading $b(x)$. The body force could be due to gravity (if the bar were placed vertically instead of horizontally as shown), a magnetic force or a thermal stress; in the one-dimensional case, we will consider body force per unit length, so the units of $b(x)$ are force/length. In addition, loads can be prescribed at the ends of the bar, where the displacement is not prescribed; these loads are called tractions and denoted by $\bar{t}$. These loads are in units of force per area, and when multiplied by the area, give the applied force.

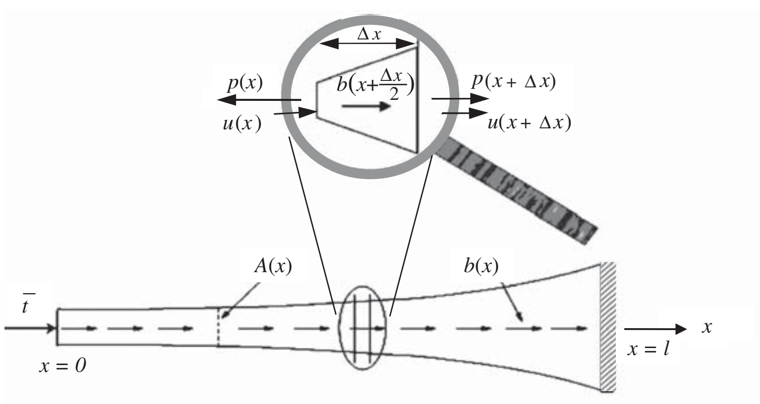

Figure 3.2 A one-dimensional stress analysis (elasticity) problem.

The bar must satisfy the following conditions:

1. It must be in equilibrium.
2. It must satisfy the elastic stress–strain law, known as Hooke's law: $\sigma(x) = E(x)\varepsilon(x)$.
3. The displacement field must be compatible.
4. It must satisfy the strain–displacement equation.

The differential equation for the bar is obtained from equilibrium of internal force $p(x)$ and external force $b(x)$ acting on the body in the axial (along the x-axis) direction. Consider equilibrium of a segment of the bar along the x-axis, as shown in Figure 3.2. Summing the forces in the x-direction gives

$$-p(x) + b\left(x + \frac{\Delta x}{2}\right)\Delta x + p(x + \Delta x) = 0.$$

Rearranging the terms in the above and dividing by Δx, we obtain

$$\frac{p(x + \Delta x) - p(x)}{\Delta x} + b\left(x + \frac{\Delta x}{2}\right) = 0.$$

If we take the limit of the above equation as $\Delta x \to 0$, the first term is the derivative $\mathrm{d}p/\mathrm{d}x$ and the second term becomes $b(x)$. Therefore, the above can be written as

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} + b(x) = 0. \tag{3.1}$$

This is the equilibrium equation expressed in terms of the internal force p. The stress is defined as the force divided by the cross-sectional area:

$$\sigma(x) = \frac{p(x)}{A(x)}, \quad \text{so} \quad p(x) = A(x)\sigma(x). \tag{3.2}$$

The strain–displacement (or kinematical) equation is obtained by applying the engineering definition of strain that we used in Chapter 2 for an infinitesimal segment of the bar. The elongation of the segment is given by $u(x + \Delta x) - u(x)$ and the original length is Δx; therefore, the strain is given by

$$\varepsilon(x) = \frac{\text{elongation}}{\text{original length}} = \frac{u(x + \Delta x) - u(x)}{\Delta x}.$$

Taking the limit of the above as $\Delta x \to 0$, we recognize that the right right-hand side is the derivative of $u(x)$. Therefore, the strain–displacement equation is

$$\varepsilon(x) = \frac{\mathrm{d}u}{\mathrm{d}x}. \tag{3.3}$$

The stress–strain law for a linear elastic material is Hooke's law, which we already saw in Chapter 2:

$$\sigma(x) = E(x)\varepsilon(x), \tag{3.4}$$

where E is Young's modulus.

Substituting (3.3) into (3.4) and the result into (3.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b=0, \qquad 0<x<l. \tag{3.5}$$

The above is a second-order ordinary differential equation. In the above equation, $u(x)$ is the dependent variable, which is the unknown function, and x is the independent variable. In (3.5) and thereafter the dependence of functions on x will be often omitted. The differential equation (3.5) is a specific form of the equilibrium equation (3.1). Equation (3.1) applies to both linear and nonlinear materials whereas (3.5) assumes linearity in the definition of the strain (3.3) and the stress–strain law (3.4). Compatibility is satisfied by requiring the displacement to be continuous. More will be said later about the degree of smoothness, or continuity, which is required.

To solve the above differential equation, we need to prescribe boundary conditions at the two ends of the bar. For the purpose of illustration, we will consider the following specific boundary conditions: at $x=l$, the displacement, $u(x=l)$, is prescribed; at $x=0$, the force per unit area, or traction, denoted by $\bar{t}$, is prescribed. These conditions are written as

$$\begin{aligned}\sigma(0)&=\left(E\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0}=\frac{p(0)}{A(0)}\equiv-\bar{t},\\ u(l)&=\bar{u}.\end{aligned} \tag{3.6}$$

Note that the superposed bars designate denote a prescribed boundary value in the above and throughout this book.

The traction $\bar{t}$ has the same units as stress (force/area), but its sign is positive when it acts in the positive x-direction regardless of which face it is acting on, whereas the stress is positive in tension and negative in compression, so that on a negative face a positive stress corresponds to a negative traction; this will be clarified in Section 3.5. Note that either the load or the displacement can be specified at a boundary point, but not both.

The governing differential equation (3.5) along with the boundary conditions (3.6) is called the *strong form* of the problem. To summarize, the strong form consists of the governing equation and the boundary conditions, which for this example are

$$\begin{aligned}&\text{(a)}\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b=0 \quad \text{on} \quad 0<x<l,\\ &\text{(b)}\quad \sigma(x=0)=\left(E\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0}=-\bar{t},\\ &\text{(c)}\quad u(x=l)=\bar{u}.\end{aligned} \tag{3.7}$$

It should be noted that $\bar{t}$, $\bar{u}$ and b are given. They are the data that describe the problem. The unknown is the displacement $u(x)$.

3.1.2 The Strong Form for Heat Conduction in One Dimension[1]

Heat flow occurs when there is a temperature difference within a body or between the body and its surrounding medium. Heat is transferred in the form of conduction, convection and thermal radiation. The heat flow through the wall of a heated room in the winter is an example of conduction. On the other hand, in convective heat transfer, the energy transfer to the body depends on the temperature difference between the surface of the body and the surrounding medium. In this Section, we will focus on heat conduction. A discussion involving convection is given in Section 3.5.

[1] Reccommended for Science and Engineering Track.

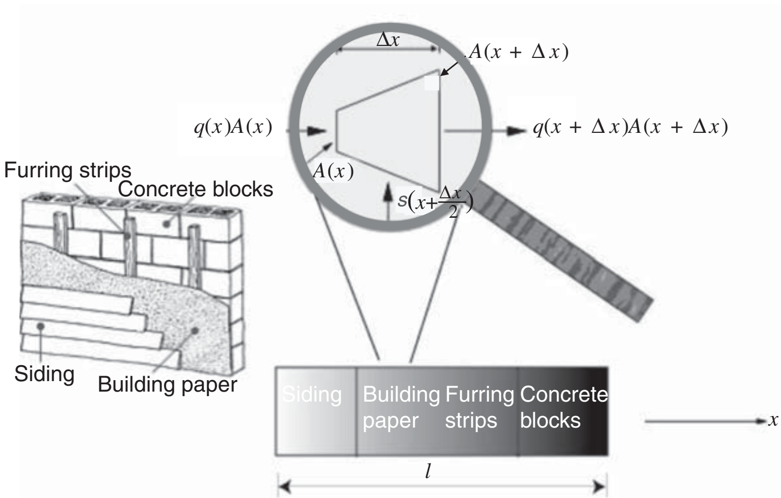

Figure 3.3 A one-dimensional heat conduction problem.

Consider a cross section of a wall of thickness l as shown in Figure 3.3. Our objective is to determine the temperature distribution. Let $A(x)$ be the area normal to the direction of heat flow and let $s(x)$ be the heat generated per unit thickness of the wall, denoted by l. This is often called a heat source. A common example of a heat source is the heat generated in an electric wire due to resistance. In the one-dimensional case, the rate of heat generation is measured in units of energy per time; in SI units, the units of energy are joules (J) per unit length (meters, m) and time (seconds, s). Recall that the unit of power is watts ($1\ \text{W} = 1\ \text{J s}^{-1}$). A heat source $s(x)$ is considered positive when heat is generated, i.e. added to the system, and negative when heat is withdrawn from the system. Heat flux, denoted by $q(x)$, is defined as a the rate of heat flow across a surface. Its units are heat rate per unit area; in SI units, W m^{-2}. It is positive when heat flows in the positive x-direction. We will consider a steady-state problem, i.e. a system that is not changing with time.

To establish the differential equation that governs the system, we consider energy balance (or conservation of energy) in a control volume of the wall. Energy balance requires that the rate of heat energy (qA) that is generated in the control volume must equal the heat energy leaving the control volume, as the temperature, and hence the energy in the control volume, is constant in a steady-state problem. The heat energy leaving the control volume is the difference between the flow in at on the left-hand side, qA, and the flow out on the right-hand side, $q(x + \Delta x)A(x + \Delta x)$. Thus, energy balance for the control volume can be written as

$$\underbrace{s(x + \Delta x/2)\Delta x}_{\text{heat generated}} + \underbrace{q(x)A(x)}_{\text{heat flow in}} - \underbrace{q(x + \Delta x)A(x + \Delta x)}_{\text{heat flow out}} = 0.$$

Note that the heat fluxes are multiplied by the area to obtain a the heat rate, whereas the source s is multiplied by the length of the segment. Rearranging terms in the above and dividing by Δx, we obtain

$$\frac{q(x + \Delta x)A(x + \Delta x) - q(x)A(x)}{\Delta x} = s(x + \Delta x/2).$$

If we take the limit of the above equation as $\Delta x \to 0$, the first term coincides with the derivative $\mathrm{d}(qA)/\mathrm{d}x$ and the second term reduces to $s(x)$. Therefore, the above can be written as

$$\frac{\mathrm{d}(qA)}{\mathrm{d}x} = s. \tag{3.8}$$

The constitutive equation for heat flow, which relates the heat flux to the temperature, is known as Fourier's law and is given by

$$q = -k\frac{\mathrm{d}T}{\mathrm{d}x}, \tag{3.9}$$

where T is the temperature and k is the thermal conductivity (which must be positive); in SI units, the dimensions of thermal conductivity are $\mathrm{W\,m^{-1}\,{}^{o}C^{-1}}$. A negative sign appears in (3.9) because the heat flows from high (hot) to low temperature (cold), i.e. opposite to the direction of the gradient of the temperature field.

Inserting (3.9) into (3.8) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + s = 0, \qquad 0 < x < l. \tag{3.10}$$

When Ak is constant, we obtain

$$Ak\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + s = 0, \qquad 0 < x < l. \tag{3.11}$$

At the two ends of the problem domain, either the flux or the temperature must be prescribed; these are the boundary conditions. We consider the specific boundary conditions of the prescribed temperature $\overline{T}$ at $x = l$ and prescribed flux $\overline{q}$ at $x = 0$. The prescribed flux $\overline{q}$ is positive if heat (energy) flows out of the bar, i.e. $q(x = 0) = -\overline{q}$. The strong form for the heat conduction problem is then given by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + s = 0 \quad \text{on} \quad 0 < x < l,\\
&-q = k\frac{\mathrm{d}T}{\mathrm{d}x} = \overline{q} \quad \text{on} \quad x = 0,\\
&T = \overline{T} \quad \text{on} \quad x = l.
\end{aligned} \tag{3.12}$$

3.1.3 Diffusion in One Dimension[2]

Diffusion is a process where a material is transported by atomic motion. Thus, in the absence of the motion of a fluid, materials in the fluid are diffused throughout the fluid by atomic motion. Examples are the diffusion of perfume into a room when a heavily perfumed person walks in, the diffusion of contaminants in a lake and the diffusion of salt into a glass of water (the water will get salty by diffusion even in the absence of fluid motion).

Diffusion also occurs in solids. One of the simplest forms of diffusion in solids occurs when two materials come in contact with each other. There are two basic mechanisms for diffusion in solids: vacancy diffusion and interstitial diffusion. Vacancy diffusion occurs primarily when the diffusing atoms are of a similar size. A diffusing atom requires a vacancy in the other solid for it to move. Interstitial diffusion, schematically depicted in Figure 3.4, occurs when a diffusing atom is small enough to move between the atoms in the other solid. This type of diffusion requires no vacancy defects.

Let c be the concentration of diffusing atoms with the dimension of atoms m^{-3}. The flux of atoms, $q(x)$ (atoms $\mathrm{m}^{-2}\ \mathrm{s}^{-1}$), is positive in the direction from higher to lower concentration. The relationship between flux and concentration is known as Fick's first law, which is given as

$$q = -k\frac{\mathrm{d}c}{\mathrm{d}x},$$

[2] Recommended for Science and Engineering Track.

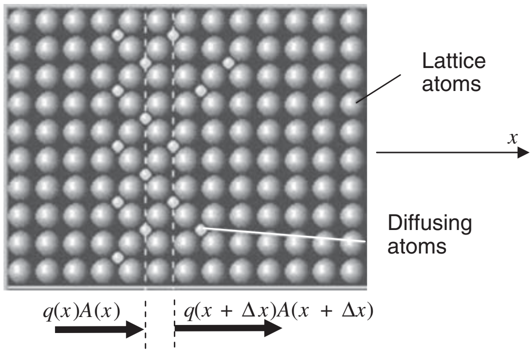

Figure 3.4 Interstitial diffusion in an atomic lattice.

where k is the diffusion coefficient, $\mathrm{m}^{-2}\ \mathrm{s}^{-1}$. The balance equation for steady-state diffusion can be developed from Figure 3.4 by the same procedures that we used to derive the heat conduction equation by imposing conservation of each species of atoms and Fick's law. The equations are identical in structure to the steady-state heat conduction equation and differ only in the constants and variables:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}c}{\mathrm{d}x}\right) = 0 \quad \text{on} \quad 0 < x < l.$$

3.2 THE WEAK FORM IN ONE DIMENSION

To develop the finite element equations, the partial differential equations must be restated in an integral form called the *weak form*. A weak form of the differential equations is equivalent to the governing equation and boundary conditions, i.e. the strong form. In many disciplines, the weak form has specific names; for example, it is called the principle of virtual work in stress analysis.

To show how weak forms are developed, we first consider the strong form of the stress analysis problem given in (3.7). We start by multiplying the governing equation (3.7a) and the traction boundary condition (3.7b) by an *arbitrary* function $w(x)$ and integrating over the domains on which they hold: for the governing equation, the pertinent domain is the interval $[0, l]$, whereas for the traction boundary condition, it is the cross-sectional area at $x = 0$ (no integral is needed because this condition only holds only at a point, but we do multiply by the area A). The resulting two equations are

$$\begin{aligned} &\text{(a)} \quad \int_0^l w\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b\right]\mathrm{d}x = 0 \qquad \forall w, \\ &\text{(b)} \quad \left(wA\left(E\frac{\mathrm{d}u}{\mathrm{d}x} + \bar{t}\right)\right)_{x=0} = 0 \qquad \forall w. \end{aligned} \tag{3.13}$$

The function $w(x)$ is called the *weight function*; in more mathematical treatments, it is also called the test function. In the above, $\forall w$ denotes that $w(x)$ is an arbitrary function, i.e. (3.13) has to hold for all functions

$w(x)$. The arbitrariness of the weight function is crucial, as otherwise a weak form is not equivalent to the strong form (see Section 3.7). The weight function can be thought of as an enforcer: whatever it multiplies is enforced to be zero by its arbitrariness.

You might have noticed that we did not enforce the boundary condition on the displacement in (3.13) by the weight function. It will be seen that it is easy to construct trial or candidate solutions $u(x)$ that satisfy this displacement boundary condition, so we will assume that all candidate solutions of Equation (3.13) satisfy this boundary condition. Similarly, you will shortly see that it is convenient to have all weight functions satisfy

$$w(l) = 0. \tag{3.14}$$

So we impose this restriction on the set of weight functions.

As you will see, in solving a weak form, a set of *admissible solutions* $u(x)$ that satisfy certain conditions is considered. These solutions are called *trial solutions*. They are also called candidate solutions.

One could use (3.13) to develop a finite element method, but because of the second derivative of $u(x)$ in the expression, very smooth trial solutions would be needed; such smooth trial solutions would be difficult to construct in more than one dimension. Furthermore, the resulting stiffness matrix would not be symmetric, because the first integral is not symmetric in $w(x)$ and $u(x)$. For this reason, we will transform (3.13) into a form containing only first derivatives. This will lead to a symmetric stiffness matrix, allow us to use less smooth solutions and will simplify the treatment of the traction boundary condition.

For convenience, we rewrite (3.13a) in the equivalent form:

$$\int_0^l w \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x + \int_0^l wb\,\mathrm{d}x = 0 \qquad \forall w. \tag{3.15}$$

To obtain a weak form in which only first derivatives appear, we first recall the rule for taking the derivative of a product:

$$\frac{\mathrm{d}}{\mathrm{d}x}(wf) = w\frac{\mathrm{d}f}{\mathrm{d}x} + f\frac{\mathrm{d}w}{\mathrm{d}x} \Rightarrow w\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(wf) - f\frac{\mathrm{d}w}{\mathrm{d}x}.$$

Integrating the above equation on the right over the domain $[0, l]$, we obtain

$$\int_0^l w\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \int_0^l \frac{\mathrm{d}}{\mathrm{d}x}(wf)\mathrm{d}x - \int_0^l f\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x.$$

The fundamental theorem of calculus states that the integral of a derivative of a function is the function itself. This theorem enables us to replace the first integral on the right-hand side by a set of boundary values and rewrite the equation as

$$\int_0^l w\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = (wf)|_0^l - \int_0^l f\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x \equiv (wf)_{x=l} - (wf)_{x=0} - \int_0^l f\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x. \tag{3.16}$$

The above formula is known as *integration by parts*. We will find that integration by parts is useful whenever we relate strong forms to weak forms.

To apply the integration by parts formula to (3.15), let $f = AE(\mathrm{d}u/\mathrm{d}x)$. Then (3.16) can be written as

$$\int_0^l w\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x = \left(wAE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Big|_0^l - \int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{3.17}$$

Using (3.17), (3.15) can be written as follows:

$$\left(wA\underbrace{E\frac{\mathrm{d}u}{\mathrm{d}x}}_{\sigma}\right)\Bigg|_0^l - \int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int_0^l wb\,\mathrm{d}x = 0 \qquad \forall w \text{ with } w(l) = 0. \tag{3.18}$$

We note that by the stress–strain law and strain–displacement equations, the underscored boundary term is the stress σ (as shown), so the above can be rewritten as

$$(wA\sigma)_{x=l} - (wA\sigma)_{x=0} - \int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int_0^l wb\,\mathrm{d}x = 0 \qquad \forall w \text{ with } w(l) = 0.$$

The first term in the above vanishes because of (3.14): this is why it is convenient to construct weight functions that vanish on prescribed displacement boundaries. Though the term looks quite insignificant, it would lead to loss of symmetry in the final equations.

From (3.13b), we can see that the second term equals $(wA\bar{t})_{x=0}$, so the above equation becomes

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = (wA\bar{t})_{x=0} + \int_0^l wb\,\mathrm{d}x \qquad \forall w \text{ with } w(l) = 0. \tag{3.19}$$

Let us recapitulate what we have done. We have multiplied the governing equation and traction boundary by an arbitrary, smooth weight function and integrated the products over the domains where they hold. We have added the expressions and transformed the integral so that the derivatives are of lower order.

We now come to the crux of this development: *We state that the trial solution that satisfies the above for all smooth* $w(x)$ *with* $w(l) = 0$ *is the solution*. So the solution is obtained as follows:

Find $u(x)$ among the *smooth* functions that satisfy $u(l) = \bar{u}$ such that

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = (wA\bar{t})_{x=0} + \int_0^l wb\,\mathrm{d}x \qquad \forall w \text{ with } w(l) = 0. \tag{3.20}$$

The above is called the *weak form*. The name originates from the fact that solutions to the weak form need not be as smooth as solutions of the strong form, i.e. they have weaker continuity requirements. This is explained later.

Understanding how a solution to a differential equation can be obtained by this rather abstract statement, and why it is a useful solution, is not easy. It takes most students considerable thought and experience to comprehend the process. To facilitate this, we will give two examples in which a solution is obtained to a specific problem.

We will show in the next section that the weak form (3.20) is equivalent to the equilibrium equation (3.7a) and traction boundary condition (3.7b). In other words, the trial solution *that satisfies (3.20) is the solution of the strong form*. The proof of this statement in Section 3.4 is a crucial step in the theory of finite elements. In getting to (3.19), we have gone through a set of mathematical steps that are correct, but we have no basis for saying that the solution to the weak form is a solution of the strong form unless we can show that (3.20) implies (3.7).

It is important to remember that the trial solutions $u(x)$ must satisfy the displacement boundary conditions (3.7c). Satisfying the displacement boundary condition is essential for the trial solutions, so these boundary conditions are often called *essential boundary conditions*. We will see in Section 3.4 that the traction boundary conditions emanate naturally from the weak form (3.20), so trial solutions need not be constructed to satisfy the traction boundary conditions. Therefore, these boundary conditions are called *natural boundary conditions*. Additional smoothness requirements on the trial solutions will be discussed in Sections 3.3 and 3.9.

A trial solution that is smooth and satisfies the essential boundary conditions is called *admissible*. Similarly, a weight function that is smooth and vanishes on essential boundaries is *admissible*. When weak forms are used to solve a problem, the trial solutions and weight functions must be admissible.

Note that in (3.20), the integral is symmetric in w and u. This will lead to a symmetric stiffness matrix. Furthermore, the highest order derivative that appears in the integral is of first order: this will have important ramifications on the construction of finite element methods.

3.3 CONTINUITY

Although we have now developed the weak form, we still have not specified how smooth the weight functions and trial solutions must be. Before examining this topic, we will examine the concept of smoothness, i.e. continuity. A function is called a C^n function if its derivatives of order j for $0 \le j \le n$ exist and are continuous functions in the entire domain. We will be concerned mainly with C^0, C^{-1} and C^1 functions. Examples of these are illustrated in Figure 3.5. As can be seen, a C^0 function is piecewise continuously differentiable, i.e. its first derivative is continuous except at selected points. The derivative of a C^0 function is a C^{-1} function. So for example, if the displacement is a C^0 function, the strain is a C^{-1} function. Similarly, if a temperature field is a C^0 function, the flux is a C^{-1} function if the conductivity is C^0. In general, the derivative of a C^n function is C^{n-1}.

The degree of smoothness of C^0, C^{-1} and C^1 functions can be remembered by some simple mnemonic devices. As can be seen from Figure 3.5, a C^{-1} function can have both kinks and jumps. A C^0 function has no jumps, i.e. discontinuities, but it has kinks. A C^1 function has no kinks or jumps. Thus, there is a progression of smoothness as the superscript increases that is summarized in Table 3.1. In the literature, jumps in the function are often called strong discontinuities, whereas kinks are called weak discontinuities.

It is worth mentioning that CAD databases for smooth surfaces usually employ functions that are at least C^1; the most common are spline functions. Otherwise, the surface would possess kinks stemming from the function description, e.g. in a car there would be kinks in the sheet metal wherever C^1 continuity is not observed. We will see that finite elements usually employ C^0 functions.

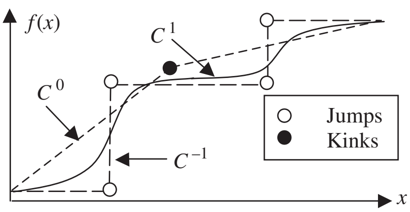

Figure 3.5 Examples of C^{-1}, C^0 and C^1 functions.

Table 3.1 Smoothness of functions.

Smoothness	Kinks	Jumps	Comments
C^{-1}	Yes	Yes	Piecewise continuous
C^0	Yes	No	Piecewise continuously differentiable
C^1	No	No	Continuously differentiable

3.4 THE EQUIVALENCE BETWEEN THE WEAK AND STRONG FORMS

In the previous section, we constructed the weak form from the strong form. To show the equivalence between the two, we will now show the converse: the weak form implies the strong form. This will insure that when we solve the weak form, then we have a solution to the strong form.

The proof that the weak form implies the strong form can be obtained by simply reversing the steps by which we obtained the weak form. So instead of using integration by parts to eliminate the second derivative of $u(x)$, we reverse the formula to obtain an integral with a higher derivative and a boundary term. For this purpose, interchange the terms in (3.17), which gives

$$\int_0^l \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \left(wAE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Big|_0^l - \int_0^l w \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x.$$

Substituting the above into (3.20) and placing the integral terms on the left-hand side and the boundary terms on the right-hand side gives

$$\int_0^l w\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b\right]\mathrm{d}x + wA(\bar{t}+\sigma)_{x=0} = 0 \qquad \forall w \text{ with } w(l) = 0. \tag{3.21}$$

The key to making the proof possible is the arbitrariness of $w(x)$. It can be assumed to be *anything we need* in order to prove the equivalence. Our selection of $w(x)$ is guided by having seen this proof before – What we will do is not immediately obvious, but you will see it works! First, we let

$$w = \psi(x)\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b\right], \tag{3.22}$$

where $\psi(x)$ is smooth, $\psi(x) > 0$ on $0 < x < l$ and $\psi(x)$ vanishes on the boundaries. An example of a function satisfying the above requirements is $\psi(x) = x(l-x)$. Because of how $\psi(x)$ is constructed, it follows that $w(l) = 0$, so the requirement that $w = 0$ on the prescribed displacement boundary, i.e. the essential boundary, is met.

Inserting (3.22) into (3.21) yields

$$\int_0^l \psi\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + b\right]^2 \mathrm{d}x = 0. \tag{3.23}$$

The boundary term vanishes because we have constructed the weight function so that $w(0) = 0$. As the integrand in (3.23) is the product of a positive function and the square of a function, it must be positive at every point in the problem domain. So the only way the equality in (3.23) is met is if the integrand is zero at every point! Hence, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b=0, \qquad 0<x<l, \tag{3.24}$$

which is precisely the differential equation in the strong form, (3.7a).

From (3.24) it follows that the integral in (3.21) vanishes, so we are left with

$$(wA(\bar{t}+\sigma))_{x=0}=0 \qquad \forall w \text{ with } w(l)=0. \tag{3.25}$$

As the weight function is arbitrary, we select it such that $w(0)=1$ and $w(l)=0$. It is very easy to construct such a function, for example, $(l-x)/l$ is a suitable weight function; any smooth function that you can draw on the interval [0, l] that vanishes at $x=l$ is also suitable.

As the cross-sectional area $A(0)\neq 0$ and $w(0)\neq 0$, it follows that

$$\sigma=-\bar{t} \quad \text{at} \quad x=0, \tag{3.26}$$

which is the natural (prescribed traction) boundary condition, Equation (3.7b).

The last remaining equation of the strong form, the displacement boundary condition (3.7c), is satisfied by all trial solutions by construction, i.e. as can be seen from (3.20) we required that $u(l)=\bar{u}$. Therefore, we can conclude that *the trial solution that satisfies the weak form satisfies the strong form.*

Another way to prove the equivalence to the strong form starting from (3.20) that is more instructive about the character of the equivalence is as follows. We first let

$$r(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b \quad \text{for} \quad 0<x<l$$

and

$$r_0=A(0)\sigma(0)+\bar{t}.$$

The variable $r(x)$ is called the residual; $r(x)$ is the error in Equation (3.7a) and r_0 is the error in the traction boundary condition (3.7b). Note that when $r(x)=0$, the equilibrium equation (3.7a) is met exactly and when $r_0=0$ the traction boundary condition (3.7b) is met exactly.

Equation (3.20) can then be written as

$$\int_0^l w(x)r(x)\,\mathrm{d}x+w(0)r_0=0 \qquad \forall w \text{ with } w(l)=0. \tag{3.27}$$

We now prove that $r(x)=0$ by contradiction. Assume that at some point $0<a<l$, $r(a)\neq 0$. Then assuming $r(x)$ is smooth, it must be nonzero in a small neighborhood of $x=a$ as shown in Figure 3.6(a). We have complete latitude in the construction of $w(x)$ as it is an arbitrary smooth function. So we construct it as shown in Figure 3.6(b). Equation (3.27) then becomes

$$\int_0^l w(x)r(x)\,\mathrm{d}x+w(0)r_0\approx\frac{1}{2}\,r(a)\delta\neq 0.$$

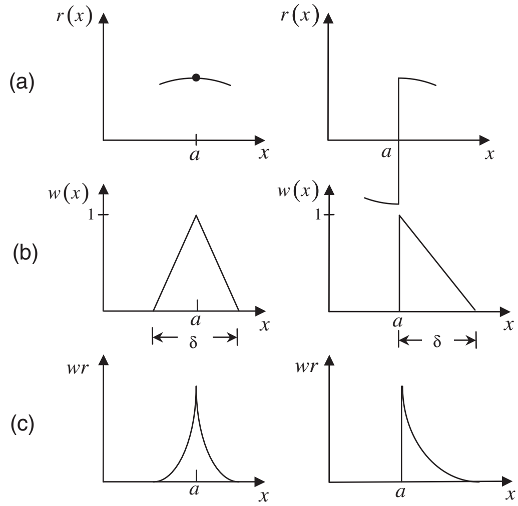

Figure 3.6 Illustration of the equivalence between the weak and strong forms: (a) an example of the residual function; (b) choice of the weight function and (c) product of residual and weight functions. On the left, the procedure is shown for a C° function; on the right for a C^{-1} function.

The above implies that (3.27) is violated, so by contradiction $r(a)$ cannot be nonzero. This can be repeated at any other point in the open interval $0 < x < l$, so it follows that $r(x) = 0$ for $0 < x < l$, i.e. the governing equation (3.27) is met. We now let $w(0) = 1$; as the integral vanishes because $r(x) = 0$ for $0 < x < l$, it follows from (3.27) that $r_0 = 0$ and hence the traction boundary condition is also met.

We can see from the above why we have said that multiplying the equation, or to be more precise the residual, by the weight function enforces the equation: because of the arbitrariness of the weight function, anything it multiplies must vanish. The proofs of the equivalence of the strong and weak forms hinge critically on the weak form holding for *any smooth function*. In the first proof (Equations (3.7)–(3.20)), we selected a special arbitrary weight function (based on foresight as to how the proof would evolve) that has to be smooth, whereas in the second proof, we used the arbitrariness and smoothness directly. The weight function in Figure 3.6(b) may not appear particularly smooth, but it is as smooth as we need for this proof.

Example 3.1

Develop the weak form for the strong form:

$$\begin{aligned}
&\text{(a)} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{Ex}\right) + 10Ax = 0, \qquad 0 < x < 2,\\
&\text{(b)} \quad u_{x=0} \equiv u(0) = 10^{-4},\\
&\text{(c)} \quad \sigma_{x=2} = \left(E\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=2} = 10.
\end{aligned} \tag{3.28}$$

Equation (3.28c) is a condition on the derivative of $u(x)$, so it is a natural boundary condition; (3.28b) is a condition on $u(x)$, so it is an essential boundary condition. Therefore, as the weight function must vanish on the essential boundaries, we consider all smooth weight functions $w(x)$ such that $w(0) = 0$. The trial solutions $u(x)$ must satisfy the essential boundary condition $u(0) = 10^{-4}$.

We start by multiplying the governing equation and the natural boundary condition over the domains where they hold by an arbitrary weight function:

$$\begin{aligned}
&\text{(a)} \quad \int_0^2 \left[w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right) + 10Ax \right] \mathrm{d}x = 0 \qquad \forall w(x),\\
&\text{(b)} \quad \left(wA\left(E\frac{du}{dx} - 10\right)\right)_{x=2} = 0 \qquad \forall w(2).
\end{aligned} \tag{3.29}$$

Next we integrate the first equation in the above by parts, exactly as we did in going from (3.13a) to (3.17):

$$\int_0^2 \left[w\left(\frac{\mathrm{d}}{\mathrm{d}x} AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\right] \mathrm{d}x = \left(wAE\frac{\mathrm{d}u}{\mathrm{d}x}\right)\Bigg|_{x=0}^{x=2} - \int_0^2 \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x. \tag{3.30}$$

We have constructed the weight functions so that $w(0) = 0$; therefore, the first term on the RHS of the above vanishes at $x = 0$. Substituting (3.30) into (3.29a) gives

$$-\int_0^2 AE\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int_0^2 10wAx\,\mathrm{d}x + \left(wAE\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=2} = 0 \qquad \forall w(x) \text{ with } w(0) = 0. \tag{3.31}$$

Substituting (3.29b) into the last term of (3.31) gives (after a change of sign)

$$\int_0^2 AE\frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x - \int_0^2 10wAx\,\mathrm{d}x - 10(wA)_{x=2} = 0 \qquad \forall w(x) \text{ with } w(0) = 0. \tag{3.32}$$

Thus, the weak form is as follows: find $u(x)$ such that for all smooth $u(x)$ with $u(0) = 10^{-4}$, such that (3.32) holds for all smooth $w(x)$ with $w(0) = 0$.

Example 3.2

Develop the weak form for the strong form:

$$\begin{aligned}
&\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0 \quad \text{on} \quad 1 < x < 3,\\
&\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=1} = 2, \qquad u(3) = 1.
\end{aligned} \tag{3.33}$$

The conditions on the weight function and trial solution can be inferred from the boundary conditions. The boundary point $x = 1$ is a natural boundary as the derivative is prescribed there, whereas the boundary $x = 3$ is an essential boundary as the solution itself is prescribed. Therefore, we require that $w(3) = 0$ and that the trial solution satisfies the essential boundary condition $u(3) = 1$.

Next we multiply the governing equation by the weight function and integrate over the problem domain; similarly, we multiply the natural boundary condition by the weight function, which yields

$$\begin{aligned}
&\text{(a)} \quad \int_1^3 w\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{d}x = 0,\\
&\text{(b)} \quad \left(w\left(\frac{\mathrm{d}u}{\mathrm{d}x} - 2\right)\right)_{x=1} = 0.
\end{aligned} \tag{3.34}$$

Integration by parts of the integrand in (3.34a) gives

$$\int_1^3 w \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x = \left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=3} - \left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=1} - \int_1^3 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x. \tag{3.35}$$

As $w(3) = 0$, the first term on the RHS in the above vanishes. Substituting (3.35) into (3.34a) gives

$$-\int_1^3 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x - \left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=1} = 0. \tag{3.36}$$

Adding (3.34b) to (3.36) gives

$$\int_1^3 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x + 2w(1) = 0. \tag{3.37}$$

So the weak form is: find a smooth function $u(x)$ with $u(3) = 1$ for which (3.37) holds for all smooth $w(x)$ with $w(3) = 0$.

To show that the weak form implies the strong form, we reverse the preceding steps. Integration by parts of the first term in (3.37) gives

$$\int_1^3 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)\Bigg|_1^3 - \int_1^3 w \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x. \tag{3.38}$$

Next we substitute (3.38) into (3.37), giving

$$\left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=3} - \left(w \frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=1} - \int_1^3 w \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x + 2w(1) = 0. \tag{3.39}$$

Since on the essential boundary, the weight function vanishes, i.e. $w(3) = 0$, the first term in the above drops out. Collecting terms and changing signs give

$$\int_1^3 w \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}\,\mathrm{d}x + \left(w\left(\frac{\mathrm{d}u}{\mathrm{d}x} - 2\right)\right)_{x=1} = 0. \tag{3.40}$$

We now use the same arguments as Equations (3.22)–(3.26). As $w(x)$ is arbitrary, let

$$w = \psi(x)\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2},$$

where

$$\psi(x) = \begin{cases} 0, & x = 1, \\ > 0, & 1 < x < 3, \\ 0, & x = 3. \end{cases}$$

Then (3.40) becomes

$$\int_1^3 \psi(x)\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right)^2 \mathrm{d}x = 0.$$

As the integrand is positive in the interval $[1, 3]$, it follows that the only way that the integrand can vanish is if

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} = 0 \quad \text{for} \quad 1 < x < 3,$$

which is the differential equation in the strong form (3.33).

Now let $w(x)$ be a smooth function that vanishes at $x = 3$ but equals one at $x = 1$. You can draw an infinite number of such functions: any curve between those points with the specified end values will do. As we already know that the integral in (3.40) vanishes, we are left with

$$\left(w\left(\frac{\mathrm{d}u}{\mathrm{d}x} - 2\right)\right)_{x=1} = 0 \quad \Rightarrow \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x} - 2\right)_{x=1} = 0,$$

so the natural boundary condition is satisfied. As the essential boundary condition is satisfied by all trial solutions, we can then conclude that the solution of the weak form is the solution to the strong form.

Example 3.3

Obtain a solution to the weak form in Example 3.1 by using trial solutions and weight functions of the form

$$u(x) = \alpha_0 + \alpha_1 x,$$
$$w(x) = \beta_0 + \beta_1 x,$$

where α_0 and α_1 are unknown parameters and β_0 and β_1 are arbitrary parameters. Assume that A is constant and $E = 10^5$. To be admissible the weight function must vanish at $x = 0$, so $\beta_0 = 0$. For the trial solution to be admissible, it must satisfy the essential boundary condition $u(0) = 10^{-4}$, so $\alpha_0 = 10^{-4}$.

From this simplification, it follows that only one unknown parameter and one arbitrary parameter remain, and

$$\begin{aligned} u(x) &= 10^{-4} + \alpha_1 x, &\qquad \frac{\mathrm{d}u(x)}{\mathrm{d}x} &= \alpha_1, \\ w(x) &= \beta_1 x, &\qquad \frac{\mathrm{d}w}{\mathrm{d}x} &= \beta_1. \end{aligned} \tag{3.41}$$

Substituting the above into the weak form (3.32) yields

$$\int_0^2 \beta_1 \alpha_1 E \,\mathrm{d}x - \int_0^2 \beta_1 x \, 10 \,\mathrm{d}x - (\beta_1 x \, 10)_{x=2} = 0.$$

Evaluating the integrals and factoring out β_1 gives

$$\beta_1(2\alpha_1 E - 20 - 20) = 0.$$

As the above must hold for all β_1, it follows that the term in the parentheses must vanish, so $\alpha_1 = 20/E = 2 \times 10^{-4}$. Substituting this result into (3.41) gives the weak solution, which we indicate by superscript 'lin' as it is obtained from linear trial solutions: $u^{\text{lin}} = 10^{-4}(1 + 2x)$ and $\sigma^{\text{lin}} = 20$ (the stress-strain law must be used to obtain the stresses). The results are shown in Figure 3.7 and compared to the exact solution given by

$$u^{\text{ex}}(x) = 10^{-4}(1 + 3x - x^3/6), \qquad \sigma^{\text{ex}}(x) = 10(3 - x^2/2).$$

Observe that even this very simple linear approximation for a trial solution gives a reasonably accurate result, but it is not exact. We will see the same lack of exactness in finite element solutions.

Repeat the above with quadratic trial solutions and weight functions

$$u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2, \qquad w(x) = \beta_0 + \beta_1 x + \beta_2 x^2.$$

As before, because of the conditions on the essential boundaries, $\alpha_0 = 10^{-4}$ and $\beta_0 = 0$. Substituting the above fields with the given values of α_0 and β_0 into the weak form gives

$$\int_0^2 (\beta_1 + 2\beta_2 x)(E(\alpha_1 + 2\alpha_2 x))\mathrm{d}x - \int_0^2 (\beta_1 x + \beta_2 x^2)10\,\mathrm{d}x - ((\beta_1 x + \beta_2 x^2)\ 10)_{x=2} = 0.$$

Integrating, factoring out β_1, β_2 and rearranging the terms gives

$$\beta_1[E(2\alpha_1 + 4\alpha_2) - 40] + \beta_2\left(\left(4\alpha_1 + \frac{32\alpha_2}{3}\right)E - \frac{200}{3}\right) = 0.$$

As the above must hold for arbitrary weight functions, it must hold for arbitrary β_1 and β_2. Therefore, the coefficients of β_1 and β_2 must vanish (recall the scalar product theorem), which gives the following linear algebraic equation in α_1 and α_2:

$$E\begin{bmatrix} 2 & 4 \\ 4 & \dfrac{32}{3} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 40 \\ \dfrac{200}{3} \end{bmatrix}.$$

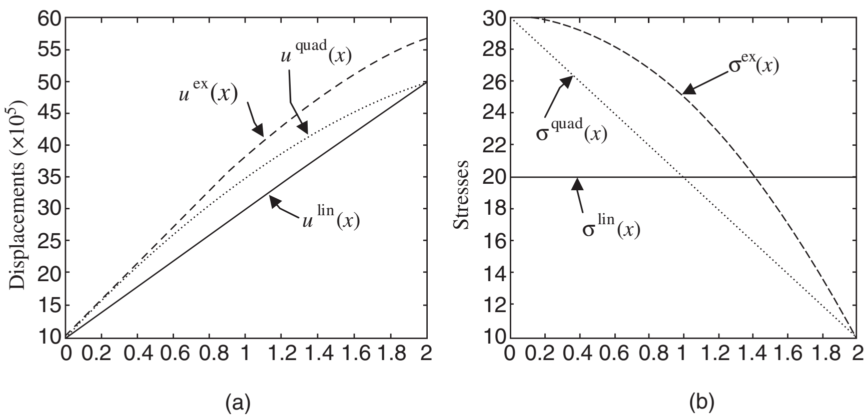

Figure 3.7 Comparison of linear (lin) and quadratic (quad) approximations to the exact solution of (a) displacements and (b) stresses.

The solution is $\alpha_1 = 3 \times 10^{-4}$ and $\alpha_2 = -0.5 \times 10^{-4}$. The resulting displacements and stresses are

$$u^{\text{quad}} = 10^{-4}(1 + 3x - 0.5x^2), \qquad \sigma^{\text{quad}} = 10(3 - x).$$

The weak solution is shown in Figure 3.7, from which you can see that the two-parameter, quadratic trial solution matches the exact solution more closely than the one-parameter linear trial solution.

3.5 ONE-DIMENSIONAL STRESS ANALYSIS WITH ARBITRARY BOUNDARY CONDITIONS

3.5.1 Strong Form for One-Dimensional Stress Analysis

We will now consider a more general situation, where instead of specifying a stress boundary condition at $x = 0$ and a displacement boundary condition at $x = l$, displacement and stress boundary conditions can be prescribed at either end. For this purpose, we will need a more general notation for the boundaries.

The boundary of the one-dimensional domain, which consists of two end points, is denoted by Γ. The portion of the boundary where the displacements are prescribed is denoted by Γ_u ; the boundary where the traction is prescribed is denoted by Γ_t. In this general notation, both Γ_u and Γ_t can be empty sets (no points), one point or two points. *The traction and displacement both cannot be prescribed at the same boundary point*. Physically, this can be seen to be impossible by considering a bar such as that in Figure 3.2. If we could prescribe both the displacement and the force on the right-hand side, this would mean that the deformation of the bar is independent of the applied force. It would also mean that the material properties have no effect on the force–displacement behavior of the bar. Obviously, this is physically unrealistic, so any boundary point is either a prescribed traction or a prescribed displacement boundary. We write this as $\Gamma_t \cap \Gamma_u = 0$. We will see from subsequent examples that this can be generalized to other systems: Natural boundary conditions and essential boundary conditions cannot be applied at the same boundary points.

We will often call boundaries with essential boundary conditions essential boundaries; similarly, boundaries with natural boundary conditions will be called natural boundaries. We can then say that a boundary cannot be both a natural and an essential boundary. It also follows from the theory of boundary value problems that one type of boundary condition is needed at each boundary point, i.e. we cannot have any boundary at which neither an essential nor a natural boundary condition is applied. Thus, any boundary is either an essential boundary or a natural boundary and their union is the entire boundary. Mathematically, this can be written as $\Gamma_t \cup \Gamma_u = \Gamma$.

To summarize the above, at any boundary, either the function or its derivative must be specified, but we cannot specify both at the same boundary. So any boundary must be an essential boundary or a natural boundary, but it cannot be both. These conditions are very important and can be mathematically expressed by the two conditions that we have stated above:

$$\boxed{\Gamma_t \cup \Gamma_u = \Gamma, \qquad \Gamma_t \cap \Gamma_u = 0.} \tag{3.42}$$

The two boundaries are said to be *complementary*: the essential boundary plus its complement, the natural boundary, constitute the total boundary, and vice versa.

Using the above notation, we summarize the strong form for one-dimensional stress analysis (3.7) in Box 3.1.

Box 3.1. Strong form for 1D stress analysis

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b=0, \qquad 0<x<l,\\
&\sigma n = En\frac{\mathrm{d}u}{\mathrm{d}x}=\bar{t} \quad \text{on} \quad \Gamma_t,\\
&u=\bar{u} \quad \text{on} \quad \Gamma_u.
\end{aligned}
\tag{3.43}
$$

In the above, we have added a unit normal to the body and denoted it by n; as can be seen from Figure 3.2, $n=-1$ at $x=0$ and $n=+1$ at $x=l$. This trick enables us to write the boundary condition in terms of the tractions applied at either end. For example, when a positive force per unit area is applied at the left-hand end of the bar in Figure 3.2, the stress at that end is negative, i.e. compressive, and $\sigma n=-\sigma=\bar{t}$. At any right-hand boundary point, $n=+1$ and so $\sigma n=\sigma=\bar{t}$.

3.5.2 Weak Form for One-Dimensional Stress Analysis

In this section, we will develop the weak form for one-dimensional stress analysis (3.43), with arbitrary boundary conditions. We first rewrite the formula for integration by parts in the notation introduced in Section 3.2:

$$
\int_\Omega w\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x=(wfn)|_\Gamma-\int_\Omega f\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x=(wfn)|_{\Gamma_u}+(wfn)|_{\Gamma_t}-\int_\Omega f\frac{\mathrm{d}w}{\mathrm{d}x}\mathrm{d}x. \tag{3.44}
$$

In the above, the subscript Ω on the integral indicates that the integral is evaluated over the one-dimensional problem domain, i.e. the notation Ω indicates any limits of integration, such as $[0,l]$, $[a,b]$. The subscript Γ indicates that the preceding quantity is evaluated at all boundary points, whereas the subscripts Γ_u and Γ_t indicate that the preceding quantities are evaluated on the prescribed displacement and traction boundaries, respectively. The second equality follows from the complementarity of the traction and displacement boundaries: Since, as indicated by (3.42), the total boundary is the sum of the traction and displacement boundaries, the boundary term can be expressed as the sum of the traction and displacement boundaries.

The weight functions are constructed so that $w=0$ on Γ_u, and the trial solutions are constructed so that $u=\bar{u}$ on Γ_u.

We multiply the first two equations in the strong form (3.43) by the weight function and integrate over the domains over which they hold: the domain Ω for the differential equation and the domain Γ_t for the traction boundary condition. This gives

$$
\begin{aligned}
&\text{(a)} \quad \int_\Omega w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right)+b\right)\mathrm{d}x=0 \qquad \forall w,\\
&\text{(b)} \quad (wA(\bar{t}-\sigma n))|_{\Gamma_t}=0 \qquad \forall w.
\end{aligned}
\tag{3.45}
$$

Denoting $f=AE(\mathrm{d}u/\mathrm{d}x)$ and using integration by parts (3.44) of the first term in (3.45a) and combining with (3.45b) yields

$$
(wA\sigma n)|_{\Gamma_u}+(wA\bar{t})|_{\Gamma_t}-\int_\Omega \frac{\mathrm{d}w}{\mathrm{d}x}AE\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x+\int_\Omega wb\,\mathrm{d}x=0 \qquad \forall w \text{ with } w=0 \text{ on } \Gamma_u. \tag{3.46}
$$

The boundary term on Γ_u vanishes because $w|_{\Gamma_u} = 0$. The weak form then becomes

$$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = (wA\bar{t})|_{\Gamma_t} + \int_{\Omega} wb\,\mathrm{d}x \qquad \forall w \text{ with } w = 0 \text{ on } \Gamma_u.$$

At this point, we introduce some new notation, so we will not need to keep repeating the phrase '$u(x)$ is smooth enough and satisfies the essential boundary condition'. For this purpose, we will denote the set of all functions that are smooth enough by H^1. H^1 functions are C^0 continuous. Mathematically, this is expressed as $H^1 \subset C^0$. However, not all C^0 functions are suitable trial solutions. We will further elaborate on this in Section 3.9; H^1 is a space of functions with square integrable derivatives.

We denote the set of all functions that are *admissible* trial solutions by U, where

$$U = \{u(x) | u(x) \in H^1,\ u = \bar{u} \text{ on } \Gamma_u\}. \tag{3.47}$$

Any function in the set U has to satisfy all conditions that follow the vertical bar. Thus, the above denotes the set of all functions that are smooth enough (the first condition after the bar) and satisfy the essential boundary condition (the condition after the comma). Thus, we can indicate that a function $u(x)$ is an admissible trial solution by stating that $u(x)$ is in the set U, or $u(x) \in U$.

We will similarly denote the set of all *admissible* weight functions by

$$U_0 = \{w(x) | w(x) \in H^1,\ w = 0 \text{ on } \Gamma_u\}. \tag{3.48}$$

Notice that this set of functions is identical to U, except that the weight functions must vanish on the essential boundaries. This space is distinguished from U by the subscript nought.

Such sets of functions are often called *function spaces*, or just *spaces*. The function space H^1 contains an infinite number of functions. Therefore, it is called an infinite-dimensional set. For a discussion of various spaces, the reader may wish to consult Ciarlet (1978), Oden and Reddy (1978) and Hughes (1987).

With these definitions, we can write the weak form ((3.45), (3.47) and (3.48)) as in Box 3.2.

Box 3.2. Weak form for 1D stress analysis

Find $u(x) \in U$ such that

$$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = (wA\bar{t})|_{\Gamma_t} + \int_{\Omega} wb\,\mathrm{d}x \qquad \forall w \in U_0. \tag{3.49}$$

Note that the functions $w(x)$ and $u(x)$ appear symmetrically in the first integral in (3.49), whereas they do not in (3.45a). In (3.49), both the trial solutions and weight functions appear as first derivatives, whereas in the first integral in (3.45a), the weight functions appear directly and the trial solution appears as a second derivative. It will be seen that consequently (3.49) leads to a symmetric stiffness matrix and a set of symmetric linear algebraic equations, whereas (3.45a) does not.

3.6 ONE-DIMENSIONAL HEAT CONDUCTION WITH ARBITRARY BOUNDARY CONDITIONS[3]

3.6.1 Strong Form for Heat Conduction in One Dimension with Arbitrary Boundary Conditions

Following the same procedure as in Section 3.5.1, the portion of the boundary where the temperature is prescribed, i.e. the essential boundary, is denoted by Γ_T and the boundary where the flux is prescribed is

[3]Recommended for Science and Engineering Track.

denoted by Γ_q ; these are the boundaries with natural boundary conditions. These boundaries are complementary, so

$$\boxed{\Gamma_q \cup \Gamma_T = \Gamma, \qquad \Gamma_q \cap \Gamma_T = 0.} \tag{3.50}$$

With the unit normal used in (3.43), we can express the natural boundary condition as $qn = \overline{q}$. For example, positive flux q causes heat inflow (negative $\overline{q}$) on the left boundary point where $qn = -q = \overline{q}$ and heat outflow (positive $\overline{q}$) on the right boundary point where $qn = q = \overline{q}$.

We can then rewrite the strong form (3.12) as shown in Box. 3.3.

Box 3.3. Strong form for 1D heat conduction problems

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right) + s = 0 \quad \text{on} \quad \Omega,\\
&qn = -kn\frac{\mathrm{d}T}{\mathrm{d}x} = \overline{q} \quad \text{on} \quad \Gamma_q,\\
&T = \overline{T} \quad \text{on} \quad \Gamma_T.
\end{aligned} \tag{3.51}$$

3.6.2 Weak Form for Heat Conduction in One Dimension with Arbitrary Boundary Conditions

We again multiply the first two equations in the strong form (3.51) by the weight function and integrate over the domains over which they hold, the domain Ω for the differential equation and the domain Γ_q for the flux boundary condition, which yields

$$\begin{aligned}
&\text{(a)} \quad \int_{\Omega} w\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x + \int_{\Omega} ws\,\mathrm{d}x = 0 \qquad \forall w,\\
&\text{(b)} \quad (wA(qn - \overline{q}))|_{\Gamma_q} = 0 \qquad \forall w.
\end{aligned} \tag{3.52}$$

Using integration by parts of the first term in (3.52a) gives

$$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x}Ak\frac{\mathrm{d}T}{\mathrm{d}x}\mathrm{d}x = \left(wAk\frac{\mathrm{d}T}{\mathrm{d}x}n\right)\bigg|_{\Gamma} + \int_{\Omega} ws\,\mathrm{d}x \qquad \forall w \text{ with } w = 0 \text{ on } \Gamma_T. \tag{3.53}$$

Recalling that $w = 0$ on Γ_T and combining (3.53) with (3.52b) gives

Box 3.4: Weak form for 1D heat conduction problems
Find $T(x) \in U$ such that

$$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x}Ak\frac{\mathrm{d}T}{\mathrm{d}x}\mathrm{d}x = -(wA\overline{q})\Big|_{\Gamma_q} + \int_{\Omega} ws\,\mathrm{d}x \qquad \forall w \in U_0. \tag{3.54}$$

Notice the similarity between (3.54) and (3.49).

3.7 TWO-POINT BOUNDARY VALUE PROBLEM WITH GENERALIZED BOUNDARY CONDITIONS[4]

3.7.1 Strong Form for Two-Point Boundary Value Problems with Generalized Boundary Conditions

The equations developed in this chapter for heat conduction, diffusion and elasticity problems are all of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)+f=0 \quad \text{on} \quad \Omega. \tag{3.55}$$

Such one-dimensional problems are called *two-point boundary value* problems. Table 3.2 gives the particular meanings of the above variables and parameters for several applications. The natural boundary conditions can also be generalized as (based on Becker *et al*. (1981))

$$\left(\kappa n\frac{\mathrm{d}\theta}{\mathrm{d}x}-\overline{\Phi}\right)+\beta(\theta-\overline{\theta})=0 \quad \text{on} \quad \Gamma_\Phi. \tag{3.56}$$

Equation (3.56) is a natural boundary condition because the derivative of the solution appears in it. (3.56) reduces to the standard natural boundary conditions considered in the previous sections when $\beta(x)=0$. Notice that the essential boundary condition can be recovered as a limiting case of (3.56) when $\beta(x)$ is a penalty parameter, i.e. a large number (see Chapter 2). In this case, $\Gamma\equiv\Gamma_\Phi$and Equation (3.56) is called a generalized boundary condition.

An example of the above generalized boundary condition is an elastic bar with a spring attached as shown in Figure 3.8. In this case, $\beta(l)=k$ and (3.56) reduces to

$$\left(E(l)n(l)\frac{\mathrm{d}u}{\mathrm{d}x}(l)-\bar{t}\right)+k(u(l)-\overline{u})=0 \quad \text{at} \quad x=l, \tag{3.57}$$

where $\beta(l)=k$ is the spring constant. If the spring stiffness is set to a very large value, the above boundary condition enforces $u(l)=\overline{u}$; if we let $k=0$, the above boundary condition corresponds to a prescribed traction boundary. In practice, such generalized boundary conditions (3.57) are often used to model the influence of the surroundings. For example, if the bar is a simplified model of a building and its foundation, the spring can represent the stiffness of the soil.

Table 3.2 Conversion table for alternate physical equations of the general form (3.55) and (3.56).

Field/parameter	Elasticity	Heat conduction	Diffusion
θ	u	T	c
κ	E	k	k
f	b	s	s
$\bar{\Phi}$	$\bar{t}$	$-\bar{q}$	$-\bar{q}$
$\bar{\theta}$	$\bar{u}$	$\bar{T}$	$\bar{c}$
Γ_Φ	Γ_t	Γ_q	Γ_q
Γ_θ	Γ_u	Γ_T	Γ_c
β	k	h	h

[4]Recommended for Advanced Track.

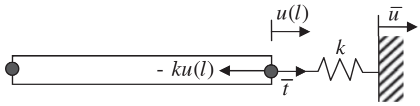

Figure 3.8 An example of the generalized boundary for elasticity problem.

Another example of the application of this boundary condition is convective heat transfer, where energy is transferred between the surface of the wall and the surrounding medium. Suppose convective heat transfer occurs at $x = l$. Let $T(l)$ be the wall temperature at $x = l$ and $\overline{T}$ be the temperature in the medium. Then the flux at the boundary $x = l$ is given by $q(l) = h(T(l) - \overline{T})$, so $\beta(l) = h$ and the boundary condition is

$$kn\frac{\mathrm{d}u}{\mathrm{d}x} + h(T(l) - \overline{T}) = 0, \tag{3.58}$$

where h is convection coefficient, which has dimensions of W m^{-2} $^{\circ}$C^{-1}. Note that when the convection coefficient is very large, the temperature $\overline{T}$ is immediately felt at $x = l$ and thus the essential boundary condition is again enforced as a limiting case of the natural boundary condition.

There are two approaches to deal with the boundary condition (3.56). We will call them the *penalty* and *partition* methods. In the penalty method, the essential boundary condition is enforced as a limiting case of the natural boundary condition by equating $\beta(x)$ to a penalty parameter. The resulting strong form for the penalty method is given in Box. 3.5.

Box 3.5. General strong form for 1D problems-penalty method

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left(A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) + f = 0 \quad \text{on} \quad \Omega, \\ &\left(\kappa n\frac{\mathrm{d}\theta}{\mathrm{d}x} - \overline{\Phi}\right) + \beta(\theta - \overline{\theta}) = 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{3.59}$$

In the partition approach, the total boundary is partitioned into the natural boundary, Γ_Φ, and the complementary essential boundary, Γ_θ. The natural boundary condition has the generalized form defined by Equation (3.56). The resulting strong form for the partition method is summarized in Box 3.6.

Box 3.6. General strong form for 1D problems-partition method

$$\begin{aligned} &(\mathrm{a}) \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) + f = 0 \quad \text{on} \quad \Omega, \\ &(\mathrm{b}) \quad \left(\kappa n\frac{\mathrm{d}\theta}{\mathrm{d}x} - \overline{\Phi}\right) + \beta(\theta - \overline{\theta}) = 0 \quad \text{on} \quad \Gamma_\Phi, \\ &(\mathrm{c}) \quad \theta = \overline{\theta} \quad \text{on} \quad \Gamma_\theta. \end{aligned} \tag{3.60}$$

3.7.2 Weak Form for Two-Point Boundary Value Problems with Generalized Boundary Conditions

In this section, we will derive the general weak form for two-point boundary value problems. Both the penalty and partition methods described in Section 3.7.1 will be considered. To obtain the general weak form for the penalty method, we multiply the two equations in the strong form (3.59) by the weight function and integrate over the domains over which they hold: the domain Ω for the differential equation and the domain Γ for the generalized boundary condition.

$$\begin{aligned}
&\text{(a)} \quad \int_{\Omega} w\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)+f\right)\mathrm{d}x = 0 \qquad \forall w,\\
&\text{(b)} \quad wA\left(\left(\kappa n\frac{\mathrm{d}\theta}{\mathrm{d}x}-\overline{\Phi}\right)+\beta(\theta-\overline{\theta})\right)\Big|_{\Gamma} = 0 \qquad \forall w.
\end{aligned} \tag{3.61}$$

After integrating by parts the first term in (3.61a) and adding (3.61b), the general weak form for 1D problems is summarized in Box 3.7.

> BOX 3.7. General weak form for 1D problems-penalty method
> Find $\theta(x) \in H^1$ such that
>
> $$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x}A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\mathrm{d}x - \int_{\Omega} wf\,\mathrm{d}x - wA(\overline{\Phi}-\beta(\theta-\overline{\theta}))\big|_{\Gamma} = 0 \qquad \forall w \in H^1. \tag{3.62}$$

Note that in the penalty method, $\Gamma_\Phi \equiv \Gamma$, the weight function is arbitrary on Γ, i.e. $\forall w(x) \in H^1$, and the solution is not *a priori* enforced to vanish on the essential boundary, i.e. $\theta(x) \in H^1$. The essential boundary condition is obtained as a limiting case of the natural boundary condition by making $\beta(x)$ very large, i.e. a penalty parameter.

In the partition method, the general weak form for one-dimensional problems is given in Box 3.8.

> Box 3.8. General weak form for 1D problems-partition method
> Find $\theta(x) \in U$ such that
>
> $$\int_{\Omega} \frac{\mathrm{d}w}{\mathrm{d}x}A\kappa\frac{\mathrm{d}\theta}{\mathrm{d}x}\mathrm{d}x - \int_{\Omega} wf\,\mathrm{d}x - wA(\overline{\Phi}-\beta(\theta-\overline{\theta}))\Big|_{\Gamma_\Phi} = 0 \qquad \forall w \in U_0, \tag{3.63}$$

where U and U_0 are given in (3.47) and (3.48), respectively. Notice that in the partition approach, the weight function vanishes on the essential boundary, Γ_θ, i.e., $\forall w \in U_0$. The boundaries Γ_θ and Γ_Φ are complementary.

3.8 ADVECTION–DIFFUSION[5]

In many situations, a substance is both transported and diffused through a medium. For example, a pollutant in an aquifer is dispersed by both diffusion and the movement of the water in the aquifer. In cooling ponds for power plants, heat energy moves through the pond by both diffusion and transport due to motion of the water. If sugar is added to a cup of coffee, it will disperse throughout the cup by diffusion; dispersal is accelerated by stirring, which advects the sugar. The dispersal due to motion of the fluid has several names besides advection: convection and transport are two other widely used names.

[5]Recommnded for Advanced Track.

3.8.1 Strong Form of Advection–Diffusion Equation

Consider the one-dimensional advection–diffusion of a species in a one-dimensional model of cross-sectional area $A(x)$, it could be a pipe or an aquifer; the concentration of the species or energy is denoted by $\theta(x)$. In an aquifer, the flow may extend to a large distance normal to the plane, so we consider a unit depth, where depth is the dimension perpendicular to the plane. In a pipe, $A(x)$ is simply the cross-sectional area. The velocity of the fluid is denoted by $v(x)$, and it is assumed to be constant in the cross section at each point along the axis, i.e. for each x. A source $s(x)$ is considered; it may be positive or negative. The latter indicates decay or destruction of the species. For example, in the transport of a radioactive contaminant, $s(x)$ is the change in a particular isotope, which may decrease due to decay or increase due to formation. The fluid is assumed to be incompressible, which has some ramifications that you will see later.

The conservation principle states that the species (be it a material, an energy or a state) is conserved in each control volume Δx. Therefore, the amount of species entering minus the amount of leaving equals the amount produced (a negative volume when the species decays). In this case, we have two mechanisms for inflow and outflow, the advection, which is $(Av\theta)_x$, and diffusion, which is $q(x)$. The conservation principle can then be expressed as

$$(Av\theta)_x + (Aq)_x - (Av\theta)_{x+\Delta x} - (Aq)_{x+\Delta x} + \Delta x s_{x+\Delta x/2} = 0.$$

Dividing by Δx and taking the limit $\Delta x \to 0$, we obtain (after a change of sign)

$$\frac{\mathrm{d}(Av\theta)}{\mathrm{d}x} + \frac{\mathrm{d}(Aq)}{\mathrm{d}x} - s = 0. \tag{3.64}$$

We now consider the incompressibility of the fluid. For an incompressible fluid, the volume of material entering a control volume equals the volume of material leaving, which gives

$$(Av)_x = (Av)_{x+\Delta x}.$$

Putting the right-hand side on the left-hand side, dividing by Δx and letting $\Delta x \to 0$, we obtain

$$\frac{\mathrm{d}(Av)}{\mathrm{d}x} = 0. \tag{3.65}$$

If we use the derivative product rule on the first term of (3.64), we obtain

$$\frac{\mathrm{d}(Av\theta)}{\mathrm{d}x} = \frac{\mathrm{d}(Av)}{\mathrm{d}x}\theta + Av\frac{\mathrm{d}\theta}{\mathrm{d}x}, \tag{3.66}$$

where the first term on the RHS vanishes by (3.65), so substituting (3.66) into (3.64) yields

$$Av\frac{\mathrm{d}\theta}{\mathrm{d}x} + \frac{\mathrm{d}(Aq)}{\mathrm{d}x} - s = 0. \tag{3.67}$$

This is the conservation equation for a species in a moving incompressible fluid. If the diffusion is linear, Fick's first law holds, so

$$q = -k\frac{\mathrm{d}\theta}{\mathrm{d}x}, \tag{3.68}$$

where k is the diffusivity. Substituting (3.68) into (3.67) gives

$$\boxed{Av\frac{\mathrm{d}\theta}{\mathrm{d}x}-\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)-s=0.} \tag{3.69}$$

The above is called the advection–diffusion equation. The first term accounts for the advection (sometimes called the transport) of the material. The second term accounts for the diffusion. The third term is the source term.

We consider the usual essential and natural boundary conditions

$$\begin{aligned}&\text{(a)}\quad \theta=\overline{\theta}\quad \text{on}\quad \Gamma_\theta,\\ &\text{(b)}\quad -k\frac{\mathrm{d}\theta}{\mathrm{d}x}n=qn=\overline{q}\quad \text{on}\quad \Gamma_q,\end{aligned} \tag{3.70}$$

where Γ_θ and Γ_q are complementary, see (3.50).

The advection–diffusion equation is important in its own right, but it is also a model for many other equations. Equations similar to the advection–diffusion equation are found throughout the field of computational fluid dynamics. For example, the vorticity equation is of this form. If we replace θ by v, then the second term in (3.66) corresponds to the transport term in the Navier–Stokes equations, which are the fundamental equations of fluid dynamics.

3.8.2 Weak Form of Advection–Diffusion Equation

We obtain the weak form of (3.69) by multiplying the governing equation by an arbitrary weight function $w(x)$ and integrating over the domain. Similarly, the weak statement of the natural boundary conditions is obtained by multiplying (3.70b) with the weight function and the area A. The resulting weak equations are

$$\begin{aligned}&\text{(a)}\quad \int_\Omega w\left(Av\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)-\frac{\mathrm{d}}{\mathrm{d}x}\left(Ak\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)-s\right)\mathrm{d}x=0\qquad \forall w,\\ &\text{(b)}\quad Aw\left(kn\frac{\mathrm{d}\theta}{\mathrm{d}x}+\overline{q}\right)\Big|_{\Gamma_q}=0\qquad \forall w.\end{aligned} \tag{3.71}$$

The spaces of trial solution and weight function are exactly as before, see (3.47) and (3.48).

We can see that the second term in Equation (3.71a) is unsymmetric in w and θ and involves a second derivative, which we want to avoid as it would require smoother trial solutions than is convenient. We can reduce the order of the derivatives by integration by parts.

The first term in (3.71a) is puzzling as it involves a first derivative only, but it is not symmetric. It turns out that we cannot make this term symmetric via integration by parts, as the integrand then becomes $(\mathrm{d}w/\mathrm{d}x)Av\theta$: In this case, integration by parts just switches the derivative from the trial solution to the weight function. So we leave this term as it is.

Integration by parts of the second term in (3.71a) and combining with (3. 71b) gives

$$\int_\Omega wAv\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)\mathrm{d}x+\int_\Omega \frac{\mathrm{d}w}{\mathrm{d}x}Ak\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)\mathrm{d}x-\int_\Omega ws\,\mathrm{d}x+(Aw\overline{q})\Big|_{\Gamma_q}=0, \tag{3.72}$$

The weak form is then as follows: find the trial solution $\theta(x)\in U$ such that (3.72) holds for all $w(x)\in U_0$.

We will not prove that the weak form implies the strong form; the procedure is exactly like before and consists of simply reversing the preceding steps. An important property of (3.72) is that the first term is not symmetric in $w(x)$ and $\theta(x)$. Therefore, the discrete equations for this weak form will not be symmetric.

Equation (3.72) and its boundary conditions become tricky when $k = 0$. In that case, there is no diffusion, only transport. Treatment of this special case is beyond this book, see Donea and Huerta (2002).

Instead of the flux boundary condition (3.70b), the total inflow of material at the boundary is often prescribed by the alternate boundary condition

$$(-k\frac{d\theta}{dx} + v\theta)n = \bar{q}_T. \tag{3.73}$$

Integrating the first term in (3.72) by parts and adding the product of the weight function, area A and (3.73) gives

$$-\int_\Omega \frac{dw}{dx}Av\theta\,dx + \int_\Omega \frac{dw}{dx}Ak\left(\frac{d\theta}{dx}\right)dx - \int_\Omega ws\,dx + (Aw\bar{q}_T)\Big|_{\Gamma_q} = 0. \tag{3.74}$$

The weak form then consists of Equation (3.74) together with an essential boundary condition (3.70a) and the generalized boundary condition (3.73).

3.9 MINIMUM POTENTIAL ENERGY[6]

An alternative approach for developing the finite element equations that is widely used is based on *variational principles*. The theory that deals with variational principles is called variational calculus, and at first glance it can seem quite intimidating to undergraduate students. Here we will give a simple introduction in the context of one-dimensional stress analysis and heat conduction. We will also show that the outcome of these variational principles is equivalent to the weak form for symmetric systems such as heat conduction and elasticity. Therefore, the finite element equations are also identical. Finally, we will show how variational principles can be developed from weak forms. The variational principle corresponding to the weak form for elasticity is called the *theorem of minimum potential energy*. This theorem is stated in Box 3.9.

> Box 3.9. Theorem of minimum potential energy
> The solution of the strong form is the minimizer of
>
> $$W(u(x)) \text{ for } \forall u(x) \in U \text{ where } W(u(x)) = \underbrace{\frac{1}{2}\int_\Omega AE\left(\frac{du}{dx}\right)^2 dx}_{W_{\text{int}}} - \underbrace{\left(\int_\Omega ub\,dx + (uA\bar{t})|_{\Gamma_t}\right)}_{W_{\text{ext}}}. \tag{3.75}$$

In elasticity, W is the potential energy of the system. We have indicated by the subscripts 'int' and 'ext' that the first term is physically the internal energy and the second term the external energy.

We will now show that the minimizer of $W(u(x))$ corresponds to the weak form, which we already know implies the strong form. Showing that the equation for the minimizer of $W(u(x))$ is the weak form implies that the minimizer is the solution, as we have already shown that the solution to the weak form is the solution of the strong form.

One of the major intellectual hurdles in learning variational principles is to understand the meaning of $W(u(x))$. $W(u(x))$ is *a function of a function*. Such a function of a function is called a *functional*. We will now examine how $W(u(x))$ varies as the function $u(x)$ is changed (or varied). An infinitesimal change in a function is called a *variation of the function* and denoted by $\delta u(x) \equiv \zeta w(x)$, where $w(x)$ is an arbitrary function (we will use both symbols) and $0 < \zeta \ll 1$, i.e. it is a very small positive number.

[6]Recommended for Structural Mechanics and Advanced Tracks.

The corresponding change in the functional is called the *variation in the functional* and denoted by δW, which is defined by

$$\delta W = W(u(x) + \zeta w(x)) - W(u(x)) \equiv W(u(x) + \delta u(x)) - W(u(x)). \tag{3.76}$$

This equation is analogous to the definition of a differential except that in the latter one considers a change in the independent variable see Oden and Reddy (1983) and Reddy (2000) for details on variational calculus. A differential gives the change in a function due to a change of the independent variable. A variation of a functional gives the change in a functional due to a change in the function. If you replace 'function' by 'functional' and 'independent variable' by 'function' in the first sentence, you have the second sentence.

From the statement of minimum potential energy given in Box 3.9, it is clear that the function $u(x) + \zeta w(x)$ must still be in U. To meet this condition, $w(x)$ must be smooth and vanish on the essential boundaries, i.e.

$$w(x) \in U_0. \tag{3.77}$$

Let us evaluate the variation of the first term in $\delta W_{\rm int}$. From the definition of the variation of a functional, Equation (3.76), it follows that

$$\begin{aligned} \delta W_{\rm int} &= \frac{1}{2}\int_\Omega AE\left(\frac{\mathrm{d}u}{\mathrm{d}x} + \zeta\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x - \frac{1}{2}\int_\Omega AE\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega AE\left(\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + 2\zeta\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x} + \zeta^2\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right) \mathrm{d}x - \frac{1}{2}\int_\Omega AE\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x. \end{aligned} \tag{3.78}$$

The first and fourth terms in the above cancel. The third term can be neglected because ζ is small, so its square is a second-order term. We are left with

$$\delta W_{\rm int} = \zeta\int_\Omega AE\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x. \tag{3.79}$$

The variation in the external work is evaluated by using the definition of a variation and the second term in Equation (3.75); we divide it into the parts due to the body force and traction for clarity. This gives

$$\begin{aligned} \delta W_{ext}^{\Omega} &= \int_\Omega (u + \zeta w)b\,dx - \int_\Omega ub\,dx = \zeta\int_\Omega wb\,dx \\ \delta W_{ext}^{\Gamma} &= (u + \zeta w)A\bar{t}|_{\Gamma_t} - (u\bar{t})A|_{\Gamma_t} = \zeta(wA\bar{t})|_{\Gamma_t} \end{aligned} \tag{3.80}$$

$$\delta W_{ext} = \delta W_{ext}^{\Omega} + \delta W_{ext}^{\Gamma} = \zeta\left(\int_\Omega wb\,dx + (wA\bar{t})|_{\Gamma_t}\right) \tag{3.81}$$

At the minimum of $W(u(x))$, the variation of the functional must vanish, just as the differentials or the derivatives of a function vanish at a minimum of a function. This is expressed as $\delta W = 0$. Thus, we have

$$0 = \delta W = \delta W_{\rm int} - \delta W_{\rm ext}. \tag{3.82}$$

Substituting (3.79)–(3.81) into the above and dividing by ζ yields the following: for $u(x) \in U$,

$$\delta W/\zeta = \int_\Omega AE\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x - \int_\Omega wb\,\mathrm{d}x - (wA\bar{t})\Big|_{\Gamma_t} = 0, \qquad w(x) \in U_0. \tag{3.83}$$

Do you recognize the above? It is precisely the statement of the weak form, Equation (3.49) that we developed in Section 3.6. Also recall that we have shown in Section 3.4 that the weak form implies the strong form, so it follows that the minimizer of the potential energy functional gives the strong form.

To be precise, we have only shown that a stationary point of the energy corresponds to the strong form. It can also be shown that the stationary point is a minimizer, see Equation (3.75) or Becker, Carey and Oden (1981, pp. 60–62).

In most books on variational principles, the change in the function $u(x)$, instead of being denoted by $\zeta w(x)$, is denoted by $\delta u(x)$. Equation (3.83) is then written as follows. Find $u \in U$ such that

$$\delta W = \int_\Omega AE\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}(\delta u)}{\mathrm{d}x}\right)\mathrm{d}x - \int_\Omega \delta u b\,\mathrm{d}x - (\delta u A\,\bar{t})\Big|_{\Gamma_t} = 0 \qquad \forall \delta u \in U_0. \tag{3.84}$$

This can be further simplified by using the strain–displacement equation and the stress-strain law in the first terms in the first integrand in (3.84), which gives

$$\delta W = \underbrace{\int_\Omega A\sigma\,\delta\varepsilon dx}_{\delta W_{\mathrm{int}}} - \underbrace{\left(\int_\Omega b\delta u dx + (\bar{t}A\delta u\,)|_{\Gamma_t}\right)}_{\delta W_{ext}} = 0 \tag{3.85}$$

The above is called the *principle of virtual work*: the admissible displacement field ($u \in U$) for which the variation in the internal work δW_{int} equals the variation in the external work δW_{ext} for all $\forall \delta u \in U_0$ satisfies equilibrium and the natural boundary conditions. Note that (3.85) is identical to the weak forms (3.49) and (3.83), just the nomenclature is different.

A very interesting feature of the minimum potential energy principle is its relationship to the energy of the system. Consider the term W_{int} in Equation (3.75). Substituting the strain–displacement equation (3.3) and Hooke's law (3.4) enables us to write it as

$$W_{\mathrm{int}} = \int_\Omega w_{\mathrm{int}} A\,\mathrm{d}x = \frac{1}{2}\int_\Omega AE\varepsilon^2\,\mathrm{d}x. \tag{3.86}$$

If we examine a graph of a linear law, Figure 3.9, we can see that the energy per unit volume is $w_{\mathrm{int}} = (1/2)E\varepsilon^2$. Thus, W_{int}, the integral of the energy density over the volume, is the total *internal energy*

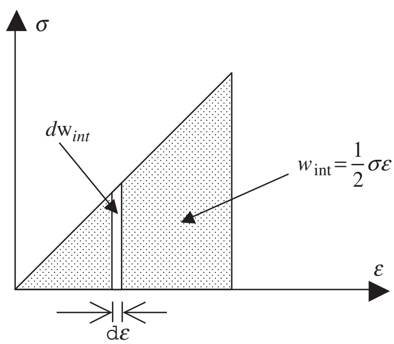

Figure 3.9 Definition of internal energy density or strain energy density w_{int}.

of the system, which is why the subscript 'int', which is short for 'internal', is appended to this term. This energy is also called the strain energy, which is the potential energy that is stored in a body when it is deformed. This energy can be recovered when the body is unloaded. Think of a metal ruler that is bent or a spring that is compressed; when the force is released, they spring back releasing the stored energy. The second term is also an energy, as the two terms that comprise W_{ext} are products of force (b or $\bar{t}$) and displacement u; in any case, it has to be an energy for the equation to be dimensionally consistent.

We can rewrite the functional in Equation (3.75) as

$$W = W_{\text{int}} - W_{\text{ext}} \tag{3.87}$$

by using the definitions underscored, and the variational principle is $\delta W = 0$. This clarifies the physical meaning of the principle of minimum potential energy: the solution is the minimizer (i.e. a stationary point) of the potential energy W among all admissible displacement functions.

Many finite element texts use the theorem of minimum potential energy as a means for formulating finite element methods. The natural question that emerges in these approaches to teaching finite elements is: How did this theorem come about and how can corresponding principles be developed for other differential equations? In fact, the development of variational principles took many years and was a topic of intense research in the eighteenth and nineteenth centuries. Variational principles cannot be constructed by simple rules like we have used for weak forms. However, some weak forms can be converted to variational principles, and in the next section, we show how to construct a variational principle for 1D stress analysis and heat conduction.

An attractive feature of the potential energy theorem is that it holds for any elastic system. Thus, if we write the energy for any other system, we can quickly derive finite element equations for that system; this will be seen in Chapter 10 for beams. Variational principles are also very useful in the study of the accuracy and convergence of finite elements.

The disadvantage of variational approach is that there are many systems to which they are not readily applicable. Simple variational principles cannot be developed for the advection–diffusion equation for which we developed a weak form in Section 3.7 by the same straightforward procedure as for the other equations. Variational principles can only be developed for systems that are self-adjoint. The weak form for the advection–diffusion equation is not symmetric, and it is not a self-adjoint system (see Becker, Carey and Oden (1981) for definition of self-adjoint systems).

Variational principles identical to those for elasticity apply to heat transfer and other diffusion equations. This is not surprising, as the equations are identical except for the parameters. As an example, the variational principle for heat conduction is given in Box 3.10.

Box 3.10. Variational principle for heat conduction

$$\text{Let } W(T(x)) = \underbrace{\frac{1}{2}\int_{\Omega} Ak\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)^2 \mathrm{d}x}_{W_{int}} - \underbrace{\left(\int_{\Omega} Ts\,\mathrm{d}x - (TA\bar{q})\Big|_{\Gamma_q}\right)}_{W_{\text{ext}}},$$

then the solution of the strong form of (3.51) is the minimizer of $W(T(x))$ for $\forall T(x) \in U$.

The functional in this variational principle is not a physical energy; in fact, the temperature itself corresponds to the physical energy. However, the functional is often called an energy even for diffusion equations; we will call it a *mathematical energy*. The proof of the equivalence of this principle to the weak form (and hence to the strong form) of the heat conduction equations just involves replacing the symbols in (3.78)–(3.83) according to Table 3.2; the mathematics is identical regardless of the symbols.

3.10 INTEGRABILITY[7]

So far we have left the issue of the smoothness of the weight functions and trial solutions rather nebulous. We will now define the degree of smoothness required in weak forms more precisely. Many readers may want to skip this material on an initial reading, as the rest of the book is quite understandable without an understanding of this material.

The degree of smoothness that is required in the weight and trial functions is determined by how smooth they need to be so that the integrals in the weak form, such as (3.54), can be evaluated. This is called the *integrability of the weak form*. If the weight and trial functions are too rough, then the integrals cannot be evaluated, so then obviously the weak form is not usable.

We next roughly examine how smooth is smooth enough. If you look at a C^{-1} function that is not singular (does not become infinite), you can see that it is obviously integrable, as the area under such a function is well defined. Even the derivative of a C^{-1} function is integrable, for at a point of discontinuity $x = a$ of magnitude p, the derivative is the Dirac delta function $p\delta(x - a)$. By the definition of a Dirac delta function (See Appendix A5),

$$\int_{x_1}^{x_2} p\delta(x - a)\,\mathrm{d}x = p \quad \text{if} \quad x_1 \leq a \leq x_2.$$

So the integral of the derivative of a C^{-1} function is well defined. However, the product of the derivatives of the weight and trial functions appears in the weak form. If both of these functions are C^{-1}, and the discontinuities occur at the same point, say $x = a$, then the weak form will contain the term $\int_{x_1}^{x_2} p^2\delta(x - a)^2\,\mathrm{d}x$. The integrand here can be thought of as 'infinity squared': there is no meaningful way to obtain this integral. So C^{-1} continuity of the weight and trial functions is not sufficient.

On the contrary, if the weight and trial functions are C^0 and not singular, then the derivatives are C^{-1} and the integrand will be the product of two C^{-1} functions. You can sketch some functions and see that the product of the derivatives of two C^{-1} functions will also be C^{-1} as long as the functions are bounded (do not become infinite). Since a bounded C^{-1} function is integrable, C^0 continuity is smooth enough for the weight and trial functions.

This continuity requirement can also be justified physically. For example, in stress analysis, a C^{-1} displacement field would have gaps or overlaps at the points of discontinuity of the function. This would violate compatibility of the displacement field. Although gaps are dealt with in more advanced methods to model fracture, they are not within the scope of the methods that we are developing here. Similarly in heat conduction, a C^{-1} temperature field would entail an infinite heat flux at the points of discontinuity, which is not physically reasonable. Thus, the notions of required smoothness, which arise from the integrability of the weak form, also have a physical basis.

In mathematical treatments of the finite element method, a more precise description of the required degree of smoothness is made: the weight and trial functions are required to possess square integrable derivatives. A derivative of a function $u(x)$ is called *square integrable* if $W_{\mathrm{int}}(\theta)$, defined as

$$W_{\mathrm{int}}(\theta) = \frac{1}{2}\int_{\Omega} \kappa A \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \mathrm{d}x, \tag{3.88}$$

is bounded, i.e. $W_{\mathrm{int}}(\theta) < \infty$. The value of $\sqrt{W_{\mathrm{int}}(\theta)}$ is often called an energy norm. For heat conduction, $\theta = T$ and $\kappa(x) = k(x) > 0$. In elasticity, $\kappa(x) = E(x) > 0$ and $\theta = u$ and (3.88) corresponds to the strain energy, which appears in the principle of minimum potential energy.

It can be proven that H^1 is a subspace of C^0, i.e. $H^1 \subset C^0$, so any function in H^1 is also a C^0 function. However, the converse is not true: C^0 functions that are not in H^1 exist. An example of a function that is C^0,

[7]Recommended for Advanced Track.

but not H^1, is examined in Problem 3.8. However, such functions are usually not of the kind found in standard finite element analysis (except in fracture mechanics), so most readers will find that the specification of the required degree of smoothness by C^0 continuity is sufficient.

REFERENCES

Becker, E. B., Carey, G. F. and Oden, J.T. (1981) *Finite Elements: An introduction*, vol. 1, Prentice Hall, Englewood Cliffs, NJ.

Ciarlet, P.G. (1978) *The Finite Element Method for Elliptic Problems*, North-Holland, New York.

Donea, J. and Huerta, A. (2002) *Finite Element Methods for Flow Problems*, John Wiley & Sons, Ltd, Chichester.

Hughes, T. J. R. (1987) *The Finite Element Method*, Prentice Hall, Englewood Cliffs, NJ.

Oden, J. T. and Reddy, J.N. (1978) *An Introduction to the Mathematical Theory of Finite Elements*, Academic Press, New York.

Oden, J. T. and Reddy, J. N. (1983) *Variational Methods in Theoretical Mechanics*, 2nd ed., Springer-Verlag, New York.

Reddy, J. N. (2002) *Energy Principles and Variational Methods in Applied Mechanics*, 2nd ed., John Wiley, New York.

Problems

Problem 3.1

Show that the weak form of

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(AE\frac{\mathrm{d}u}{\mathrm{d}x}\right) + 2x = 0 \quad \text{on} \quad 1 < x < 3,\\
&\sigma(1) = \left(E\frac{du}{dx}\right)_{x=1} = 0.1,\\
&u(3) = 0.001
\end{aligned}$$

is given by

$$\int_1^3 \frac{\mathrm{d}w}{\mathrm{d}x} AE \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = -0.1(wA)_{x=1} + \int_1^3 2xw\,\mathrm{d}x \qquad \forall w \text{ with } w(3) = 0.$$

Problem 3.2

Show that the weak form in Problem 3.1 implies the strong form.

Problem 3.3

Consider a trial (candidate) solution of the form $u(x) = \alpha_0 + \alpha_1(x-3)$ and a weight function of the same form. Obtain a solution to the weak form in Problem 3.1. Check the equilibrium equation in the strong form in Problem 3.1; is it satisfied?

Check the natural boundary condition; is it satisfied?

Problem 3.4

Repeat Problem 3.3 with the trial solution $u(x) = \alpha_0 + \alpha_1(x-3) + \alpha_2(x-3)^2$.

Problem 3.5

Obtain the weak form for the equations of heat conduction with the boundary conditions $T(0) = 100$ and $\overline{q}(10) = hT$. The condition on the right is a convection condition.

Problem 3.6

Given the strong form for the heat conduction problem in a circular plate:

$$k\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right) + rs = 0, \qquad 0 < r \le R.$$

$$\text{natural boundary condition :} \quad \frac{\mathrm{d}T}{\mathrm{d}r}(r=0) = 0,$$

$$\text{essential boundary condition :} \quad T(r=R) = 0,$$

where R is the total radius of the plate, s is the heat source per unit length along the plate radius, T is the temperature and k is the conductivity. Assume that k, s and R are given:

a. Construct the weak form for the above strong form.
b. Use quadratic trial (candidate) solutions of the form $T = \alpha_0 + \alpha_1 r + \alpha_2 r^2$ and weight functions of the same form to obtain a solution of the weak form.
c. Solve the differential equation with the boundary conditions and show that the temperature distribution along the radius is given by

$$T = \frac{s}{4k}(R^2 - r^2).$$

Problem 3.7

Given the strong form for the circular bar in torsion (Figure 3.10):

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(JG\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + m = 0, \qquad 0 \le x \le l,$$

$$\text{natural boundary condition :} \quad M(x=l) = \left(JG\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)_l = \overline{M},$$

$$\text{essential boundary condition :} \quad \phi(x=0) = \overline{\phi},$$

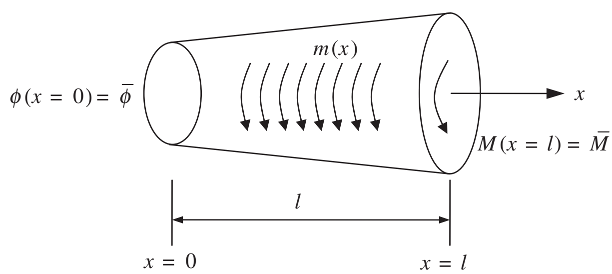

Figure 3.10 Cylindrical bar in torsion of Problem 3.7.

where $m(x)$ is a distributed moment per unit length, M is the torsion moment, ϕ is the angle of rotation, G is the shear modulus and J is the polar moment of inertia given by $J = \pi C^4/2$, where C is the radius of the circular shaft.

a. Construct the weak form for the circular bar in torsion.
b. Assume that $m(x) = 0$ and integrate the differential equation given above. Find the integration constants using boundary conditions.

Problem 3.8

Consider a problem on $0 \leq x \leq l$ which has a solution of the form

$$u = \begin{cases} -\left(\frac{1}{2}\right)^{\lambda} \frac{x}{l}, & x \leq \frac{l}{2}, \\ \left(\frac{x}{l} - \frac{1}{2}\right)^{\lambda} - \left(\frac{1}{2}\right)^{\lambda} \frac{x}{l}, & x > \frac{l}{2}. \end{cases}$$

a. Show that for $\lambda > 0$ the solution u is C^0 in the interval $0 \leq x \leq l$.
b. Show that for $0 < \lambda \leq 1/2$ the solution u is not in H^1.

Problem 3.9

Consider an elastic bar with a variable distributed spring $p(x)$ along its length as shown in Figure 3.11. The distributed spring imposes an axial force on the bar in proportion to the displacement.

Consider a bar of length l, cross-sectional area $A(x)$, Young's modulus $E(x)$ with body force $b(x)$ and boundary conditions as shown in Figure 3.11.

a. Construct the strong form.
b. Construct the weak form.

Problem 3.10

Consider an elastic bar in Figure 3.2. The bar is subjected to a temperature field $T(x)$. The temperature causes expansion of the bar and the stress-strain law is

$$\sigma(x) = E(x)(\varepsilon(x) - \alpha(x)T(x)),$$

where α is the coefficient of thermal expansion, which may be a function of x.

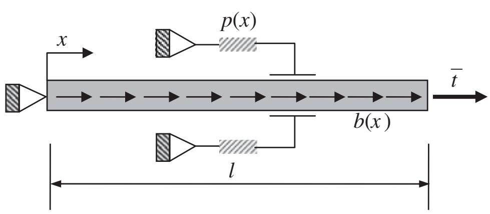

Figure 3.11 Elastic bar with distributed springs of Problem 3.9.

a. Develop the strong form by replacing the standard Hooke's law with the above in the equilibrium equation; use the boundary conditions given in Problem 3.1.

b. Construct the weak form for (3.43) when the above law holds.

Problem 3.11

Find the weak form for the following strong form:

$$\kappa \frac{d^2u}{dx^2} - \lambda u + 2x^2 = 0, \qquad \kappa, \lambda \text{ are constants, } 0 < x < 1,$$

subject to $u(0) = 1,\ u(1) = -2$.

Problem 3.12

The motion of an electric charge flux q_V is proportional to the voltage gradient. This is described by Ohm's law:

$$q_V = -k_V \frac{dV}{dx},$$

where k_V is electric conductivity and V is the voltage. Denote Q_V as the electric charge source.
Construct the strong form by imposing the condition that the electric charge is conserved.

Problem 3.13

Find the weak form for the following strong form:

$$x \frac{d^2u}{dx^2} + \frac{du}{dx} - x = 0, \qquad 0 \le x \le 1,$$

subject to $u(0) = u(1) = 0$.

Problem 3.14

Consider a bar in Figure 3.12 subjected to linear body force $b(x) = cx$. The bar has a constant cross-sectional area A and Young's modulus E. Assume quadratic trial solution and weight function

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2, \qquad w(x) = \beta_1 + \beta_2 x + \beta_3 x^2,$$

where α_i are undetermined parameters.

a. For what value of α_I is $u(x)$ kinematically admissible?

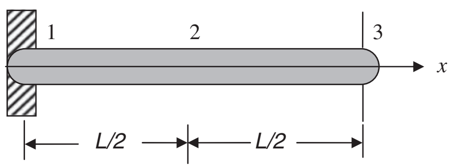

Figure 3.12 Elastic bar subjected to linear body force of Problem 3.14.

b. Using the weak form, set up the equations for α_I and solve them. To obtain the equations, express the principle of virtual work in the form $\beta_2(\cdots) + \beta_3(\cdots) = 0$. By the scalar product theorem, each of the parenthesized terms, i.e. the coefficients of β_I, must vanish.
c. Solve the problem in Figure 3.12 using two 2-node elements considered in Chapter 2 of equal size. Approximate the external load at node 2 by integrating the body force from $x = L/4$ to $x = 3L/4$. Likewise, compute the external at node 3 by integrating the body force from $x = 3L/4$ to $x = L$.

Problem 3.15

Consider the bar in Problem 3.14.

a. Using an approximate solution of the form $u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, determine $u(x)$ by the theorem of minimum potential energy. *Hint*: after enforcing admissibility, substitute the above trial solution into (3.75) and minimize with respect to independent parameters.
b. Compare the solution obtained in part (a) to an exact solution of the equation $E\dfrac{d^2u}{dx^2} + cx = 0$.
c. Does $\sigma(L) = 0$ for the approximate solutions?
d. Check whether the stress obtained from $u(x)$ by $\sigma = E\dfrac{du}{dx}$ satisfies the equilibrium.

4

Approximation of Trial Solutions, Weight Functions and Gauss Quadrature for One-Dimensional Problems

We now consider the next important ingredient of the finite element method (FEM): the construction of the approximations. In Chapter 3, we derived weak forms for the elasticity and heat conduction problems in one dimension. The weak forms involve weight functions and trial solutions for the temperature, displacements, solute concentrations and so on. In the FEM, the weight functions and trial solutions are constructed by subdividing the domain of the problem into elements and constructing functions within each element. These functions have to be carefully chosen so that the FEM is *convergent*: *The accuracy of a correctly developed FEM improves with mesh refinement*, i.e. as element size, denoted by h, decreases, the solution tends to the correct solution. This property of the FEM is of great practical importance, as mesh refinement is used by practitioners to control the quality of the finite element solutions.

For example, the accuracy of a solution is often checked by rerunning the same problem with a finer mesh; if the difference between the coarse and fine mesh solutions is small, it can be inferred that the coarse mesh solution is quite accurate. On the contrary, if a solution changes markedly with refinement of the mesh, the coarse mesh solution is inaccurate, and even the finer mesh may still be inadequate.

Although the mathematical theory of convergence is beyond the scope of the book, loosely speaking, the two *necessary* conditions for convergence of the FEM are continuity and completeness. This can schematically be expressed as

$$\boxed{\text{Continuity}} + \boxed{\text{Completeness}} \rightarrow \boxed{\text{Convergence}}$$

By *continuity* we mean that the trial solutions and weight functions are sufficiently smooth. The degree of smoothness that is required depends on the order of the derivatives that appear in the weak form. For the second-order differential equations considered in Chapter 3, where the derivatives in the weak form are first derivatives, we have seen that the weight functions and trial solutions must be C^0 continuous.

Completeness is a mathematical term that refers to the capability of a series of functions to approximate a given smooth function with arbitrary accuracy. For convergence of the FEM, it is sufficient that as the element sizes approach zero, the trial solutions and weight functions and their derivatives up to and

A First Course in Finite Elements J. Fish and T. Belytschko