Engineering Optimization

Two Phase Simplex Method

Example 6.11:

$$\text{minimize } f = -x_1 - 2x_2 + 2x_3$$

$$3x_1 + 2x_2 - 2x_3 + x_4 = 12$$

$$2x_1 + 3x_2 - 3x_3 - x_5 = 6 \quad \Rightarrow \quad 2x_1 + 3x_2 - 3x_3 - x_5 + x_6 = 6$$

$$x_i \ge 0; \quad i = 1 \text{ to } 5$$

$$w = x_6 = 6 - 2x_1 - 3x_2 + 3x_3 + x_5$$

Initial tableau: x_6 is identified to be replaced with x_2 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**	*Ratio*
x_4	3	2	–2	1	0	0	12	$\frac{12}{2} = 6$
x_6	2	3	–3	0	–1	1	6	$\frac{6}{3} = 2$
Cost	–1	–2	2	0	0	0	$f - 0$	
Artificial cost	**–2**	**–3**	**3**	0	**1**	0	$w - 6$	

Second tableau: End of Phase I. Begin Phase II. x_4 is identified to be replaced with x_5 in the basic set.

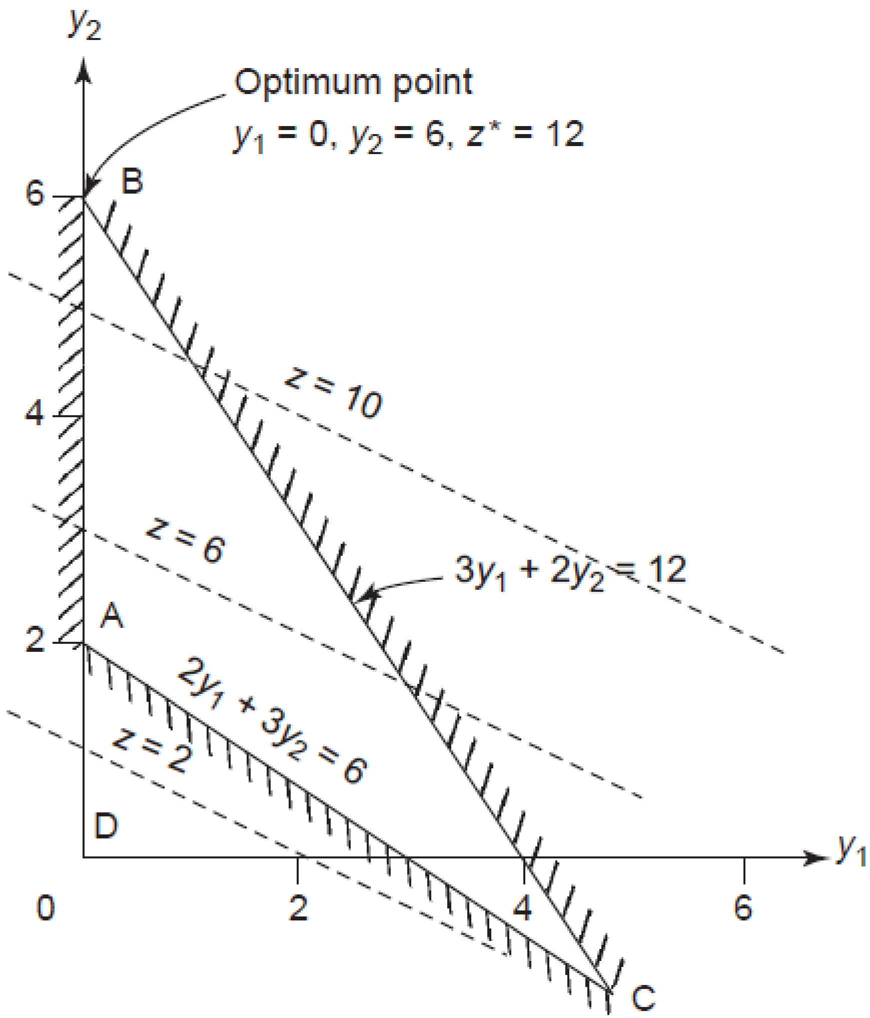
Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**	*Ratio*
x_4	$\frac{5}{3}$	0	0	1	$\boxed{\frac{2}{3}}$	$-\frac{2}{3}$	8	$\frac{8}{2/3} = 12$
x_2	$\frac{2}{3}$	1	−1	0	$-\frac{1}{3}$	$\frac{1}{3}$	2	Negative
Cost	$\frac{1}{3}$	0	0	0	$\boxed{-\frac{2}{3}}$	$\frac{2}{3}$	$f + 4$	
Artificial cost	0	0	0	0	**0**	**1**	$w - 0$	

Third tableau: Reduced cost coefficients in nonbasic columns are nonnegative; the third tableau gives the optimum solution. End of Phase II.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_5	$\frac{5}{2}$	0	0	$\frac{3}{2}$	1	−1	12
x_2	$\frac{3}{2}$	1	−1	$\frac{1}{2}$	0	0	6
Cost	**2**	0	0	**1**	0	0	$f + 12$

x5 = 12, x2 = 6, x1 = x3 = x4 = 0, and f = -12

y1 = 0, y2 = 6, and z = 12



Example 6.12 : **Use of Artificial Variables for Equality Constraints**

$$\text{maximize } z = x_1 + 4x_2$$

$$x_1 + 2x_2 \le 5,$$
$$2x_1 + x_2 = 4,$$
$$x_1 - x_2 \ge 3,$$
$$x_1, x_2 \ge 0.$$

$$\text{minimize } f = -x_1 - 4x_2$$

$$x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + x_2 + x_5 = 4$$
$$x_1 - x_2 - x_4 + x_6 = 3$$
$$x_i \ge 0; \quad i = 1 \text{ to } 6$$

Here *x3 is a slack variable, x4 is a surplus variable, and x5 and x6 are artificial variables.*

Initial tableau: x_5 is identified to be replaced with x_1 in the basic set.

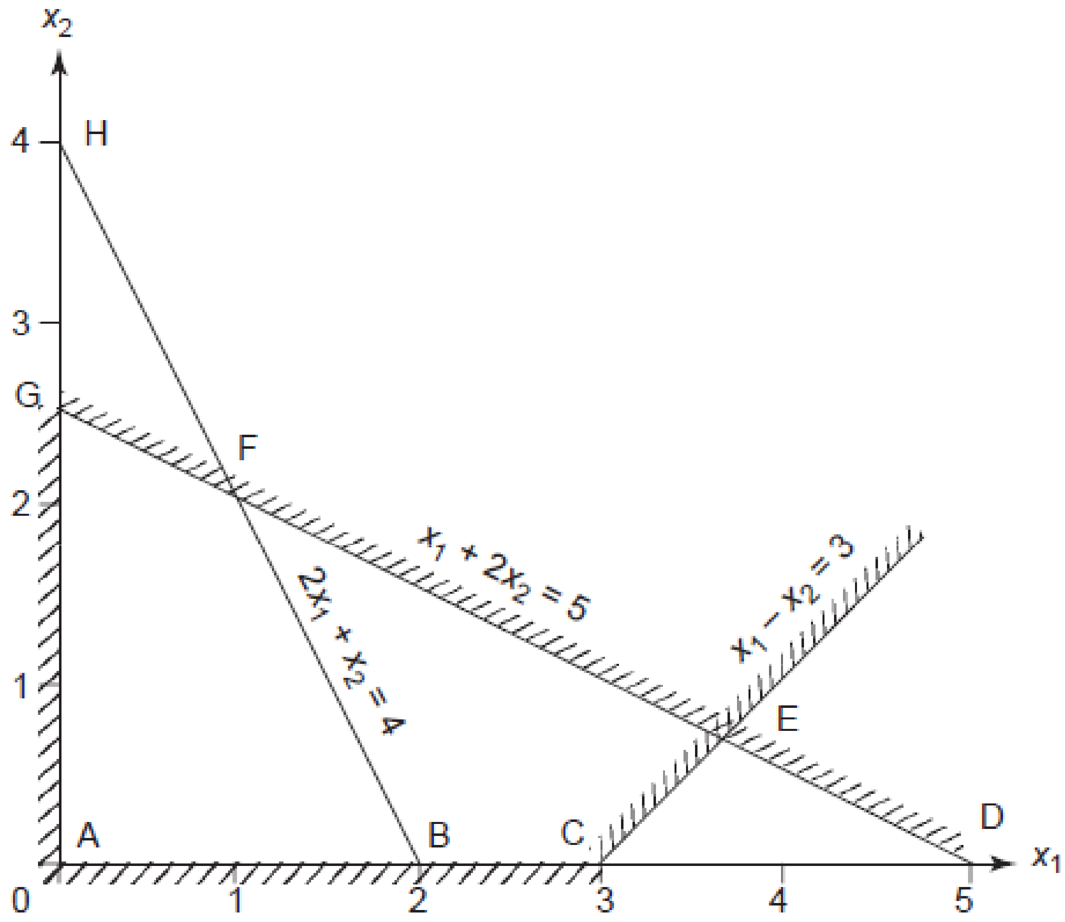
Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**	*Ratio*
x_3	1	2	1	0	0	0	5	$\frac{5}{1} = 5$
x_5	2	1	0	0	1	0	4	$\frac{4}{2} = 2$
x_6	1	–1	0	–1	0	1	3	$\frac{3}{1} = 3$
Cost	–1	–4	0	0	0	0	$f - 0$	
Artificial cost	**–3**	0	0	**1**	0	0	$w - 7$	

Second tableau: End of Phase I.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_3	0	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	3
x_1	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	2
x_6	0	$-\frac{3}{2}$	0	−1	$-\frac{1}{2}$	1	1
Cost	0	$-\frac{7}{2}$	0	0	$\frac{1}{2}$	0	$f + 2$
Artificial cost	0	$\frac{3}{2}$	0	**1**	$\frac{3}{2}$	0	$w - 1$

the artificial cost function is not zero (*w = 1).*
Therefore there is no feasible solution to the original problem.

(Infeasible Problem)



x_2
4
H
3
G
F
2
1
$x_1 + 2x_2 = 5$
$2x_1 + x_2 = 4$
$x_1 - x_2 = 3$
E
A
B
C
D
x_1
0
1
2
3
4
5

Example 6.13: **Use of Artificial Variables**

$$\text{maximize } z = 3x_1 - 2x_2$$

$$\text{subject to } x_1 - x_2 \geq 0,$$

$$x_1 + x_2 \geq 2,$$

$$x_1, x_2 \geq 0.$$

$$\text{minimize } f = -3x_1 + 2x_2$$

$$-x_1 + x_2 + x_3 = 0$$

$$x_1 + x_2 - x_4 + x_5 = 2$$

$$x_i \geq 0; \qquad i = 1 \text{ to } 5$$

where *x3 is a slack variable, x4 is a surplus variable, and x5 is an artificial variable*

Initial tableau: x_5 is identified to be replaced with x_1 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	**b**	*Ratio*
x_3	–1	1	1	0	0	0	Negative
x_5	1	1	0	–1	1	2	$\frac{2}{1} = 2$
Cost	–3	2	0	0	0	$f - 0$	
Artificial cost	**–1**	**–1**	0	**1**	0	$w - 2$	

x3 = 0 and x5 = 2

degenerate basic feasible solution

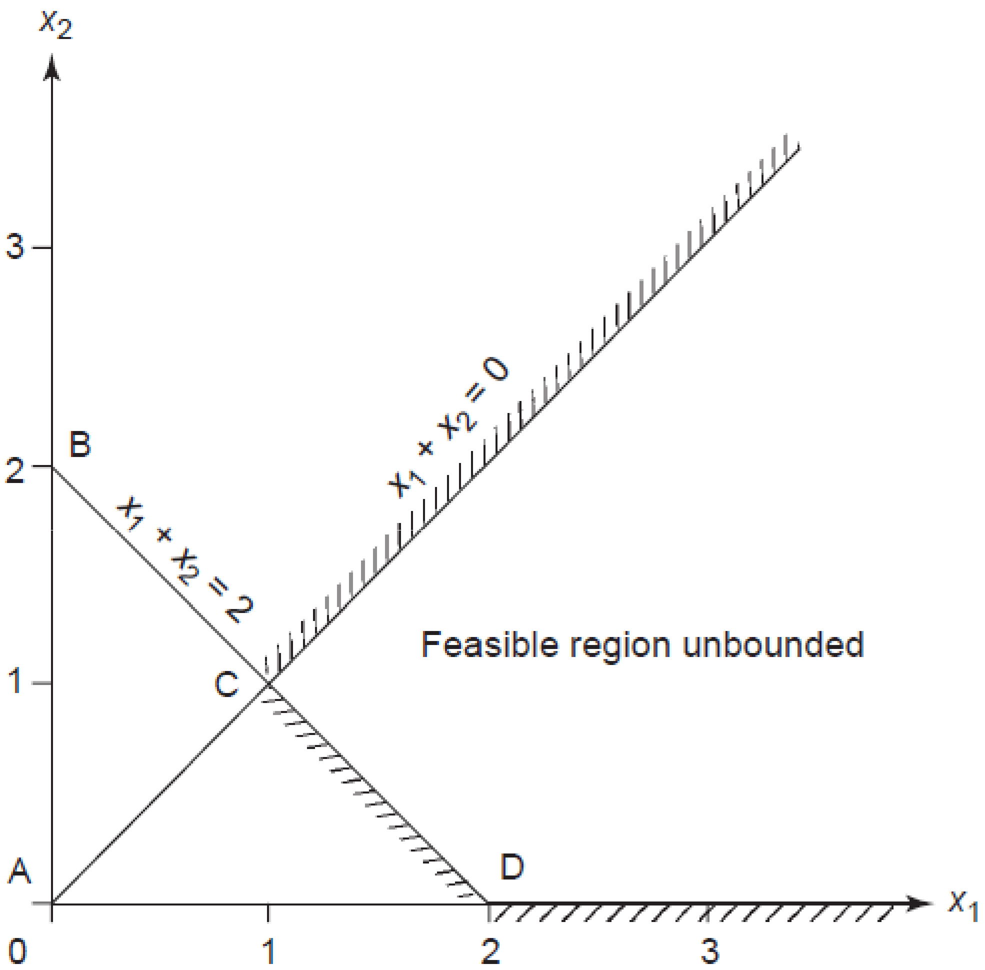
Second tableau: End of Phase I. End of Phase II.

Basic↓	x_1	x_2	x_3	x_4	x_5	**b**	*Ratio*
x_3	0	2	1	–1	1	2	Negative
x_1	1	1	0	–1	1	2	Negative
Cost	0	**5**	0	**–3**	**3**	$f + 6$	
Artificial cost	0	0	0	0	**1**	$w - 0$	

x1 = 2, x3 = 2, and other variables are zero

the reduced cost coefficient *c4 is negative, but the pivot element cannot be determined,*

Problem is unbounded



x_2
3
2
1
0
1
2
3
x_1
B
C
A
D
$x_1 + x_2 = 2$
$x_1 + x_2 = 0$
Feasible region unbounded

Example 6.14: **Implications of Degenerate Basic Feasible Solution**

$$\text{maximize } z = x_1 + 4x_2$$

$$x_1 + 2x_2 \leq 5,$$

$$2x_1 + x_2 \leq 4$$

$$2x_1 + x_2 \geq 4,$$

$$x_1 - x_2 \geq 1,$$

$$x_1, x_2 \geq 0.$$

$$\text{minimize } f = -x_1 - 4x_2$$

$$x_1 + 2x_2 + x_3 = 5$$

$$2x_1 + x_2 + x_4 = 4$$

$$2x_1 + x_2 - x_5 + x_7 = 4$$

$$x_1 - x_2 - x_6 + x_8 = 1$$

$$x_i \geq 0; \quad i = 1 \text{ to } 8$$

where *x3 and x4 are slack variables, x5 and x6 are surplus variables, and x7 and x8 are* artificial variables

Initial tableau: x_8 is identified to be replaced with x_1 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**	*Ratio*
x_3	1	2	1	0	0	0	0	0	5	$\frac{5}{1} = 5$
x_4	2	1	0	1	0	0	0	0	4	$\frac{4}{2} = 2$
x_7	2	1	0	0	–1	0	1	0	4	$\frac{4}{2} = 2$
x_8	1	–1	0	0	0	–1	0	1	1	$\frac{1}{1} = 1$
Cost	–1	–4	0	0	0	0	0	0	$f - 0$	
Artificial	–3	0	0	0	**1**	**1**	0	0	$w - 5$	

Second tableau: x_7 is identified to be replaced with x_2 in the basic set.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**	*Ratio*
x_3	0	3	1	0	0	1	0	–1	4	$\frac{4}{3}$
x_4	0	3	0	1	0	2	0	–2	2	$\frac{2}{3}$
x_7	0	3	0	0	–1	2	1	–2	2	$\frac{2}{3}$
x_1	1	–1	0	0	0	–1	0	1	1	Negative
Cost	0	–5	0	0	0	–1	0	1	$f + 1$	
Artificial	0	–3	0	0	**1**	**–2**	0	**3**	$w - 2$	

Third tableau: x_4 is identified to be replaced with x_5 in the basic set. End of Phase I.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**	*Ratio*
x_3	0	0	1	0	1	−1	−1	1	2	$\frac{2}{1}=2$
x_4	0	0	0	1	$\boxed{1}$	0	−1	0	0	$\frac{0}{1}=0$
x_2	0	1	0	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	Negative
x_1	1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	Negative
Cost	0	0	0	0	$\boxed{-\frac{5}{3}}$	$\frac{7}{3}$	$\frac{5}{3}$	$-\frac{7}{3}$	$f+\frac{13}{3}$	
Artificial	0	0	0	0	0	0	1	1	$w-0$	

Final tableau: End of Phase II.

Basic↓	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	**b**
x_3	0	0	1	–1	0	–1	0	1	2
x_5	0	0	0	1	1	0	–1	0	0
x_2	0	1	0	$\frac{1}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_1	1	0	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{3}$
Cost	0	0	0	$\frac{5}{3}$	0	$\frac{7}{3}$	0	$-\frac{7}{3}$	$f+\frac{13}{3}$

basic variables: $x_1 = \frac{5}{3}, x_2 = \frac{2}{3}, x_3 = 2, x_5 = 0$

nonbasic variables: $x_4 = x_6 = x_7 = x_8 = 0$

optimum cost function: $f = -\frac{3}{13}$ or $z = \frac{13}{3}$

It is theoretically possible for the Simplex method to fail by cycling between two degenerate basic feasible solutions.
in practice this usually does not happen.