A Mathematical Model of Human Eye Movement - A Tutorial

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In the last semester, I had a course named Neuromuscular Systems Control by Dr. Mehdi Delrobai. I had chosen this subject to make a tutorial for the project of this course because it is so close to my interest.

In my tutorial I want to explain and find an ocular biomechanical model for different eye movements step by step.

The system of the human eye consists of a semi-solid sphere or globe containing vitreous matter constituting the lens and supporting fluids. This globe is rotated within a retaining orbit by six ocular muscles. The muscles themselves are divided into three pairs, each of which is primarily responsible for one of several types of rotary movement.
Three main axes of rotation:

\[
\begin{align*}
\text{x-axis} & \rightarrow \text{superior and inferior oblique muscles} \quad \Rightarrow \text{to produce cyclorotations of the eye} \\
\text{y-axis} & \rightarrow \text{superior and inferior rectus muscles} \quad \Rightarrow \text{to produce vertical rotation} \\
\text{z-axis} & \rightarrow \text{medial and lateral rectus muscles} \quad \Rightarrow \text{to produce horizontal rotations}
\end{align*}
\]
Figure 3: Human eye movement terminology, rotation axes and extraocular muscles illustrated. Top: Secondary positions for right eye: (A) Adduction, (B) Abduction, (C) Supraduction and (D) Infraduction. Bottom-Left: Right eye showing the axes and the different rotation directions used. Listing plane is illustrated, it is the Y-Z plane. Bottom-Right: Left eye muscles shown from (A) Superior view, (B) Anterior view. The six muscles are shown along with the annulus of Zinn and the trochlea. The names and abbreviations of each muscle is shown in Table 1 along with their actions.

**Component Structures of the Oculomotor Plant**

The agonist and antagonist muscles are here assumed to be identical, although in reality their anatomy differs in some details of exact position and dimension. The convention will be adopted of supplementing parameter subscripts with 1 to identify agonist and 2 to identify antagonist components. For example, $\ell_{t1}$ designates the agonist tendon length and $\ell_{t2}$ designates antagonist tendon length. Both muscles are assumed to have the same nonzero mass $M$. The muscle bodies have lengths $\ell_{m1}$ and $\ell_{m2}$. Taken together with the muscle tendon lengths $\ell_{t1}$ and $\ell_{t2}$, the muscle-tendon pairs occupy total paths of length $\ell_{tm1}$ and $\ell_{tm2}$. When the eye is looking straightforward, it is said to be in the primary position and in this state the muscles are at their primary length $\ell_{mp}$. 
The muscles in the end of themselves are somewhat complex. First, as already observed the muscles are each attached to the globe by spring-like tendons; the contraction of either muscle produces a force which is exerted on the globe via its respective tendon. If the series tendons of the muscle were treated as linear springs, then they would be characterized by simple spring constants; however, as will be discussed later, the muscle tendons in reality behave as non-linear springs where the spring coefficient $K_t$ is understood to be a function of $F_t$, tendon force.
Second, we note from Figure 5 that each ocular muscle is composed of three parallel components in series with the aforementioned tendons. Namely, each muscle has a passive viscosity with parameter $B_{pm}$, a passive elasticity which produces a force denoted by $F_{pe}$, and an active contractile component $F_{act}$. The parameter $B_{pm}$ describes the muscle’s resistance to changes in its velocity of contraction. The passive muscle elasticity is understood to behave like a non-linear spring as does the tendon elasticity, and so the force resisting extension, $F_{pe}$, is a function of muscle length. The active contractile component, $F_{act}$, receives neural stimulation and induces contraction of the muscle fiber mass; this has the effect of exerting a force that tends to stretch the associated muscle tendon.
The Kinematics of the Oculomotor Plant

the plant model developed herein is a kinematic model which attempts to explain eye movements within the framework of Newtonian mechanics. Muscle forces are exerted on the globe via the stretching of the muscle tendon; this tendon force is translated into torque in the rotational system of the globe and orbit. For rotations up to 30° or 40°, the moment arms of the ocular muscles are approximately equal to the radius of the globe. For example, if the agonist tendon is stretched to produce a linear force $F_{t1}$, then the resulting torque on the globe is given by $r\cdot F_{t1}$. Since muscles and tendons do not resist compression, the agonist and antagonist muscle-tendon pairs only pull on the globe and do not push it. If a net torque is exerted on the globe sufficient to overcome the inertial resistance of the globe and the visco-elastic resistance of the orbit, then a rotation of the globe by $\theta$ radians results. A direct application of the rotational version of Newton's Second Law of Motion results in the following balance of torques equation:

$$J_G\ddot{\theta}+B_G\dot{\theta}+K_G\theta=r\cdot F_{t1}-r\cdot F_{t2} \tag{1}$$

Another body which experiences an acceleration is the muscle mass in both agonist and antagonist components. The inertial resistance of the muscle mass as well as the passive viscous resistance incurred are balanced against the net force of the tendon, contractile element, and passive elastic element. In the case of the agonist and antagonist, the linear version of Newton's Second Law leads to the equations

$$M_{l1}\ddot{l}_{m1}+B_{pm}\dot{l}_{m1}=F_{t1}-F_{act1}-F_{pe1}(l_{m1}) \tag{2}$$

and

$$M_{l2}\ddot{l}_{m2}+B_{pm}\dot{l}_{m2}=F_{t2}-F_{act2}-F_{pe2}(l_{m2}) \tag{3}$$

where for the moment we interpret all quantities strictly in terms of linear force. However, as previously mentioned, the model parameters used in this work are given in terms of units of rotational motion; accordingly. Equations 2 and 3 will be modified to accommodate parameters specified in the rotational frame of reference.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>1.24 cm</td>
<td>globe radius</td>
</tr>
<tr>
<td>$J_g$</td>
<td>$6 \times 10^{-5} \text{gts}^2/\degree$</td>
<td>globe rotational inertia</td>
</tr>
<tr>
<td>$B_g$</td>
<td>.0158 gt s/\degree</td>
<td>globe/orbit viscosity</td>
</tr>
<tr>
<td>$K_g$</td>
<td>.79 gt/\degree</td>
<td>globe/orbit elasticity</td>
</tr>
<tr>
<td>$B_{pm}$</td>
<td>.06 gt s/\degree</td>
<td>passive muscle viscosity</td>
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<tr>
<td>$M$</td>
<td>.748 g</td>
<td>muscle mass</td>
</tr>
<tr>
<td>$F_{\text{max}}$</td>
<td>100 gt</td>
<td>maximum isometric force</td>
</tr>
<tr>
<td>$l_{\text{mp}}$</td>
<td>4.0 cm</td>
<td>primary muscle length</td>
</tr>
<tr>
<td>$l_{\text{opt}}$</td>
<td>4.65 cm</td>
<td>optimal muscle length</td>
</tr>
<tr>
<td>$l_{\text{ms}}$</td>
<td>3.7 cm</td>
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</tr>
<tr>
<td>$l_{\text{mc}}$</td>
<td>4.8 cm</td>
<td>force level at which passive muscle elasticity becomes linear</td>
</tr>
<tr>
<td>$k_{\text{me}}$</td>
<td>.0387/\degree</td>
<td>muscle exponential shape parameter</td>
</tr>
<tr>
<td>$k_{\text{pm}}$</td>
<td>.9 gt/\degree</td>
<td>linear passive muscle elasticity</td>
</tr>
<tr>
<td>$k_{\text{ml}}$</td>
<td>.126 gt/\degree</td>
<td>minimum passive muscle elasticity</td>
</tr>
<tr>
<td>$F_{\text{mc}}$</td>
<td>20 gt</td>
<td>force level at which passive muscle elasticity becomes linear</td>
</tr>
<tr>
<td>$w$</td>
<td>.5</td>
<td>width of force-length curve</td>
</tr>
<tr>
<td>$V_{\text{max}}$</td>
<td>5689 °/s</td>
<td>max muscle velocity</td>
</tr>
<tr>
<td>$k_s$</td>
<td>2.5 gt/\degree</td>
<td>linear tendon elasticity</td>
</tr>
<tr>
<td>$k_{\text{dl}}$</td>
<td>1.5 gt/\degree</td>
<td>minimum tendon elasticity</td>
</tr>
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<td>$k_{\text{t}}$</td>
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</tr>
<tr>
<td>$l_{\text{ts}}$</td>
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</tr>
<tr>
<td>$l_{\text{tc}}$</td>
<td>.532 cm</td>
<td>tendon length at which tendon elasticity becomes linear</td>
</tr>
<tr>
<td>$F_{\text{tc}}$</td>
<td>30 gt</td>
<td>force level at which tendon elasticity becomes linear</td>
</tr>
</tbody>
</table>

Table 1: Parameter Values for the Complete Model
Formulation of Equations and Parameters

The standard unit of linear force is the Newton, N, where 1N = 1kg.m/s², or the dyne, where 1dyne = 1g.cm/s². Also, it is usually mathematically convenient to express angular displacements and rotational movements in radians; if the amount of rotation of the eye in radians is Θ, then the equivalent amount of linear displacement in the associated muscle-tendon paths is given by \( x = r\Theta \). Recall that the units of torque are N.m or dyne.cm; for example, if some force \( F \) is applied tangentially to a spherical mass of radius \( r \), the resulting torque is given by \( rF \). If we momentarily assume that the muscle tendons behave as simple linear springs with spring constant \( K_s \), and if \( J_G \), \( B_G \), and \( K_G \) denote the parameters for globe inertia, globe viscosity, and globe elasticity, respectively, in terms of standard units of force, torque and rotation, then the balance of torques in Equation 1 is given by

\[
J_G\ddot{\Theta} + B_G\dot{\Theta} + K_G\Theta = rK_s(\Delta l_{t1}-\Delta l_{t2}) \tag{4}
\]

\[
\hat{J}_G\ddot{x} + \hat{B}_G\dot{x} + \hat{K}_G x = K_s(\Delta l_{t1}-\Delta l_{t2}) \tag{5}
\]

Where

\[
\hat{J}_G = J_G/r^2, \quad \hat{B}_G = B_G/r^2, \quad \hat{K}_G = K_G/r^2 \tag{6}
\]

the proper units are g/s² for \( K_s \), g.cm² for \( J_G \), g.cm³/s for \( B_G \), and g.cm³/s² for \( K_G \). However, a convention peculiar to much of the oculomotor literature is that of reporting forces in the unit of grams tension, gt, where 1gt=1g.980 cm/s² = 980 dynes. Hence, the unit of grams tension is just the introduction of a different scale to linear force.

After scaling the torques of Equation 4 accordingly and dividing both sides by the globe radius \( r \)

\[
\frac{J_G}{980r}\ddot{\Theta} + \frac{B_G}{980r}\dot{\Theta} + \frac{K_G}{980r}\Theta = \frac{K_G}{980}(\Delta l_{t1}-\Delta l_{t2}) \tag{7}
\]
Furthermore, it is often customary to report rotations of the eye in degrees (°) as is perhaps more intuitive. Also, the change in tendon length is converted to equivalent angular displacement in degrees.

Equation 7

\[ \Theta = \frac{\pi \Theta^\circ}{180} \]

the net change in agonist and antagonist tendon lengths

We can now translate Equation 1 into parameters and variables having units consistent the general oculomotor literature. The parameters for globe inertia, globe viscosity and elasticity, and muscle tendon elasticity are henceforth denoted by \( J_g \), \( B_g \), \( K_g \), and \( K_s \), respectively. The balance of torques equation for the globe dynamics is expressed uniformly in gt thus:

\[ J_g \ddot{\Theta}^\circ + B_g \dot{\Theta}^\circ + K_g \Theta^\circ = K_s \Delta \Theta_t^\circ \quad (9) \]

Where

\[ J_g = \frac{J_G}{980r(180/\pi)} \quad \text{in gt.s}^2/\circ, \quad (10) \]

\[ B_g = \frac{B_G}{980r(180/\pi)} \quad \text{in gt.s}/\circ, \quad (11) \]

\[ K_g = \frac{K_G}{980r(180/\pi)} \quad \text{in gt}/\circ, \quad (12) \]

\[ K_s = \frac{K_s}{980(180/\pi)} \quad \text{in gt}/\circ, \quad (13) \]
and where we have temporarily assumed that the muscle tendon behaves linearly as a simple spring with spring constant $K_s$. Note carefully that the derivations of $K_s$ and $K_g$ differ due to the different places of their appearance in Equation 9. If we make no simplifying assumption as to the linearity of the muscle tendon, the following equation summarizes the globe dynamics:

$$J_G \ddot{\Theta} + B_G \dot{\Theta} + K_G \Theta = F_{t1} - F_{t2}$$  \hspace{1cm} (14)

Immediately, it should be emphasized that although $F_n$ and $F_{t2}$ are linear forces, Equation 14 is indeed valid for the rotational frame of reference; the multiplying factor $r^*$ which would translate the linear forces into torques is already accounted for in the derivation of the parameters on the left-hand side of Equation 7. With regard to the globe itself, the radius and density are taken to be $r^* = 1.24$ cm and $\rho = 1.0$. Hence, $J_G$, the rotational inertia in standard units is calculated to be approximately $4.17$ g cm$^2$, or equivalently, $4.17$ dynes cm sec$^2$; based upon Equation 10 this results in a value for $J_g$ of approximately $6 \times 10^{-5}$ gt $\cdot$ s$^2/\circ$. In addition, the previous reports show the values $B_g = .0158$ gt $\cdot$ s/° and $K_g = .79$ gt/°. they also mention an empirically determined value for tendon stiffness $K_s = 2.5$ gt/°; this value will be used to tentatively suggest values for parameters describing the non-linearity of the tendon. $M$ is approximated to be $0.748$ g based on an average primary length for the lateral and medial rectus of $4$ cm, an average cross-sectional area of $1.7$ cm, and an average human muscle density of $1.100$ g/ml.

The internal dynamics governed by Equations 2 and 3 involve forces and displacements with respect to linear motion. However, expressing all parameters in gt relative to angular displacements in degrees necessitates some modification of these equations. We assume that the forces $F_{t1}$, $F_{t2}$, $F_{act1}$, $F_{act2}$, $F_{pe1}$, and $F_{pe2}$ are reported in gt and that the parameter $B_{pm}$ is specified with the units gt $\cdot$ s/° as is $B_g$. The product of $M$ and muscle acceleration results in a force quantity in dynes, hence this term must be scaled down by the factor of 980 to be consistent with units of gt. Also, the velocity of linear muscle displacement must be converted to its angular equivalent in degrees so as to be consistent with $B_{pm}$.
The following equations for agonist and antagonist muscle dynamics result:

$$\frac{M}{980}\dot{l}_{m1} + B_{pm}(\frac{180}{\pi}l_{m1}) = F_{t1} - F_{act1} - F_{pe1}(l_{m1}) \quad (15)$$

$$\frac{M}{980}\dot{l}_{m2} + B_{pm}(\frac{180}{\pi}l_{m2}) = F_{t2} - F_{act2} - F_{pe2}(l_{m2}) \quad (16)$$

All forces are expressed in g. $B_{pm}$ is a parameter which seems to have critical influence but for which an exact value has not been established. In this work we take a representative value of 0.06 g s/° for $B_{pm}$. 
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