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Stability and stiffness analysis of a steel frame with an oblique beam using method of least work

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A R T I C L E I N F O

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A B S T R A C T

For the first time, this paper investigates the stability and stiffness analysis of a single column in a specific case with the connected oblique beam. In this case, the modification factors are analytically derived such that the stiffness of oblique beam is included in the calculation of the well- known parameters G_T and G_B. The effective length factor for such column can be obtained by these modified G_T and G_B. It is noted that the effective length factor of the column in the mentioned specific case is assessed for the first time in this research. In the following, a single span- one story steel frame is investigated to determine the lateral stiffness of the frame. In this frame, the applied lateral load and the structural frame are not on the same plane. Accordingly, it is also focused on the investigation of the effects of hinge existed in the beam as well as the changing of column base connection from fixed to hinged forms. The structure is considered as a 3- dimensional steel frame for the analysis upon the least work principle. All effective factors are taken into account including axial and shear loads as well as bending and torsional moments. At the end of the analysis, a relation is obtained through response surface method. The lateral stiffness can be calculated by the derived relation based on the specifications of the steel frame such as geometrical properties of the employed sections, specification of the used material and deviation angle of the beam.

1. Introduction

Stability analysis is one of the most important kinds of analyses which should be considered by engineers for analyzing the structures [1]. The elements of steel frames extensively used in buildings due to their structural efficiency have often small sectional areas because of the high strength of steel. This fact results in the higher probability of the global and local bucklings of elements in steel frames. Therefore, stability analysis is one of the primary concerns in design process of steel structures and developing a practical, effective and reliable approach to examine the stability of steel frames is a crucial challenge for engineers.

The effective length method has been ordinarily used by engineers for over 50 years in stability analysis of the columns in steel structures. The deficiencies of this method has resulted in presenting a new approach, so called “direct analysis method” in the last version of American Institute of Steel Constructions (ASIC) [2]. A problem with the effective length method is that if the beam connected to the column is oblique, then no solution is found in the literature for obtaining the effective length factor of the column. Although few studies on columns with elastic and rigid oblique restraint have been done, but all of these studies have been done with numerical methods like finite element method; therefore, they can only be considered for certain and not any cases. Of the mentioned researches, it can be referred to Trahair and Rasmussen researches [3,4,5,6]. In the first section of this research, it is attempted to calculate effective length factor based on the mathematical methods for the cases of oblique beam connected to the column. Providing appropriate lateral stiffness for the structure has significant effects on its seismic performance. This stiffness can be easily calculated if the lateral load and frame are all in the same plane. However, it is not straightforward while the beams connected to the columns are oblique.

In the second section of this research, lateral stiffness is calculated for a one-span one story steel frame considering that the beam connecting two columns is oblique, and the lateral load and frame are not on a plane.

In this section the important question is that while the desired responses can be obtained easily through finite element softwares, why the structures should be assessed by this method? The answer is that despite the heaviness of presented formulas, they have undeniable advantages in comparison to the numerical methods. That is in the latter, the response of each problem is obtained as per specified input values; therefore, the behavior trend of the structure cannot be investigated

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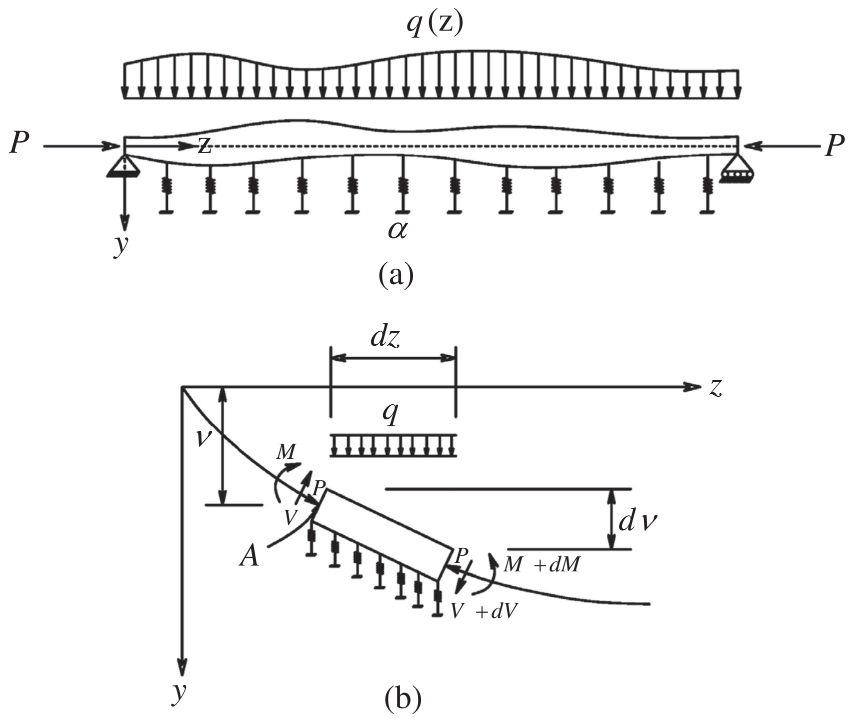

Fig. 1. The parameters used in the differential equation of buckling

well by these methods. The presented method offers a proper understanding of the behavior of the structures with rigid connection of oblique beam to column, even though with a one-story one- span structure. This understanding is much more valuable than the response of a particular problem. Moreover, the relations presented in the final section of this article have been obtained by the aid of these complicated formulas. While intensive calculation effort is needed each time in solving the problem by numerical methods, it is enough only once in the suggested method.

2. Solving the differential equation of buckling for a column

Consider a structural element subjected to an axial load P and a distributed load, q on the elastic bed as shown in Fig. 1(a). The governing differential equation of buckling for this element is generally expressed as follows:

$$EI_x \nu^{iv} + P\nu'' + \alpha\nu = q \tag{1}$$

where E is elastic modulus; I_x is moment of inertia; alpha is bed's elastic modulus and v is lateral displacement. This equation considers all factors which are effective on the stability of the element. Its parameters are presented in Fig. 1.

However, the columns, which are ordinarily used in the structures, are not located on the elastic beds. Moreover, no distributed load is applied along the column. Therefore, Eq. (1) is re-written for such columns as follows:

$$EI_x \nu^{iv} + P\nu'' = 0 \tag{2}$$

Solving the above differential equation results in the following solution:

$$\nu = A + Bz + C\sin kz + D\cos kz, \quad k^2 = \frac{P}{EI} \tag{3}$$

Obviously, four boundary conditions are needed for obtaining accurate values of ν. In this article, the buckling of columns is investigated in the steel frame and therefore no ideal assumption (such as hinged, roller or fixed end) should be considered for their boundaries. Fig. 2 is used for better understanding of the investigated column.

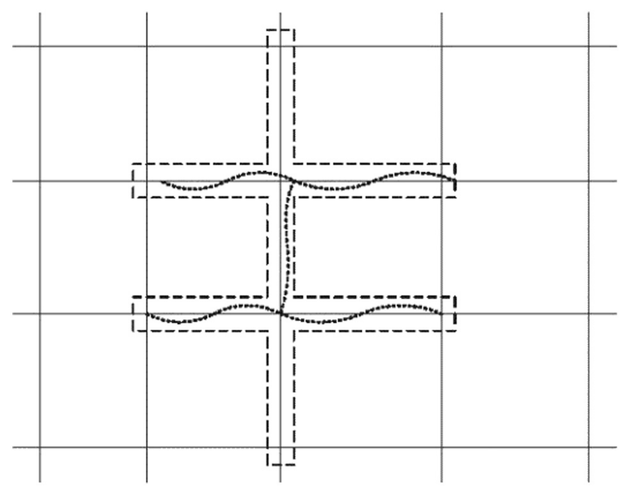

Fig. 2. Configuration of the element used in the steel frame for buckling analysis.

Fig. 3 is applied to express the boundary conditions in the problem. In this figure, both ends of the studied element can have either torsion or transmission movement. It should be noted that the springs used in each ends indicate the stiffness of the front and back beams at the end of column. In the other words, each spring contain the stiffness of two beams.

Boundary conditions of the considered problem are presented as follows:

$$\text{at } z = L \rightarrow \quad \begin{aligned} -EI\upsilon''' - P\upsilon' &= \beta_B \upsilon \\ -EI\upsilon'' &= -\alpha_B \upsilon' \end{aligned}$$

$$\text{at } z = 0 \rightarrow \quad \begin{aligned} -EI\upsilon''' - P\upsilon' &= \beta_T \upsilon \\ -EI\upsilon'' &= -\alpha_T \upsilon' \end{aligned}$$

The first line shows the equalization of the bending moment of column end with the moment created in the torsional spring. The second line shows the equalization of the shear force of column end with the force created in the transitive spring. The parameters used the

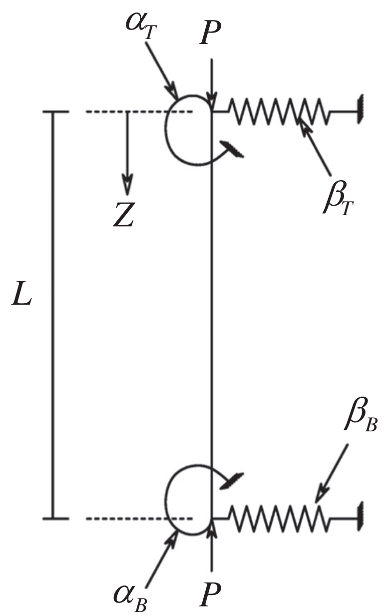

Fig. 3. The boundary conditions considered for the element under buckling.

mentioned boundary conditions are as follows:

α_T: stiffness of the top torsional spring
α_B: stiffness of the bottom torsional spring
β_T: stiffness of the top transmission spring
β_B: stiffness of the bottom transmission spring.

Now, the aforementioned boundary conditions are inserted in Eq. (3) to find the constants, A to Z. Furthermore, several new parameters are defined after the insertion and writing the equations in the matrix forms presented as follows:

$$\begin{bmatrix} T_T & (kL)^2 & 0 & T_T \\ 0 & R_T & R_T kL & (kL)^2 \\ T_B & T_B-(kL)^2 & T_B\sin(kL) & T_B\cos(kL) \\ 0 & R_B & R_B kL\cos(kL)-(kL)^2\sin(kL) & -R_B kL\sin(kL)-(kL)^2\cos(kL) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

where, $k=\sqrt{\frac{P}{EI}}$, $T_B=\frac{\beta_B L^3}{EI}$, $T_T=\frac{\beta_T L^3}{EI}$, $R_B=\frac{\alpha_B L}{EI}$ and $R_T=\frac{\alpha_T L}{EI}$.

Matrix determinant of the factors should be zero in order to obtain non-trivial responses from above equations. Therefore it is given as:

$$\begin{vmatrix} T_T & (kL)^2 & 0 & T_T \\ 0 & R_T & R_T kL & (kL)^2 \\ T_B & T_B-(kL)^2 & T_B\sin(kL) & T_B\cos(kL) \\ 0 & R_B & R_B kL\cos(kL)-(kL)^2\sin(kL) & -R_B kL\sin(kL)-(kL)^2\cos(kL) \end{vmatrix} = 0 \quad (5)$$

The resulting expansion of the above determinant is as follows:

$$\begin{aligned} & T_B\times kL^6\times\sin(kL)+T_T\times kL^6\times\sin(kL)-R_B\times T_B\times kL^5\times\cos(kL) \\ & -R_B\times T_T\times kL^5\times\cos(kL)-R_T\times T_B\times kL^5\times\cos(kL)-R_T\times T_T\times kL^5\times\cos(kL) \\ & -T_T\times T_B\times kL^4\times\sin(kL)-R_B\times R_T\times T_B\times kL^4\times\sin(kL) \\ & -R_B\times R_T\times T_T\times kL^4\times\sin(kL)+R_T\times T_B\times T_T\times kL^2\times\sin(kL) \\ & -R_T\times T_B\times T_T\times kL^2\times\sin(kL)-R_B\times R_T\times T_B\times T_T\times kL \\ & +R_B\times T_B\times T_T\times kL^3\times\cos(kL)+R_T\times T_B\times T_T\times kL^3\times\cos(kL) \\ & -R_B\times R_T\times T_B\times T_T\times kL\times\cos(kL)^2-R_B\times R_T\times T_B\times T_T\times kL\times\sin(kL)^2 \\ & +R_B\times R_T\times T_B\times T_T\times kL^2\times\sin(kL)+2\times R_B\times R_T\times T_B\times T_T\times kL\times\cos(kL)=0 \end{aligned} \quad (6)$$

Evidently, Eq. (6) cannot be used for practical problems, because of its complexity; therefore, two assumptions are considered to simplify the trend of problem for solving and obtaining the determinant result:

1– The top of column moves only against its bottom, while the bottom of the column is completely fixed. In the other words, the transitional displacement of the column is limited to the relative movement between its top and bottom. By this assumption: $T_T=0, T_B=\infty$.
2– The beams connected to the column experience deformation in the form of double curvatures. Consequently, torsional angles are the same at the ends near and far from the column. Fig. 4 well illustrate this situation for better understanding.

In order to apply the first assumption, the third line is primarily divided by T_B and then the values of T_T and T_B are considered as 0 and ∞, respectively. Finally, Eq. (5) is obtained in the following form:

$$\begin{vmatrix} 0 & (kL)^2 & 0 & 0 \\ 0 & R_T & R_T kL & (kL)^2 \\ 1 & 1 & \sin(kL) & \cos(kL) \\ 0 & R_B & R_B kL\cos(kL)-(kL)^2\sin(kL) & -R_B kL\sin(kL)-(kL)^2\cos(kL) \end{vmatrix} = 0 \quad (7)$$

By expanding the determinant through MATLAB software, the above formula becomes:

$$-kL^2\times(R_B\times kL^3\times\cos(kL)-kL^4\times\sin(kL)+R_T\times kL^3\times\cos(kL)+R_B\times R_T\times kL^2\times\sin(kL))=0 \quad (8)$$

After simplification, Eq. (8) will be re-written as follows:

$$(R_B+R_T)kL+[R_T R_B-(kL)^2]\tan(kL)=0 \quad (9)$$

If the beams connected to the column are all perpendicular to it, then considering Fig. 4, the following equations can be presented:

$$\alpha_T=\frac{6EI_{BT}}{L_{BT}},\quad \alpha_B=\frac{6EI_{BB}}{L_{BB}} \quad (10)$$

$$R_B=\frac{\alpha_B L_C}{EI_C}=\frac{6EI_{BB}}{L_{BB}}\frac{L_C}{EI_C}=6\left(\frac{I_{BB}/L_{BB}}{I_C/L_C}\right)=\frac{6}{G_B} \quad (11)$$

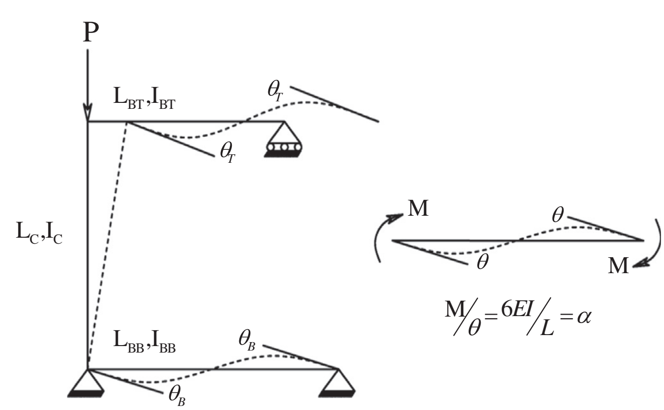

Fig. 4. Second assumption for simplifying the solution of differential equation.

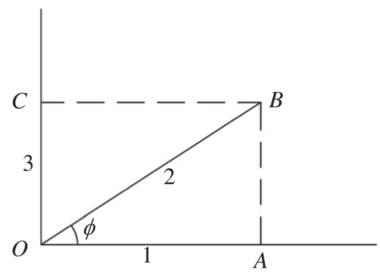

Fig. 5. View of the element of oblique beam and virtual elements substituted in the plan.

$$R_T = \frac{\alpha_T L_C}{EI_C} = \frac{6EI_{BT}}{L_{BT}}\frac{L_C}{EI_C} = 6\left(\frac{I_{BT}/L_{BT}}{I_C/L_C}\right) = \frac{6}{G_T} \tag{12}$$

By substituting Eqs. (11) and (12) into Eq. (9), the following equation is obtained:

$$\frac{kL}{\tan KL} - \frac{(\mathrm{kL})^2 G_T G_B - 36}{6(\mathrm{G}_T + \mathrm{G}_B)} = 0 \tag{13}$$

However, concerning general cases in which the beams connected to the column can be oblique, the following expression is used to obtain the relations.

According to Figs. 5–8, one can write as the following relations:

$$\tan(\theta_1) = \frac{AT_1}{OA} \tag{14}$$

$$\tan(\theta_2) = \frac{BT_2}{OB} \tag{15}$$

$$\tan(\theta_3) = \frac{CT_3}{OC} \tag{16}$$

$$OA = OB\cos(\phi), \quad OC = OB\sin(\phi) \tag{17}$$

Besides, from Fig. 9, M_1 and M_3 have the forms written as:

$$M_1 = M_2\cos(\phi) \tag{18}$$

$$M_3 = M_2\sin(\phi) \tag{19}$$

The formula obtained from the relations of strength of material is as follows:

$$\theta_2 = \frac{M_2 L_2}{EI_2} \tag{20}$$

Considering small deformations, it can be written:

$$\tan(\theta_1) = \theta_1, \quad \tan(\theta_2) = \theta_2, \quad \tan(\theta_3) = \theta_3$$

$$AT_1 = BT_2 = CT_3$$

Now, by substituting the Eqs. (17) and (18) into Eq. (20), θ_1 is written as:

$$\theta_1 = \frac{M_1 L_2}{EI_2\cos^2(\phi)} \tag{21}$$

According to the relation, the stiffness of frame is multiplied by the second power of the cosinus of the beam deviation angle. Based on the

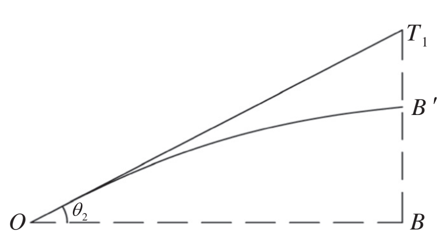

Fig. 6. Deformed view of real element, the second element of Fig. 5.

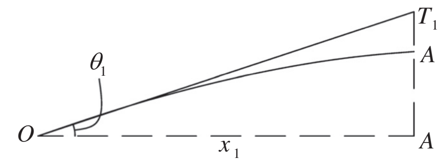

Fig. 7. Deformed view of virtual element, the first element of Fig. 5.

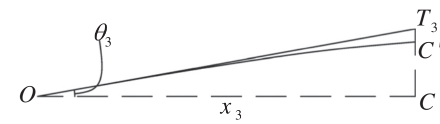

Fig. 8. Deformed view of virtual element, the third element of Fig. 5.

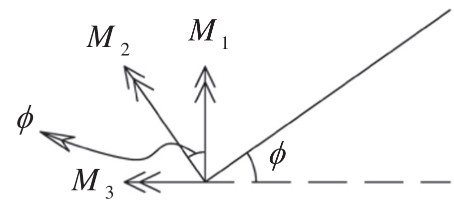

Fig. 9. View of the moments formed in the real and virtual elements.

mentioned expression, it can be written as:

$$\alpha_T = \frac{3(EI)_{BTL}\cos^2\phi_{TL}}{L_{BTL}} + \frac{3(EI)_{BTR}\cos^2\phi_{TR}}{L_{BTR}} \tag{22}$$

$$\alpha_B = \frac{3(EI)_{BBL}\cos^2\phi_{BL}}{L_{BBL}} + \frac{3(EI)_{BBR}\cos^2\phi_{BR}}{L_{BBR}} \tag{23}$$

The parameters utilized in the Eqs. (22) and (23) are as follows:

$(EI)_{BTL}$: bending rigidity of the top beam and left side of the column
L_{BTL}: length of the top beam and left side of the column
ϕ_{TL}: deviation angle of the top beam and left side of the column.

Other parameters are also entitled in the same way as well.

It should be noted that the effect of transitional spring has been totally removed in the studied problem due to the application of the first assumption and therefore there is no need to modify the axial stiffness of the beam.

Now, it can be written as:

$$R_T = \frac{\alpha_T L_C}{EI_C} = 3\cos^2\phi_{TL}\frac{\left(\frac{I_{BTL}}{L_{BTL}}\right)}{\left(\frac{I_C}{L_C}\right)} + 3\cos^2\phi_{TR}\frac{\left(\frac{I_{BTR}}{L_{BTR}}\right)}{\left(\frac{I_C}{L_C}\right)} \tag{24}$$

$$R_B = \frac{\alpha_B L_C}{EI_C} = 3\cos^2\phi_{BL}\frac{\left(\frac{I_{BBL}}{L_{BBL}}\right)}{\left(\frac{I_C}{L_C}\right)} + 3\cos^2\phi_{BR}\frac{\left(\frac{I_{BBR}}{L_{BBR}}\right)}{\left(\frac{I_C}{L_C}\right)} \tag{25}$$

Finally, in order to formulate the equation more easily, new definitions are given for G_T and G_B as follows:

$$G_T = \frac{\sum\left(\frac{I_C}{L_C}\right)}{\sum\left(\frac{I_{BT}}{L_{BT}}\cos^2\phi_T\right)} \tag{26}$$

$$G_B = \frac{\sum\left(\frac{I_C}{L_C}\right)}{\sum\left(\frac{I_{BB}}{L_{BB}}\cos^2\phi_B\right)} \tag{27}$$

Eqs. (24) and (25) are re-written using the above mentioned definitions as follows:

Table 1
Effective length factors obtained from the developed formula compared with those obtained from Trahair and Rasmussen research [5].

Deviation angle (degrees)	Effective length factor	
	Results from reference no. [5]	Results from developed formula in this research
0	1.540	1.536
10	1.555	1.549
20	1.591	1.594
30	1.650	1.667
40	1.740	1.800

$$R_T = \frac{6}{G_T} \tag{11-repeat}$$

$$R_B = \frac{6}{G_L} \tag{12-repeat}$$

Considering the Eqs. (11) and (12), only the factors G_T and G_B should be modified for solving the problems in which the beam(s) connected to the column is/are oblique. The curves, presented by Julian and Lorenz [2] for obtaining the length factor effective in G_T and G_B, can be used after modifying the mentioned parameters.

Table 1 compares the effective length factors obtained from Table 3 in the Trahair and Rasmussen research [5] and those calculated in this research by alignment chart using G_T and G_B presented in Eqs. (26) and (27). It is noticeable that the results presented in the Trahair and Rasmussen research are in terms of critical buckling load [5]. The critical loads presented in the Trahair and Rasmussen research have been converted to the effective length factor through the Eqs. (5), (6) and (38) presented in their article [5]. Regarding the effective length factors compared in Table 1, the results obtained from the developed formula in this research are reliable for practical purposes.

The deviation angle in the above table is limited to 40 degrees because for deviation angle greater than 45 degrees, buckling of column will occur about other principal axis of column section. It is noticeable that the boundary conditions are identical at the top and bottom of column in the Trahair and Rasmussen research [5]. Moreover G_{top} and G_{bot} can be calculated from $k_{\alpha y}$ presented in their research.

3. Lateral stiffness of the steel frame with oblique beam

This section focuses on the investigation of lateral stiffness variations in a one-story one-span steel frame in which the lateral load applied to the frame and structural frame are not on one plane.

This investigation is conducted in 6 different cases:

1. Columns to ground connections are clamped; and columns to beam connections are rigid.
2. Columns to ground connections are clamped; column to beam connection is hinged near the load; and other column to beam connection is rigid.
3. Columns to ground connections are clamped; column to beam connection is hinged far from the load; and other column to beam connection is rigid.
4. Columns to ground connections are clamped; both columns to beam connections are hinged.
5. Column to ground connection is hinged near the load; other column to ground connection is clamped; and both columns to beam connections are rigid.
6. Column to ground is hinged far from the load; other column to ground connection is clamped; and both columns to beam connections are rigid.

The aforementioned cases are analyzed through the least work method [7,8].

3.1. First case (columns to ground and columns to beam connections are rigid)

Determination the degree of indeterminacy:

$$\text{D. I.} = (6m + r) - (6n + c) = 6$$
$$m = 3, \quad r = 12, \quad n = 4, \quad c = 0$$

Therefore, the calculations are expected to be ended in a sixth order equation set. Regarding Fig. 11, R_{x_1}, R_{y_1}, R_{z_1}, M_{x_1}, M_{y_1} and $M_{z_1}$are considered as the main unknowns of the problem.

Now, the equilibrium equations are expressed based on Fig. 11 as follows:

$$\sum F_x = 0 \quad \rightarrow \quad R_{x_1} + R_{x_2} + P = 0 \quad \rightarrow \quad R_{x_2} = -(R_{x_1} + P) \tag{28}$$

$$\sum F_y = 0 \quad \rightarrow \quad R_{y_1} + R_{y_2} = 0 \quad \rightarrow \quad R_{y_2} = -R_{y_1} \tag{29}$$

$$\sum F_z = 0 \quad \rightarrow \quad R_{z_1} + R_{z_2} = 0 \quad \rightarrow \quad R_{z_2} = -R_{z_1} \tag{30}$$

$$\sum M_{x_D} = 0 \quad \rightarrow \quad -R_{z_1} b + M_{x_2} + M_{x_1} = 0 \quad \rightarrow \quad M_{x_2} = R_{z_1} b - M_{x_1} \tag{31}$$

$$\sum M_{y_D} = 0 \quad \rightarrow \quad PL_1 + R_{z_1} a + M_{y_2} + M_{y_1} = 0 \quad \rightarrow \quad M_{y_2} = -PL_1 - R_{z_1} a - M_{y_1} \tag{32}$$

$$\sum M_{z_D} = 0 \quad \rightarrow \quad R_{x_1} b - R_{y_1} a + M_{z_2} + M_{z_1} = 0 \quad \rightarrow \quad M_{z_2} = R_{y_1} a - R_{x_1} b - M_{z_1} \tag{33}$$

Now, strain energy of the elements is obtained and that of the left far column (far from load) is as follows:

$$U_1 = \int_0^{L_1} \frac{M_x^2(z)dz}{2EI_{x_1}} + \int_0^{L_1} \frac{M_y^2(z)dz}{2EI_{y_1}} + \int_0^{L_1} \frac{M_z^2(z)dz}{2GJ_1} + \int_0^{L_1} \frac{V_x^2(z)dz}{2GA_{s_{x1}}} + \int_0^{L_1} \frac{V_y^2(z)dz}{2GA_{s_{y1}}} + \int_0^{L_1} \frac{N^2(z)dz}{2EA_1} \tag{34}$$

For the left far column, it can be written as:

$$M_x(z) = M_{x_1} + R_{y_1} z \tag{35}$$

$$M_y(z) = M_{y_1} - R_{x_1} z \tag{36}$$

Eqs. (35) and (36) are substituted into Eq. (34) form which the integration is obtained as follows:

$$U_1 = \frac{M_{x_1}^2 L_1}{2EI_{x_1}} + \frac{R_{y_1}^2 L_1^3}{6EI_{x_1}} + \frac{M_{x_1} R_{y_1} L_1^2}{2EI_{x_1}} + \frac{M_{y_1}^2 L_1}{2EI_{y_1}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_1}} - \frac{M_{y_1} R_{x_1} L_1^2}{2EI_{y_1}} + \frac{M_{z_1}^2 L_1}{2GJ_1} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x1}}} + \frac{R_{y_1}^2 L_1}{2GA_{s_{y1}}} + \frac{R_{z_1}^2 L_1}{2EA_1} \tag{37}$$

Strain energy of the beam element is:

$$U_2 = \int_0^{L_2} \frac{M_z^2(x)dx}{2EI_{z_2}} + \int_0^{L_2} \frac{M_y^2(x)dx}{2EI_{y_2}} + \int_0^{L_2} \frac{M_x^2(x)dx}{2GJ_2} + \int_0^{L_2} \frac{V_z^2(x)dx}{2GA_{s_{z2}}} + \int_0^{L_2} \frac{V_y^2(x)dx}{2GA_{s_{y2}}} + \int_0^{L_2} \frac{N^2(x)dx}{2EA_2} \tag{38}$$

Similarly, for the beam element it can be written as:

$$M_{y_B}(x) = M_{y_B} + F_{z_B} x \tag{39}$$

$$M_{z_B}(x) = M_{z_B} - F_{y_B} x \tag{40}$$

Eqs. (39) and (40) are substituted into Eq. (38) and integrated as follows:

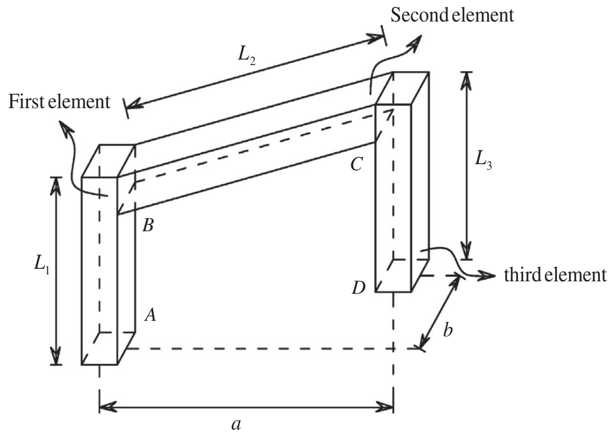

Fig. 10. Geometry of the studied problem.

$$U_2 = \frac{M_{zB}^2 L_2}{2EI_{z_2}} + \frac{F_{yB}^2 L_2^3}{6EI_{z_2}} - \frac{M_{zB} F_{yB} L_2^2}{2EI_{z_2}} + \frac{M_{yB}^2 L_2}{2EI_{y_2}} + \frac{F_{zB}^2 L_2^3}{6EI_{y_2}} + \frac{M_{yB} F_{zB} L_2^2}{2EI_{y_2}} + \frac{M_{xB}^2 L_2}{2GJ_2} + \frac{F_{zB}^2 L_2}{2GA_{s_{z2}}} + \frac{F_{yB}^2 L_2}{2GA_{s_{y2}}} + \frac{F_{xB}^2 L_2}{2EA_2} \tag{41}$$

All forces or moments are transferred from the left far column to the beam due to the presence of rigid connection between these two elements. Therefore, (with respect to Fig. 10) it can be written as:

$$F_{zB} = R_{z_1} \tag{42}$$

$$F_{xB} = R_{x_1}\cos(\alpha) + R_{y_1}\sin(\alpha) \tag{43}$$

$$F_{yB} = -R_{x_1}\sin(\alpha) + R_{y_1}\cos(\alpha) \tag{44}$$

$$M_{zB} = M_{z_1} \tag{45}$$

$$M_{xB} = M_{xC}\cos(\alpha) + M_{yC}\sin(\alpha) \tag{46}$$

$$M_{yB} = -M_{xC}\sin(\alpha) + M_{yC}\cos(\alpha) \tag{47}$$

It can be written for the left far column at the end of B as:

$$M_{xC} = M_{x_1} - R_{y_1}L_1 \tag{48}$$

$$M_{yC} = M_{y_1} - R_{x_1}L_1 \tag{49}$$

By substituting Eqs. (48)–(49) into Eqs. (42)–(47) and then the result in Eq. (40), it is given as:

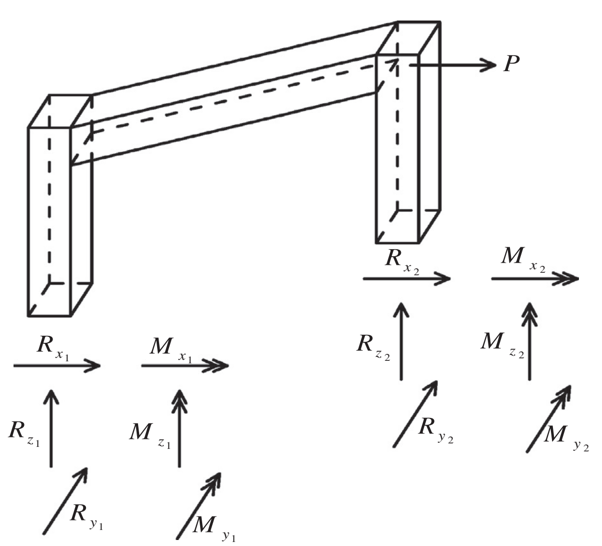

Fig. 11. Lateral load applied to the steel frame and the supporting reactions.

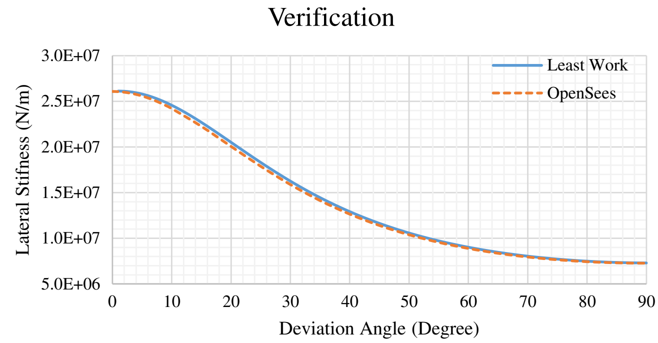

Fig. 12. Verifying the results obtained from OpenSees.

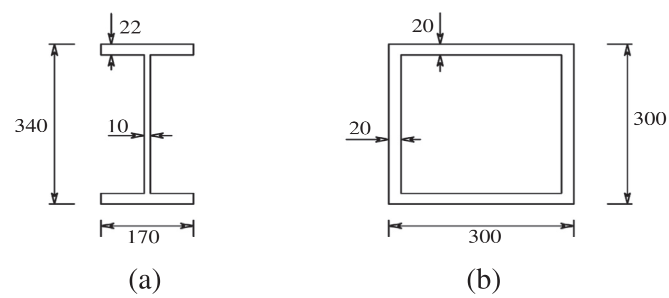

Fig. 13. The sections used for a) beam and b) columns, the values as per mm.

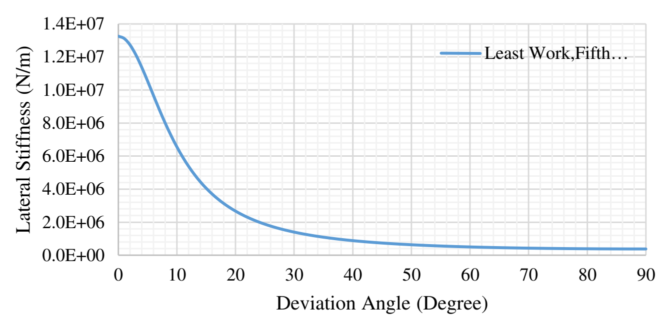

Fig. 14. Changing of lateral stiffness for the fifth case.

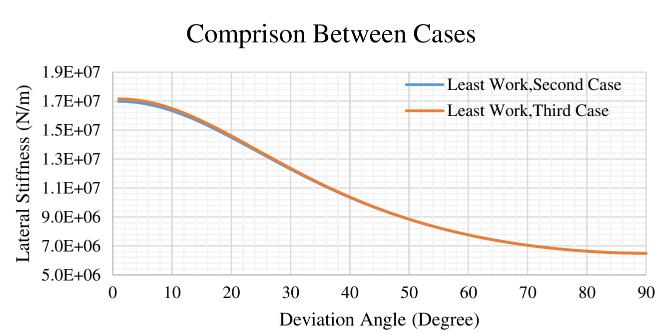

Fig. 15. Lateral stiffness of the steel frame in the second and third cases.

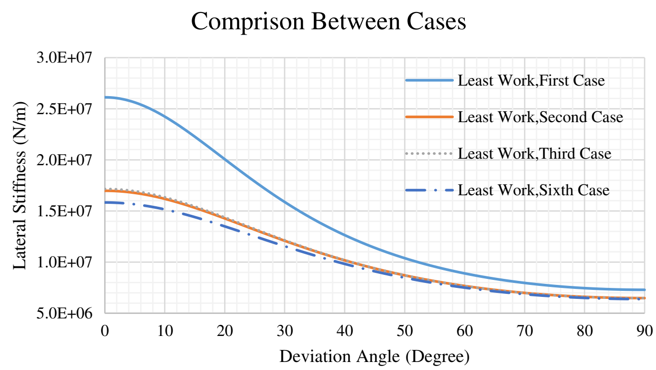

Fig. 16. Variations of lateral stiffness in different cases.

$$\begin{aligned}U_2 = & \frac{M_{y_1}^2 \cos^2(\alpha) L_2}{2EI_{y_2}} + \frac{R_{x_1}^2 L_1^2 \cos^2(\alpha) L_2}{2EI_{y_2}} - \frac{M_{y_1} R_{x_1} L_1 \cos^2(\alpha) L_2}{EI_{y_2}} + \frac{M_{x_1}^2 \sin^2(\alpha) L_2}{2EI_{y_2}} \\ & + \frac{R_{y_1}^2 L_1^2 \sin^2(\alpha) L_2}{2EI_{y_2}} \\ & + \frac{M_{x_1} R_{y_1} L_1 \sin^2(\alpha) L_2}{EI_{y_2}} - \frac{M_{x_1} M_{y_1} \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} + \frac{M_{x_1} R_{x_1} L_1 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} \\ & - \frac{M_{y_1} R_{y_1} L_1 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} \\ & + \frac{R_{x_1} R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} + \frac{R_{z_1}^2 L_2^3}{6EI_{y_2}} + \frac{M_{y_1} R_{z_1} \cos(\alpha) L_2^2}{2EI_{y_2}} - \frac{R_{x_1} R_{z_1} L_1 \cos(\alpha) L_2^2}{2EI_{y_2}} \\ & - \frac{M_{x_1} R_{z_1} \sin(\alpha) L_2^2}{2EI_{y_2}} \\ & - \frac{R_{y_1} R_{z_1} L_1 \sin(\alpha) L_2^2}{2EI_{y_2}} + \frac{M_{z_1}^2 L_2}{2EI_{z_2}} + \frac{R_{y_1}^2 \cos^2(\alpha) L_2^3}{6EI_{z_2}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2^3}{6EI_{z_2}} \\ & - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} \\ & - \frac{M_{z_1} R_{y_1} \cos(\alpha) L_2^2}{2EI_{z_2}} + \frac{M_{z_1} R_{x_1} \sin(\alpha) L_2^2}{2EI_{z_2}} + \frac{M_{y_1}^2 \sin^2(\alpha) L_2}{2GJ_2} + \frac{R_{x_1}^2 L_1^2 \sin^2(\alpha) L_2}{2GJ_2} \\ & - \frac{M_{y_1} R_{x_1} L_1 \sin^2(\alpha) L_2}{GJ_2} \\ & + \frac{M_{x_1}^2 \cos^2(\alpha) L_2}{2GJ_2} + \frac{R_{y_1}^2 L_1^2 \cos^2(\alpha) L_2}{2GJ_2} + \frac{M_{x_1} R_{y_1} L_1 \cos^2(\alpha) L_2}{GJ_2} + \frac{M_{x_1} M_{y_1} \sin(\alpha)\cos(\alpha) L_2}{GJ_2} \\ & - \frac{M_{x_1} R_{x_1} L_1 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} + \frac{M_{y_1} R_{y_1} L_1 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} - \frac{R_{x_1} R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} \\ & + \frac{R_{y_1}^2 \cos^2(\alpha) L_2}{2GA_{sy_2}} \\ & + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{2GA_{sy_2}} - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{R_{z_1}^2 L_2}{2GA_{sz_2}} + \frac{R_{y_1}^2 \sin^2(\alpha) L_2}{2EA_2} + \frac{R_{x_1}^2 \cos^2(\alpha) L_2}{2EA_2} \\ & - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2}{EA_2}\end{aligned} \tag{50}$$

Strain energy of the right near column (near to load) is calculated as follows:

$$\begin{aligned}U_3 = & \int_0^{L_3} \frac{M_x^2(z)dz}{2EI_{x_3}} + \int_0^{L_3} \frac{M_y^2(z)dz}{2EI_{y_3}} + \int_0^{L_3} \frac{M_z^2(z)dz}{2GJ_3} + \int_0^{L_3} \frac{V_x^2(z)dz}{2GA_{sx_3}} \\ & + \int_0^{L_3} \frac{V_y^2(z)dz}{2GA_{sy_3}} + \int_0^{L_3} \frac{N^2(z)dz}{2EA_3}\end{aligned} \tag{51}$$

It is written for the right near column element as:

$$M_x(z) = M_{x_2} + R_{y_2} z \tag{52}$$

$$M_y(z) = M_{y_2} - R_{x_2} z \tag{53}$$

Eqs. (52) and (53) are substituted into Eq. (51) from which the integration is obtained as:

$$\begin{aligned}U_3 = & \frac{M_{x_2}^2 L_1}{2EI_{x_3}} + \frac{R_{y_2}^2 L_1^3}{6EI_{x_3}} + \frac{M_{x_2} R_{y_2} L_1^2}{2EI_{x_3}} + \frac{M_{y_2}^2 L_1}{2EI_{y_3}} + \frac{R_{x_2}^2 L_1^3}{6EI_{y_3}} - \frac{M_{y_2} R_{x_2} L_1^2}{2EI_{y_3}} \\ & + \frac{M_{z_2}^2 L_1}{2GJ_3} + \frac{R_{x_2}^2 L_1}{2GA_{sx_3}} + \frac{R_{y_2}^2 L_1}{2GA_{sy_3}} + \frac{R_{z_2}^2 L_1}{2EA_3}\end{aligned} \tag{54}$$

The following equation is obtained by substituting Eqs. (28)–(32) into Eq. (54):

$$\begin{aligned}U_3 = & \frac{R_{z_1}^2 b^2 L_1}{2EI_{x_3}} + \frac{M_{x_1}^2 L_1}{2EI_{x_3}} - \frac{M_{x_1} R_{z_1} b L_1}{EI_{x_3}} + \frac{R_{y_1}^2 L_1^3}{6EI_{x_3}} - \frac{R_{y_1} R_{z_1} b L_1^2}{2EI_{x_3}} + \frac{M_{x_1} R_{y_1} L_1^2}{2EI_{x_3}} \\ & + \frac{R_{z_1}^2 a^2 L_1}{2EI_{y_3}} + \frac{M_{y_1}^2 L_1}{2EI_{y_3}} + \frac{P R_{z_1} a L_1^2}{2EI_{y_3}} + \frac{M_{y_1} P L_1^2}{2EI_{y_3}} + \frac{M_{y_1} R_{z_1} a L_1}{EI_{y_3}} + \frac{P^2 L_1^3}{6EI_{y_3}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_3}} \\ & - \frac{P R_{x_1} L_1^3}{6EI_{y_3}} - \frac{R_{x_1} R_{z_1} a L_1^2}{2EI_{y_3}} - \frac{M_{y_1} R_{x_1} L_1^2}{2EI_{y_3}} + \frac{R_{y_1}^2 a^2 L_1}{2GJ_3} + \frac{R_{x_1}^2 b^2 L_1}{2GJ_3} + \frac{M_{z_1}^2 L_1}{2GJ_3} \\ & - \frac{R_{x_1} R_{y_1} a b L_1}{GJ_3} \\ & - \frac{M_{z_1} R_{y_1} a L_1}{GJ_3} + \frac{M_{z_1} R_{x_1} b L_1}{GJ_3} + \frac{P^2 L_1}{2GA_{sx_3}} + \frac{R_{x_1}^2 L_1}{2GA_{sx_3}} + \frac{P R_{x_1} L_1}{GA_{sx_3}} + \frac{R_{y_1}^2 L_1}{2GA_{sy_3}} + \frac{R_{z_1}^2 L_1}{2EA_3}\end{aligned} \tag{55}$$

Below equations are obtained after differentiation from the strain energy of the elements with some simplifications:

$$\begin{aligned}\frac{\partial U}{\partial R_{x1}} = & R_{x1}\left(\frac{L_1^3}{3EI_{y_1}} + \frac{L_1}{GA_{sx1}} + \frac{L_1^3}{3EI_{y_3}} + \frac{b^2 L_1}{GJ_3} + \frac{L_1}{GA_{sx3}} + \frac{L_1^2 \cos^2(\alpha) L_2}{EI_{y_2}} + \frac{\sin^2(\alpha) L_2^3}{3EI_{z2}}\right. \\ & \left. + \frac{L_1^2 \sin^2(\alpha) L_2}{GJ_2} + \frac{\sin^2(\alpha) L_2}{GA_{sy_2}} + \frac{\cos^2(\alpha) L_2}{EA_2}\right) + R_{y_1}\left(-\frac{abL_1}{GJ_3} + \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}}\right. \\ & - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z2}} \\ & \left. - \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} - \frac{\sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{EA_2}\right) + R_{z1}\left(-\frac{aL_1^2}{2EI_{y_3}} - \frac{L_1 \cos(\alpha) L_2^2}{2EI_{y_2}}\right) \\ & + M_{x1}\left(\frac{L_1 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} - \frac{L_1 \sin(\alpha)\cos(\alpha) L_2}{GJ_2}\right) + M_{y_1}\left(-\frac{L_1^2}{2EI_{y_1}} - \frac{L_1^2}{2EI_{y_3}} - \frac{L_1 \cos^2(\alpha) L_2}{EI_{y_2}}\right. \\ & \left. - \frac{L_1 \sin^2(\alpha) L_2}{GJ_2}\right) + M_{z_1}\left(\frac{bL_1}{GJ_3} + \frac{\sin(\alpha) L_2^2}{2EI_{z2}}\right) + P\left(-\frac{L_1^3}{6EI_{y_3}} + \frac{L_1}{GA_{sx3}}\right)\end{aligned} \tag{56}$$

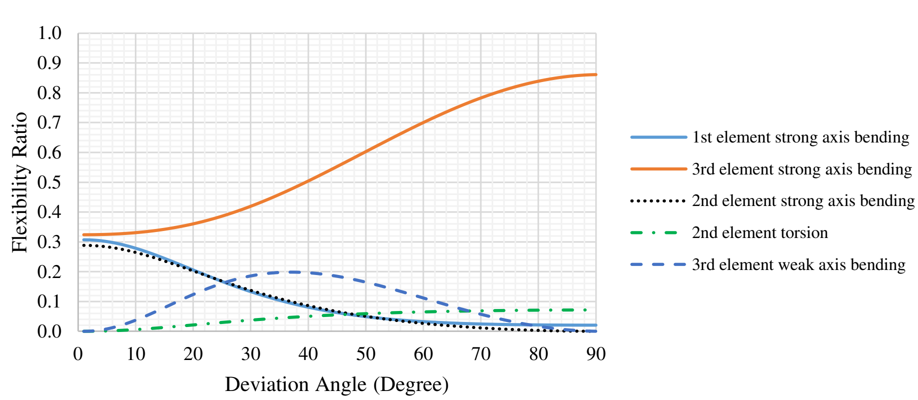

Fig. 17. Cooperation rate of internal efforts in the lateral flexibility for the first case.

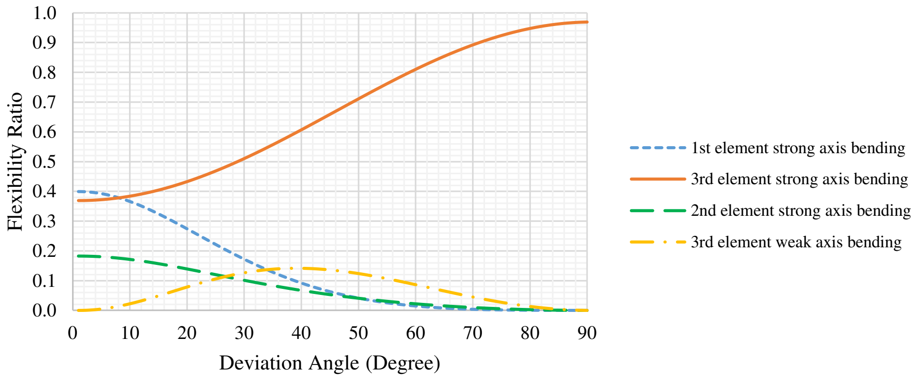

Fig. 18. Cooperation rate of internal efforts in the lateral flexibility of the steel frame in the second case.

$$\frac{\partial U}{\partial R_{y_1}} = R_{x1}\left(-\frac{abL_1}{GJ_3} + \frac{L_1^2\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} - \frac{\sin(\alpha)\cos(\alpha)L_2^3}{3EI_{z2}} - \frac{L_1^2\sin(\alpha)\cos(\alpha)L_2}{GJ_2} - \frac{\sin(\alpha)\cos(\alpha)L_2}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha)L_2}{EA_2}\right) + R_{y_1}\left(\frac{L_1^3}{3EI_{x1}} + \frac{L_1}{GA_{sy_1}} + \frac{L_1^3}{3EI_{x3}} + \frac{L_1}{GA_{sy_3}} + \frac{a^2L_1}{GJ_3} + \frac{L_1^2\sin^2(\alpha)L_2}{EI_{y_2}} + \frac{\cos^2(\alpha)L_2^3}{3EI_{z2}} + \frac{L_1^2\cos^2(\alpha)L_2}{GJ_2} + \frac{\cos^2(\alpha)L_2}{GA_{sy_2}} + \frac{\sin^2(\alpha)L_2}{EA_2}\right) + R_{z1}\left(-\frac{bL_1^2}{2EI_{x3}} - \frac{L_1\sin(\alpha)L_2^2}{2EI_{y_2}}\right) + M_{x1}\left(\frac{L_1^2}{2EI_{x1}} + \frac{L_1^2}{2EI_{x3}} + \frac{L_1\sin^2(\alpha)L_2}{EI_{y_2}} + \frac{L_1\cos^2(\alpha)L_2}{GJ_2}\right) + M_{y_1}\left(-\frac{L_1\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} + \frac{L_1\sin(\alpha)\cos(\alpha)L_2}{GJ_2}\right) + M_{z1}\left(-\frac{aL_1}{GJ_3} - \frac{\cos(\alpha)L_2^2}{2EI_{z2}}\right) \quad (57)$$

$$\frac{\partial U}{\partial R_{z1}} = R_{x1}\left(-\frac{aL_1^2}{2EI_{y_3}} - \frac{L_1\cos(\alpha)L_2^2}{2EI_{y_2}}\right) + R_{y_1}\left(-\frac{bL_1^2}{2EI_{x3}} - \frac{L_1\sin(\alpha)L_2^2}{2EI_{y_2}}\right) + R_{z_1}\left(\frac{L_1}{EA_1} + \frac{L_1}{EA_3} + \frac{b^2L_1}{EI_{x3}} + \frac{a^2L_1}{EI_{y_3}} + \frac{L_2^3}{3EI_{y_2}} + \frac{L_2}{GA_{sz2}}\right) + M_{x1}\left(-\frac{bL_1}{EI_{x3}} - \frac{\sin(\alpha)L_2^2}{2EI_{y_2}}\right) + M_{y_1}\left(\frac{aL_1}{EI_{y_3}} + \frac{\cos(\alpha)L_2^2}{2EI_{y_2}}\right) + P\left(\frac{aL_1^2}{2EI_{y_3}}\right) \quad (58)$$

$$\frac{\partial U}{\partial M_{x1}} = R_{x_1}\left(\frac{L_1\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} - \frac{L_1\sin(\alpha)\cos(\alpha)L_2}{GJ_2}\right) + R_{y_1}\left(\frac{L_1^2}{2EI_{x1}} + \frac{L_1^2}{2EI_{x3}} + \frac{L_1\sin^2(\alpha)L_2}{EI_{y_2}} + \frac{L_1\cos^2(\alpha)L_2}{GJ_2}\right) + R_{z1}\left(-\frac{bL_1}{EI_{x3}} - \frac{\sin(\alpha)L_2^2}{2EI_{y_2}}\right) + M_{x1}\left(\frac{L_1}{EI_{x1}} + \frac{L_1}{EI_{x3}} + \frac{\sin^2(\alpha)L_2}{EI_{y_2}} + \frac{\cos^2(\alpha)L_2}{GJ_2}\right) + M_{y_1}\left(-\frac{\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} + \frac{\sin(\alpha)\cos(\alpha)L_2}{GJ_2}\right) \quad (59)$$

$$\frac{\partial U}{\partial M_{y_1}} = R_{x1}\left(-\frac{L_1^2}{2EI_{y_1}} - \frac{L_1^2}{2EI_{y_3}} - \frac{L_1\cos^2(\alpha)L_2}{EI_{y_2}} - \frac{L_1\sin^2(\alpha)L_2}{GJ_2}\right) + R_{y_1}\left(-\frac{L_1\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} + \frac{L_1\sin(\alpha)\cos(\alpha)L_2}{GJ_2}\right) + R_{z_1}\left(\frac{aL_1}{EI_{y_3}} + \frac{\cos(\alpha)L_2^2}{2EI_{y_2}}\right) + M_{x1}\left(-\frac{\sin(\alpha)\cos(\alpha)L_2}{EI_{y_2}} + \frac{\sin(\alpha)\cos(\alpha)L_2}{GJ_2}\right) + M_{y_1}\left(\frac{L_1}{EI_{y_1}} + \frac{L_1}{EI_{y_3}} + \frac{\cos^2(\alpha)L_2}{EI_{y_2}} + \frac{\sin^2(\alpha)L_2}{GJ_2}\right) + P\left(\frac{L_1^2}{2EI_{y_3}}\right) \quad (60)$$

$$\frac{\partial U}{\partial M_{z_1}} = R_{x_1}\left(\frac{bL_1}{GJ_3} + \frac{\sin(\alpha)\mathrm{L}_2^2}{2EI_{z_2}}\right) + R_{y_1}\left(-\frac{aL_1}{GJ_3} - \frac{\cos(\alpha)\mathrm{L}_2^2}{2EI_{z_2}}\right) + M_{z_1}\left(\frac{L_1}{GJ_1} + \frac{L_1}{GJ_3} + \frac{L_2}{EI_{z_2}}\right) \quad (61)$$

The values of main unknowns of the problem are calculated for this case by solving the Eqs. (56–61). Therefore, the frame is analyzable and its lateral stiffness is obtained.

3.2. Second case: the connection between columns and ground is rigid; the connection between column and beam is hinged near the load and rigid far from the load

Calculating the degree of indeterminacy of the frame:

$$\mathrm{D.\,I.} = (6m + r) - (6\mathrm{n} + \mathrm{c}) = 3$$
$$\mathrm{m} = 3,\quad \mathrm{r} = 12,\quad \mathrm{n} = 4,\quad \mathrm{c} = 3(2 - 1) = 3$$

It is expected to obtain a third order equation set at the end. In this case, R_{x_1}, R_{y_1} and R_{z_1} are considered as the main unknowns of the problem. Moreover, the equivalent equations are not repeated here as they are the same as those of the first case. However, three additional conditions are written for this problem due to the presence of the hinge in this case.

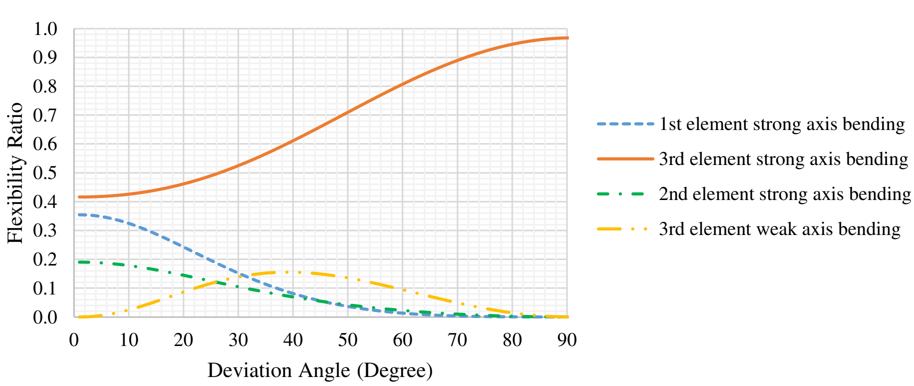

Fig. 19. Cooperation rate of internal efforts in the lateral flexibility of the steel frame in the third case.

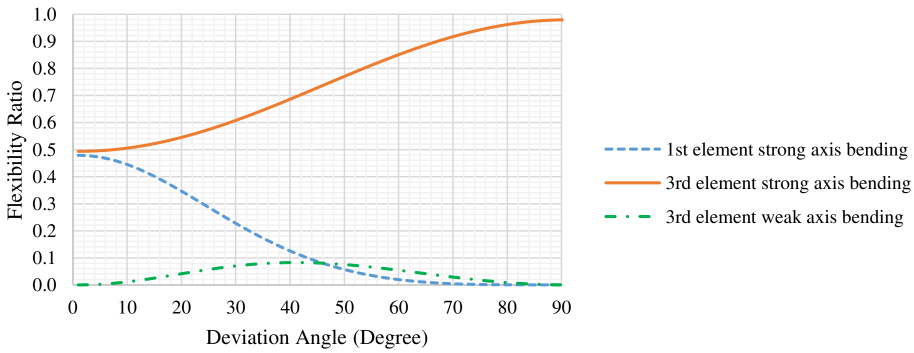

Fig. 20. Cooperation rate of internal efforts in the lateral flexibility of the steel frame in the fourth case.

$$\sum M_{x_c} = 0 \quad \rightarrow \quad M_{x_2} + R_{y_2} L_1 = 0 \quad \rightarrow \quad M_{x_2} = -R_{y_2} L_1 \tag{62}$$

$$\sum M_{y_c} = 0 \quad \rightarrow \quad M_{y_2} - R_{x_2} L_1 = 0 \quad \rightarrow \quad M_{y_2} = R_{x_2} L_1 \tag{63}$$

$$\sum M_{z_c} = 0 \quad \rightarrow \quad M_{z_1} + R_{x_1} b - R_{y_1} a = 0 \quad \rightarrow \quad M_{z_2} = 0 \tag{64}$$

Still, with respect to the Eqs. (62)–(64) and (31)−(32) it can be written:

$$M_{x_1} = R_{z_1} b - R_{y_1} L_1 \tag{65}$$

$$M_{y_1} = -PL_1 - R_{z_1} a + (P + R_{x_1}) L_1 = R_{x_1} L_1 - R_{z_1} L_2 \cos(\alpha) \tag{66}$$

$$M_{z_1} = R_{y_1} L_2 \cos(\alpha) - R_{x_1} L_2 \sin(\alpha) \tag{67}$$

Strain energy of the left far column is obtained from Eq. (34). Below formula is given by substituting Eqs. (65)–(67) into Eq. (34):

$$U_1 = \frac{R_{z1}^2 L_2^2 \sin^2(\alpha) L_1}{2EI_{x1}} - \frac{R_{y1} R_{z1} L_2 \sin(\alpha) L^2{}_1}{2EI_{x1}} + \frac{R_{y1}^2 L_1^3}{6EI_{x1}} + \frac{R_{z1}^2 L_2^2 \cos^2(\alpha) L_1}{2EI_{y1}} - \frac{R_{x1} R_{z1} L_2 \cos(\alpha) L_1^2}{2EI_{y1}} + \frac{R_{x1}^2 L_1^3}{6EI_{y1}} + \frac{R_{y1}^2 L_2^2 \cos^2(\alpha) L_1}{2GJ_1} + \frac{R_{x1}^2 L_2^2 \sin^2(\alpha) L_1}{2GJ_1} - \frac{R_{x1} R_{y1} L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_1} + \frac{R_{x1}^2 L_1}{2GA_{sx1}} + \frac{R_{y1}^2 L_1}{2GA_{sy1}} + \frac{R_{z1}^2 L_1}{2A_1 E} \tag{68}$$

Strain energy of the beam element is obtained for this case using Eq. (38) as follows:

$$U_2 = \frac{F_{zB}^2 L_2^3}{6EI_{y_2}} + \frac{F_{y_2}^2 L_2^3}{6EI_{z_2}} + \frac{F_{zB}^2 L_2}{2GA_{s_{z2}}} + \frac{F_{yB}^2 L_2}{2GA_{s_{y2}}} + \frac{F_{xB}^2 L_2}{2A_2 E} \tag{69}$$

In this case, the forces are transmitted from element to beam using the point C in Fig. 10. Due to the presence of hinge at this point, it is used for simplifying the calculations. Moreover, only the forces, and not the moments, are transferred from the right near column to the beam due to the presence of hinge at the point C. Therefore, it can be written as follows:

$$F_{zB} = R_{z2} = -R_{z1} \tag{70}$$

$$F_{xB} = (P + R_{x2}) \cos(\alpha) + R_{y_2} \sin(\alpha) = -R_{x_1} \cos(\alpha) - R_{y_1} \sin(\alpha) \tag{71}$$

$$F_{yB} = -(P + R_{x2}) \sin(\alpha) + R_{y_2} \cos(\alpha) = R_{x_1} \sin(\alpha) - R_{y_1} \cos(\alpha) \tag{72}$$

By substituting the Eqs. (70)–(72) into Eq. (69), it can be written as follows:

$$U_2 = \frac{R_{x1}^2 \sin^2(\alpha) L_2^3}{6EI_{z2}} + \frac{R_{y1}^2 \cos^2(\alpha) L_2^3}{6EI_{z2}} - \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z2}} + \frac{R_{z1}^2 L_2^3}{6EI_{y_2}} + \frac{R_{x1}^2 \sin^2(\alpha) L_2}{2GA_{sy_2}} + \frac{R_{y1}^2 \cos^2(\alpha) L_2}{2GA_{sy_2}} - \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{R_{z1}^2 L_2}{2GA_{sz2}} + \frac{R_{x1}^2 \cos^2(\alpha) L_2}{2EA_2} + \frac{R_{y1}^2 \sin^2(\alpha) L_2}{2EA_2} + \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2}{EA_2} \tag{73}$$

Strain energy of the right near column is also calculated by Eq. (51) for this case. Eq. (74) is written using the equivalent equations as well as Eqs. (62)–(64) as follows:

$$U_3 = \frac{R_{y_1}^2 L_1^3}{6EI_{x_3}} + \frac{P^2 L_1^3}{6EI_{x_3}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_3}} + \frac{PR_{x_1} L_1^3}{3EI_{y_3}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x3}}} + \frac{P^2 L_1}{2GA_{s_{x3}}} + \frac{PR_{x_1} L_1}{GA_{s_{x3}}} + \frac{R_{y_1}^2 L_1}{2GA_{s_{y_3}}} + \frac{R_{z_1}^2 L_1}{2A_3 E} \tag{74}$$

After differentiating the strain energy of elements and conducting mathematical operations, the formulas are written as follows:

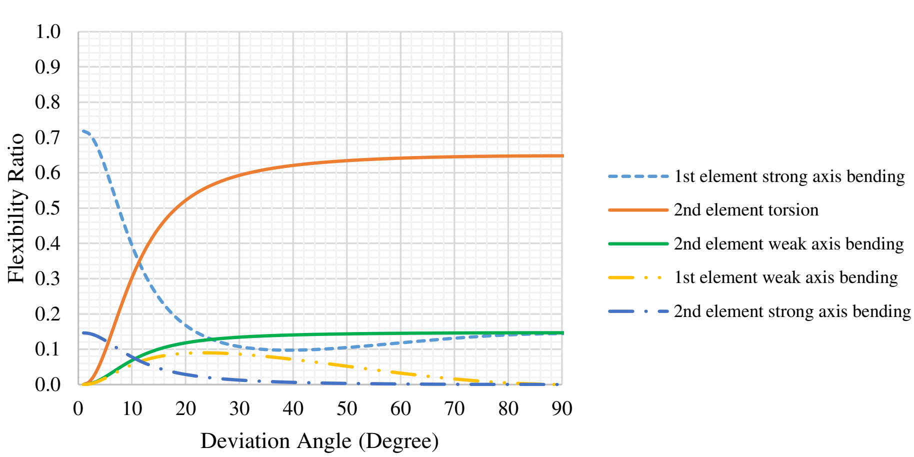

Fig. 21. Cooperation rate of internal efforts in the lateral flexibility of the steel frame in the fifth case.

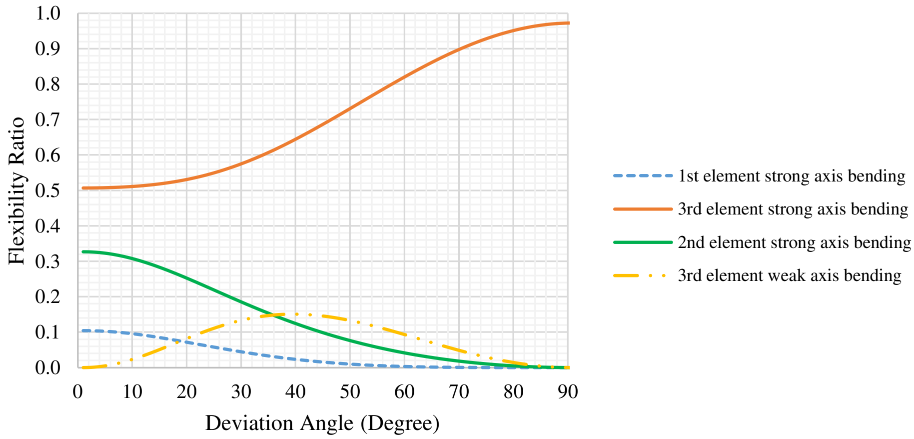

Fig. 22. Cooperation rate of internal efforts in the lateral flexibility of the steel frame in the sixth case.

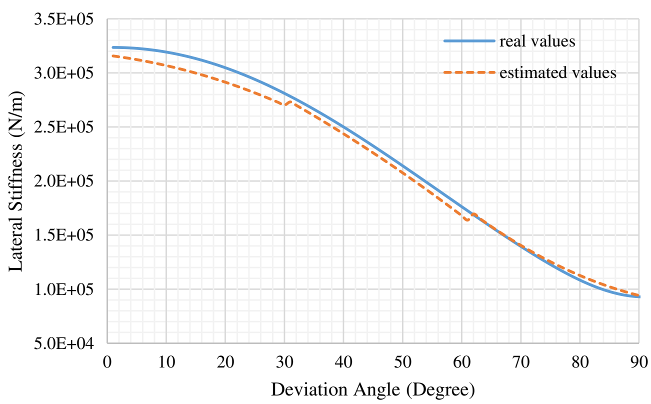

Fig. 23. The values obtained through explained equations and real values for the first case.

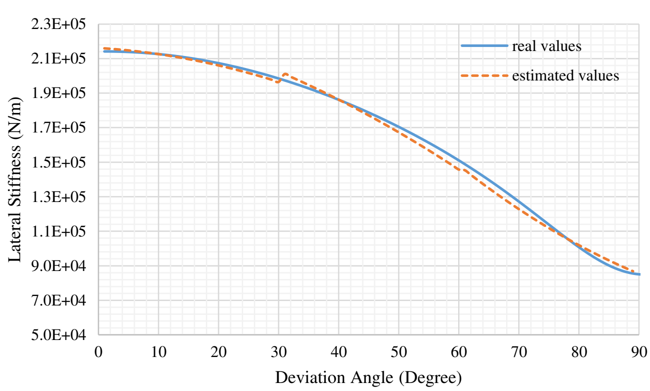

Fig. 24. The values obtained from the explained equations and real values for the second case.

$$\frac{\partial U}{\partial R_{x1}} = R_{x1}\left(\frac{L_1^3}{3EI_{y_1}} + \frac{L_2^2 \sin^2(\alpha) L_1}{GJ_1} + \frac{L_1}{GA_{sx_1}} + \frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{sx_3}} + \frac{\sin^2(\alpha) L_2^3}{3EI_{z_3}} + \frac{\sin^2(\alpha) L_2}{GA_{sy_2}} + \frac{\cos^2(\alpha) L_2}{A_2 E}\right) + R_{y_1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_1} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} - \frac{\sin(\alpha)\cos(\alpha) L_3}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{A_2 E}\right) + R_{z1}\left(-\frac{L_2\cos(\alpha) L_1^2}{2EI_{y_1}}\right) + P\left(\frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{sx_3}}\right) \tag{75}$$

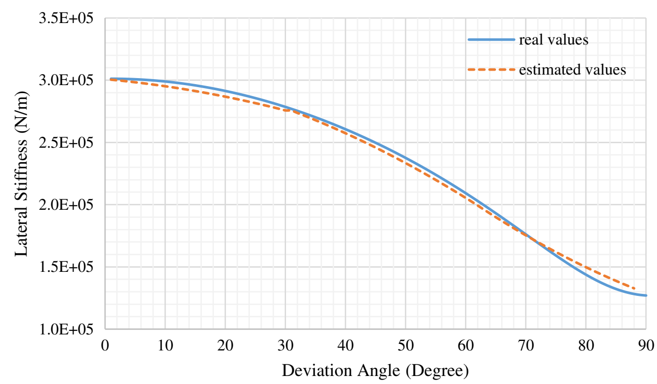

Fig. 25. The values obtained from the explained equations and real values for the third case.

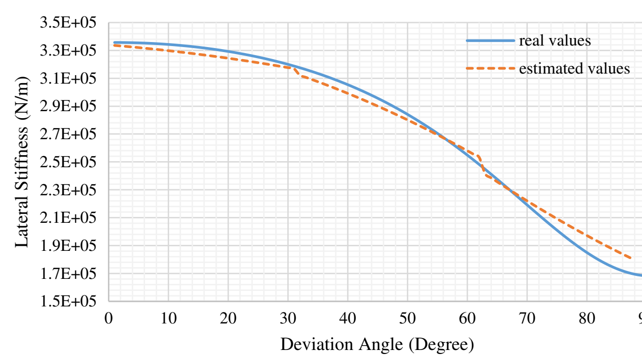

Fig. 26. The values obtained from the explained equations and real values for the fourth case.

$$\frac{\partial U}{\partial R_{y_1}} = R_{x1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_1} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} - \frac{\sin(\alpha)\cos(\alpha) L_3}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{A_2 E}\right) + R_{y_1}\left(\frac{L_1^3}{3EI_{x_1}} + \frac{L_2^2 \cos^2(\alpha) L_1}{GJ_1} + \frac{L_1}{GA_{sy_1}} + \frac{L_1^3}{3EI_{x_3}} + \frac{L_1}{GA_{sy_3}} + \frac{\cos^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{\cos^2(\alpha) L_2}{GA_{sy_2}} + \frac{\sin^2(\alpha) L_2}{A_2 E}\right) + R_{z1}\left(-\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x_1}}\right) \tag{76}$$

$$\frac{\partial U}{\partial R_{z1}} = R_{x1}\left(-\frac{L\cos(\alpha) L_1^2}{2EI_{y_1}}\right) + R_{y_1}\left(-\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x_1}}\right) + R_{z1}\left(\frac{L_2^2 \sin^2(\alpha) L_1}{EI_{x_1}} + \frac{L_2^2 \cos^2(\alpha) L_1}{EI_{y_1}} + \frac{L_1}{A_1 E} + \frac{L_1}{A_3 E} + \frac{L_2^3}{3EI_{y_2}} + \frac{L_2}{GA_{sz_2}}\right) \tag{77}$$

The considered structure is analyzed and all unknowns of the problem are calculated by solving the Eqs. (75)–(77).

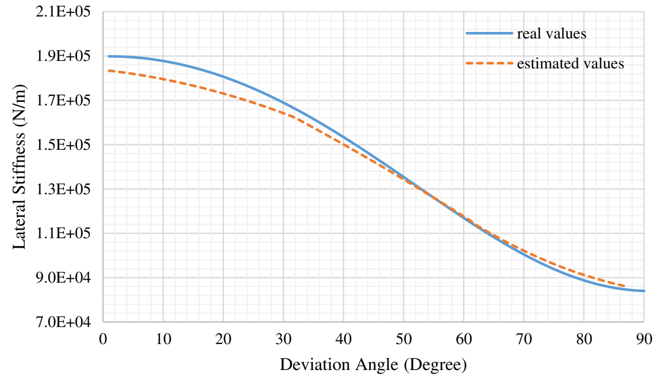

Fig. 27. The values obtained from the explained equations and real values for the sixth case.

3.3. *Third case: the connection between columns and ground is rigid; the connection between column and beam is rigid near the load and hinged far from the load*

The degree of indeterminacy of the frame is calculated as follows:

D. I. $= (6m + r) - (6n + c) = 3$
$m = 3,\ r = 12,\ n = 4,\ c = 3(2 - 1) = 3$

It is expected to obtain a third order equation set at the end. In this case R_{x_1}, R_{y_1}, R_{z_1} are considered as the main unknowns of the problem. Here, the equivalent equations are the same as those of the first case and therefore have not been repeated. However, three additional condition equations are written for the problem in this case.

$$\sum M_{xB} = 0 \quad \rightarrow \quad M_{x_1} + R_{y_1} L_1 = 0 \quad \rightarrow \quad M_{x_1} = -R_{y_1} L_1 \tag{78}$$

$$\sum M_{yB} = 0 \quad \rightarrow \quad M_{y_1} - R_{x_1} L_1 = 0 \quad \rightarrow \quad M_{y_1} = R_{x_1} L_1 \tag{79}$$

$$\sum M_{zB} = 0 \quad \rightarrow \quad M_{z_2} + R_{y_2} a - R_{x_2} b - Pb = 0 \quad \rightarrow \quad M_{z_1} = 0 \tag{80}$$

The following formulas are written considering the Eqs. (78)–(80) and (31)–(33):

$$M_{x_2} = R_{z_1} L_2 \sin(\alpha) + R_{y_1} L_1 \tag{81}$$

$$M_{y_2} = -PL_1 - R_{z_1} L_2 \cos(\alpha) - R_{x_1} L_1 \tag{82}$$

$$M_{z_2} = R_{y_1} L_2 \cos(\alpha) - R_{x_1} L_2 \sin(\alpha) \tag{83}$$

The strain energy is calculated for the right near column using Eq. (34).

Eq. (84) is written substituting the Eqs. (78)–(80) into Eq. (37) as follows:

$$U_1 = \frac{R_{y_1}^2 L_1^3}{6EI_{x_1}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_1}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x1}}} + \frac{R_{y_1}^2 L_1}{2GA_{s_{y1}}} + \frac{R_{z_1}^2 L_1}{2EA_1} \tag{84}$$

Strain energy of the beam element of the case is calculated through Eq. (38) as follows:

$$U_2 = \frac{F_{zB}^2 L_2^3}{6EI_{y_2}} + \frac{F_{yB}^2 L_2^3}{6EI_{z_2}} + \frac{F_{zB}^2 L_2}{2GA_{s_{z2}}} + \frac{F_{yB}^2 L_2}{2GA_{s_{y2}}} + \frac{F_{xB}^2 L_2}{2EA_2} \tag{85}$$

In this case, the point B of Fig. 10 is used for transferring the forces to the beam elements. Due to the presence of hinge in this point, only the forces are transmitted from the left far column to the beam.

$$F_{zB} = R_{z1} \tag{86}$$

$$F_{xB} = R_{x1} \cos(\alpha) + R_{y_1} \sin(\alpha) \tag{87}$$

$$F_{yB} = -R_{x1} \sin(\alpha) + R_{y_1} \cos(\alpha) \tag{88}$$

By substituting the Eqs. (86)–(88) into Eq. (85) it can be written as:

$$U_2 = \frac{R_{z1}^2 L_2^3}{6EI_{y_2}} + \frac{R_{y1}^2 \cos^2(\alpha) L_2^3}{6EI_{z2}} + \frac{R_{x1}^2 \sin^2(\alpha) L_2^3}{6EI_{z2}} - \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z2}} + \frac{R_{y1}^2 \cos^2(\alpha) L_2}{2GA_{sy_2}} + \frac{R_{x1}^2 \sin^2(\alpha) L_2}{2GA_{sy_2}} - \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{R_{z1}^2 L_2}{2GA_{sz_2}} + \frac{R_{y1}^2 \sin^2(\alpha) L_2}{2EA_2} + \frac{R_{x1}^2 \cos^2(\alpha) L_2}{2EA_2} - \frac{R_{x1} R_{y1} \sin(\alpha)\cos(\alpha) L_2}{EA_2} \tag{89}$$

Eq. (51) is used to obtain strain energy of the right near column in this case. Eq. (90) is written by substituting the Eqs. (28)–(30) and (81)–(83) into Eq. (51) as follows:

$$U_3 = \frac{R_{z1}^2 L_2^2 \sin^2(\alpha) L_1}{2EI_{x3}} + \frac{R_{y1}^2 L_1^3}{6EI_{x3}} + \frac{R_{y1} R_{z1} L_2 \sin(\alpha) L_1^2}{2EI_{x3}} + \frac{P^2 L_1^3}{6EI_{y_3}} + \frac{R_{z1}^2 L_2^2 \cos^2(\alpha) L_1}{2EI_{y_3}} + \frac{R_{x1}^2 L_1^3}{6EI_{y_3}} + \frac{PR_{z1} L_2 \cos(\alpha) L_1^2}{EI_{y_3}} + \frac{PR_{x1} L_1^3}{3EI_{y_3}} + \frac{R_{x1} R_{z1} L_2 \cos(\alpha) L_1^2}{2EI_{y_3}} + \frac{R_{y1}^2 L_2^2 \cos^2(\alpha) L_1}{2GJ_3} + \frac{R_{x1}^2 L_2^2 \sin^2(\alpha) L_1}{2GJ_3} - \frac{R_{x1} R_{y1} L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_3} + \frac{P^2 L_1}{2GA_{sx_3}} + \frac{R_{x1}^2 L_1}{2GA_{sx_3}} + \frac{PR_{x1} L_1}{GA_{sx_3}} + \frac{R_{y1}^2 L_1}{2GA_{sy_3}} + \frac{R_{z1}^2 L_1}{2EA_3} \tag{90}$$

The strain energy derivatives of the elements are calculated and simplified to write the formulas as follows:

$$\frac{\partial U}{\partial R_{x1}} = R_{x1}\left(\frac{L_1^3}{3EI_{y_1}} + \frac{L_1}{GA_{sx_1}} + \frac{L_1^3}{3EI_{y_3}} + \frac{L_2^2 \sin^2(\alpha) L_1}{GJ_3} + \frac{L_1}{GA_{sx_3}} + \frac{\sin^2(\alpha) L_2^3}{3EI_{z2}} + \frac{\sin^2(\alpha) L_2}{GA_{sy_2}} + \frac{\cos^2(\alpha) L_2}{EA_2}\right) + R_{y_1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_3} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z2}} - \frac{\sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{EA_2}\right) + R_{z_1}\left(\frac{L_2 \cos(\alpha) L_1^2}{2EI_{y_3}}\right) + P\left(\frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{sx_3}}\right) \tag{91}$$

$$\frac{\partial U}{\partial R_{y_1}} = R_{x_1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_3} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z2}} - \frac{\sin(\alpha)\cos(\alpha) L_2}{GA_{sy_2}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{EA_2}\right) + R_{y_1}\left(\frac{L_1^3}{3EI_{x_1}} + \frac{L_1}{GA_{sy_1}} + \frac{L_1^3}{3EI_{x_3}} + \frac{L_1}{GA_{sy_3}} + \frac{L_2^2 \cos^2(\alpha) L_1}{GJ_3} + \frac{\cos^2(\alpha) L_2^3}{3EI_{z2}} + \frac{\cos^2(\alpha) L_2}{GA_{sy_2}} + \frac{\sin^2(\alpha) L_2}{EA_2}\right) + R_{z_1}\left(\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x3}}\right) \tag{92}$$

$$\frac{\partial U}{\partial R_{z1}} = R_{x_1}\left(\frac{L_2 \cos(\alpha) L_1^2}{2EI_{y_3}}\right) + R_{y_1}\left(\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x3}}\right) + R_{z_1}\left(\frac{L_1}{EA_1} + \frac{L_1}{EA_3} + \frac{L_2^2 \sin^2(\alpha) L_1}{2EI_{x3}} + \frac{L_2^2 \cos^2(\alpha) L_1}{2EI_{y_3}} + \frac{L_2^3}{3EI_{y_2}} + \frac{L_2}{GA_{sz_2}}\right) + P\left(\frac{L_2 \cos(\alpha) L_1^2}{2EI_{y_3}}\right) \tag{93}$$

The studied structure is analyzed by solving the Eqs. (91)–(93) and all unknowns of the problem including lateral stiffness of the frame can be calculated.

3.4. *Fourth case: the connection between columns and ground is rigid; the beam connections are hinged in both sides*

The degree of indeterminacy of the frame is calculated as follows:

D. I. $= (6m + r) - (6n + c) = 1$
$m = 3,\ r = 12,\ n = 4,\ c = 3 + 3 - 1 = 5$

It is expected to obtain the first order equation set at the end. In this case, R_{x_1} is considered as the main unknown of the problem.

The equivalent equations of this case are also the same as those of the first case and therefore have not been repeated. However, 6 additional condition equations can be written due to the presence of hinges:

$$\sum M_{x_B} = 0 \quad \rightarrow \quad M_{x_1} + R_{y_1} L_1 = 0 \quad \rightarrow \quad M_{x_1} = -R_{y_1} L_1 \tag{94}$$

$$\sum M_{y_B} = 0 \quad \rightarrow \quad M_{y_1} - R_{x_1} L_1 = 0 \quad \rightarrow \quad M_{y_1} = R_{x_1} L_1 \tag{95}$$

$$\sum M_{x_C} = 0 \quad \rightarrow \quad M_{x_2} + R_{y_2} L_1 = 0 \quad \rightarrow \quad M_{x_2} = -R_{y_2} L_1 \tag{96}$$

$$\sum M_{y_C} = 0 \quad \rightarrow \quad M_{y_2} - R_{x_2} L_1 = 0 \quad \rightarrow \quad M_{y_2} = R_{x_2} L_1 \tag{97}$$

$$\sum M_{z_C} = 0 \quad \rightarrow \quad M_{z_1} - R_{y_1} a + R_{x_1} b = 0 \quad \rightarrow \quad M_{z_2} = 0 \tag{98}$$

$$\sum M_{z_B} = 0 \quad \rightarrow \quad M_{z_2} + R_{y_2} a - R_{x_2} b - Pb = 0 \quad \rightarrow \quad M_{z_1} = 0 \tag{99}$$

Substituting Eqs. (94)–(96) into Eq. (31), it can be written as follows:

$$-R_{y_2} L_1 = R_{z_1} b + R_{y_1} L_1 \tag{100}$$

Eq. (101) is written with respect to the Eqs. (29) and (100) as follows:

$$R_{z_1} = 0 \quad \rightarrow \quad R_{z_2} = 0 \tag{101}$$

Regarding the Eqs. (98) and (99), it can be written as:

$$R_{y_1} = \left(\frac{b}{a}\right) R_{x_1} \tag{102}$$

Strain energy of the left far column is calculated for this case using Eq. (34). Substituting the Eqs. (94), (95) and (102) into Eq. (34), it can be written as:

$$U_1 = \frac{R_{x_1}^2 L_1^3}{6EI_{x_1}} \tan^2(\alpha) + \frac{R_{x_1}^2 L_1^3}{6EI_{y_1}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x_1}}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{y_1}}} \tan^2(\alpha) \tag{103}$$

Strain energy of the beam element is expressed for this case from Eq. (37) as follows:

$$U_2 = \frac{F_{y_B}^2 L_2^3}{6EI_{z_2}} + \frac{F_{y_B}^2 L_2}{2GA_{s_{y_2}}} + \frac{F_{x_B}^2 L_2}{2EA_2} \tag{104}$$

In this case, only the forces, and not bending moments, are transmitted from the left far column to the beam due to the presence of hinged connection (in the connection of left far column to beam). Therefore, it can be written as:

$$F_{z_B} = R_{z_1} = 0 \tag{105}$$

$$F_{x_B} = R_{x_1} \cos(\alpha) + R_{y_1} \sin(\alpha) \tag{106}$$

$$F_{y_B} = -R_{x_1} \sin(\alpha) + R_{y_1} \cos(\alpha) \tag{107}$$

Substituting the Eqs. (105)–(107) and (102) into Eq. (104), it can be written as follows:

$$U_2 = \frac{R_{x_1}^2 \sin^2(\alpha) L_2^3}{6EI_{z_2}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2^3}{6EI_{z_2}} - \frac{R_{x_1}^2 \sin^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{2GA_{s_{y_2}}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{2GA_{s_{y_2}}} - \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{GA_{s_{y_2}}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{2EA_2} \tan^2(\alpha) + \frac{R_{x_1}^2 \cos^2(\alpha) L_2}{2EA_2} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{EA_2} \tag{108}$$

In this case, strain energy of the beam element is obtained from Eq. (51) as follows:

$$U_3 = \frac{R_{x_1}^2 L_1^3}{6EI_{x_3}} \tan^2(\alpha) + \frac{P^2 L_1^3}{6EI_{y_3}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_3}} + \frac{PR_{x_1} L_1^3}{3EI_{y_3}} + \frac{P^2 L_1}{2GA_{s_{x_3}}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x_3}}} + \frac{PR_{x_1} L_1}{GA_{s_{x_3}}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{y_3}}} \tan^2(\alpha) \tag{109}$$

After differentiation and simplification:

$$R_{x_1}\left[\frac{L_1^3}{3EI_{x_1}} \tan^2(\alpha) + \frac{L_1^3}{3EI_{y_1}} + \frac{L_1}{GA_{s_{x_1}}} + \frac{L_1}{GA_{s_{y_1}}} \tan^2(\alpha) + \frac{L_1^3}{3EI_{x_3}} \tan^2(\alpha) + \frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{s_{x_3}}} + \frac{L_1}{GA_{s_{y_3}}} \tan^2(\alpha) + \frac{\sin^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{\sin^2(\alpha) L_2^3}{3EI_{z_2}} - \frac{2\sin^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{\sin^2(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\sin^2(\alpha) L_2}{GA_{s_{y_2}}} - \frac{2\sin^2(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\sin^2(\alpha) L_2}{EA_2} \tan^2(\alpha) + \frac{\cos^2(\alpha) L_2}{EA_2} + \frac{2\sin^2(\alpha) L_2}{EA_2}\right] = -P\left[\frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{s_{x_3}}}\right] \tag{110}$$

By solving Eq. (110), the considered structure is analyzed and all needed parameters can be calculated as well.

3.5. Fifth case: the connection between column near the load and ground is rigid; the connection between another column and ground is hinged; and the connection between beam and column is rigid in both sides

Degree of indeterminacy of the frame is calculated as follows:

D. I. = $(6m + r) - (6n + c) = 3$
$m = 3, \quad r = 9, \quad n = 4, \quad c = 0$

It is expected to obtain a third order equation set at the end. In this case, R_{x_1}, R_{y_1} and R_{z_1} are considered as the main unknowns of the problem.

The equivalent equations, considered for this case are:

$$\sum F_x = 0 \quad \rightarrow \quad R_{x_1} + R_{x_2} + P = 0 \quad \rightarrow \quad R_{x_2} = -(R_{x_1} + P) \tag{111}$$

$$\sum F_y = 0 \quad \rightarrow \quad R_{y_1} + R_{y_2} = 0 \quad \rightarrow \quad R_{y_2} = -R_{y_1} \tag{112}$$

$$\sum F_z = 0 \quad \rightarrow \quad R_{z_1} + R_{z_2} = 0 \quad \rightarrow \quad R_{z_2} = -R_{z_1} \tag{113}$$

$$\sum M_{x_D} = 0 \quad \rightarrow \quad -R_{z_1} b + M_{x_1} = 0 \quad \rightarrow \quad M_{x_1} = R_{z_1} b \tag{114}$$

$$\sum M_{y_D} = 0 \quad \rightarrow \quad PL_1 + R_{z_1} a + M_{y_1} = 0 \quad \rightarrow \quad M_{y_1} = -PL_1 - R_{z_1} L_2 \cos(\alpha) \tag{115}$$

$$\sum M_{z_D} = 0 \quad \rightarrow \quad R_{x_1} b - R_{y_1} a + M_{z_1} = 0 \quad \rightarrow \quad M_{z_1} = R_{y_1} L_2 \cos(\alpha) - R_{x_1} L_2 \sin(\alpha) \tag{116}$$

Strain energy of the left far column is obtained from Eq. (34) as well. Substituting Eqs. (114)–(116) into Eq. (34), it can be written as:

$$U_1 = \frac{R_{z_1}^2 L_2^2 \sin^2(\alpha) L_1}{2EI_{x_1}} + \frac{R_{y_1}^2 L_1^3}{6EI_{x_1}} + \frac{R_{y_1} R_{z_1} L_2 \sin(\alpha) L_1^2}{2EI_{x_1}} + \frac{PL_1^3}{2EI_{y_1}} + \frac{R_{z_1}^2 L_2^2 \cos^2(\alpha) L_1}{2EI_{y_1}} + \frac{PR_{z_1} L_2 \cos(\alpha) L_1^2}{EI_{y_1}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_1}} + \frac{PR_{x_1} L_1^3}{2EI_{y_1}} + \frac{R_{x_1} R_{z_1} L_2 \cos(\alpha) L_1^2}{2EI_{y_1}} + \frac{R_{y_1}^2 L_2^2 \cos^2(\alpha) L_1}{2GJ_1} + \frac{R_{x_1}^2 L_2^2 \sin^2(\alpha) L_1}{2GJ_1} - \frac{R_{x_1} R_{y_1} L_2^2 \sin(\alpha) \cos(\alpha) L_1}{GJ_1} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x_1}}} + \frac{R_{y_1}^2 L_1}{2GA_{s_{y_1}}} + \frac{R_{z_1}^2 L_1}{2EA_1} \tag{117}$$

Strain energy of the beam element is calculated by Eq. (37). The point B of Fig. 10 is used for transmitting the lateral force to the beam elements. All forces and moments are transferred from the left far column to the beam one due to the presence of rigid connection at this point.

$$F_{z_B} = R_{z_2} = -R_{z_1} \tag{118}$$

$$F_{x_B} = (P + R_{x_2}) \cos(\alpha) + R_{y_2} \sin(\alpha) = -R_{x_1} \cos(\alpha) - R_{y_1} \sin(\alpha) \tag{119}$$

$$F_{y_B} = -(P + R_{x_2}) \sin(\alpha) + R_{y_2} \cos(\alpha) = R_{x_1} \sin(\alpha) - R_{y_1} \cos(\alpha) \tag{120}$$

$$M_{z_B} = M_{z_1} = 0 \tag{121}$$

$$M_{x_B} = M_{x_C} \cos(\alpha) + M_{y_C} \sin(\alpha) \tag{122}$$

$$M_{y_B} = -M_{x_C} \sin(\alpha) + M_{y_C} \cos(\alpha) \tag{123}$$

It can be written for the end of the right near column at the point C as follows:

$$M_{x_C} = R_{y_2} L_1 \tag{124}$$

$$M_{y_C} = -R_{x_2} L_1 \tag{125}$$

Substituting Eqs. (124) and (125) into Eqs. (122) and (123), it can be written as:

$$M_{x_B} = R_{y_2} L_1 \cos(\alpha) - R_{x_2} L_1 \sin(\alpha) = -R_{y_1} L_1 \cos(\alpha) + (P + R_{x_1}) L_1 \sin(\alpha) \tag{126}$$

$$M_{y_B} = -R_{y_2} L_1 \sin(\alpha) - R_{x_2} L_1 \cos(\alpha) = R_{y_1} L_1 \sin(\alpha) + (P + R_{x_1}) L_1 \cos(\alpha) \tag{127}$$

Eq. (128) is obtained based on the Eqs. (126)–(127) and (118)–(120) as follows:

$$U_2 = \frac{R_{y_1}^2 L_1^2 \sin^2(\alpha) L_2}{2EI_{y_2}} + \frac{P^2 L_1^2 \cos^2(\alpha) L_2}{2EI_{y_2}} + \frac{R_{x_1}^2 L_1^2 \cos^2(\alpha) L_2}{2EI_{y_2}} + \frac{P R_{x_1} L_1^2 \cos^2(\alpha) L_2}{EI_{y_2}} + \frac{P R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} + \frac{R_{x_1} R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} + \frac{R_{z_1}^2 L_2^3}{6EI_{y_2}} - \frac{R_{y_1} R_{z_1} L_1 \sin(\alpha) L_2^2}{2EI_{y_2}} - \frac{P R_{z_1} L_1 \cos(\alpha) L_2^2}{2EI_{y_2}} - \frac{R_{x_1} R_{z_1} L_1 \cos(\alpha) L_2^2}{2EI_{y_2}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2^3}{6EI_{z_2}} + \frac{R_{y_1}^2 \cos^2(\alpha) L_2^3}{6EI_{z_2}} - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} + \frac{R_{y_1}^2 L_1^2 \cos^2(\alpha) L_2}{2GJ_2} + \frac{P^2 L_1^2 \sin^2(\alpha) L_2}{2GJ_2} + \frac{R_{x_1}^2 L_1^2 \sin^2(\alpha) L_2}{2GJ_2} + \frac{P R_{x_1} L_1^2 \sin^2(\alpha) L_2}{GJ_2} - \frac{P R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} - \frac{R_{x_1} R_{y_1} L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} + \frac{R_{y_1}^2 \cos^2(\alpha) L_2}{2GA_{s_{y_2}}} + \frac{R_{x_1}^2 \sin^2(\alpha) L_2}{2GA_{s_{y_2}}} - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2}{GA_{s_{y_2}}} + \frac{R_{z_1}^2 L_2}{2GA_{s_{z_2}}} + \frac{R_{y_1}^2 \sin^2(\alpha) L_2}{2EA_2} + \frac{R_{x_1}^2 \cos^2(\alpha) L_2}{2EA_2} - \frac{R_{x_1} R_{y_1} \sin(\alpha)\cos(\alpha) L_2}{EA_2} \tag{128}$$

Strain energy of the right near column is obtained according to Eq. (51). Substituting Eqs. (111)−(113) into Eq. (51), it can be written as:

$$U_3 = \frac{R_{y_1}^2 L_1^3}{6EI_{x_3}} + \frac{P^2 L_1^3}{6EI_{y_3}} + \frac{R_{x_1}^2 L_1^3}{6EI_{y_3}} + \frac{P R_{x_1} L_1^3}{3EI_{y_3}} + \frac{P^2 L_1}{2GA_{s_{x_3}}} + \frac{R_{x_1}^2 L_1}{2GA_{s_{x_3}}} + \frac{P R_{x_1} L_1}{GA_{s_{x_3}}} + \frac{R_{y_1}^2 L_1}{2GA_{s_{y_3}}} + \frac{R_{z_1}^2 L_1}{2EA_3} \tag{129}$$

Strain energy of elements is differentiated and simplifies, and then:

$$\frac{\partial U}{\partial R_{x_1}} = R_{x_1}\left(\frac{L_1^3}{3EI_{y_1}} + \frac{L_1}{GA_{s_{x_1}}} + \frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{s_{x_3}}} + \frac{L_2^2 \sin^2(\alpha) L_1}{GJ_1} + \frac{L_1^2 \cos^2(\alpha) L_2}{EI_{y_2}} + \frac{\sin^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{L_1^2 \sin^2(\alpha) L_2}{GJ_2} + \frac{\sin^2(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\cos^2(\alpha) L_2}{EA_2}\right) + R_{y_1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_1} + \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} - \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} - \frac{\sin(\alpha)\cos(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{EA_2}\right) + R_{z_1}\left(\frac{L_2 \cos(\alpha) L_1^2}{2EI_{y_1}} - \frac{L_1 \cos(\alpha) L_2^2}{2EI_{y_2}}\right) + P\left(\frac{L_1^3}{2EI_{y_1}} + \frac{L_1^3}{3EI_{y_3}} + \frac{L_1}{GA_{s_{x_3}}} + \frac{L_1^2 \cos^2(\alpha) L_2}{EI_{y_2}} + \frac{L_1^2 \sin^2(\alpha) L_2}{GJ_2}\right) \tag{130}$$

$$\frac{\partial U}{\partial R_{y_1}} = R_{x_1}\left(-\frac{L_2^2 \sin(\alpha)\cos(\alpha) L_1}{GJ_1} + \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} - \frac{\sin(\alpha)\cos(\alpha) L_2^3}{3EI_{z_2}} - \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2} - \frac{\sin(\alpha)\cos(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\sin(\alpha)\cos(\alpha) L_2}{EA_2}\right) + R_{y_1}\left(\frac{L_1^3}{3EI_{x_1}} + \frac{L_1}{GA_{s_{y_1}}} + \frac{L_1^3}{3EI_{x_3}} + \frac{L_1}{GA_{s_{y_3}}} + \frac{L_2^2 \cos^2(\alpha) L_1}{GJ_1} + \frac{L_1^2 \sin^2(\alpha) L_2}{EI_{y_2}} + \frac{\cos^2(\alpha) L_2^3}{3EI_{z_2}} + \frac{L_1^2 \cos^2(\alpha) L_2}{GJ_2} + \frac{\cos^2(\alpha) L_2}{GA_{s_{y_2}}} + \frac{\sin^2(\alpha) L_2}{EA_2}\right) + R_{z_1}\left(\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x_1}} - \frac{L_1 \sin(\alpha) L_2^2}{2EI_{y_2}}\right) + P\left(\frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{EI_{y_2}} - \frac{L_1^2 \sin(\alpha)\cos(\alpha) L_2}{GJ_2}\right) \tag{131}$$

$$\frac{\partial U}{\partial R_{z_1}} = R_{x_1}\left(\frac{L_2 \cos(\alpha) L_1^2}{2EI_{y_1}} - \frac{L_1 \cos(\alpha) L_2^2}{2EI_{y_2}}\right) + R_{y_1}\left(\frac{L_2 \sin(\alpha) L_1^2}{2EI_{x_1}} - \frac{L_1 \sin(\alpha) L_2^2}{2EI_{y_2}}\right) + R_{z_1}\left(\frac{L_1}{EA_1} + \frac{L_1}{EA_3} + \frac{L_2^2 \sin^2(\alpha) L_1}{EI_{x_1}} + \frac{L_2^2 \cos^2(\alpha) L_1}{EI_{y_1}} + \frac{L_2^3}{3EI_{y_2}} + \frac{L_2}{GA_{s_{z_2}}}\right) + P\left(\frac{L_2 \cos(\alpha) L_1^2}{EI_{y_1}} - \frac{L_1 \cos(\alpha) L_2^2}{2EI_{y_2}}\right) \tag{132}$$

By solving Eqs. (130)−(132), the considered structure is analyzed and all unknowns of the problem including lateral stiffness of the steel frame can be calculated.

3.6. Sixth case: the connection between column far from the load and ground is rigid, between other column and ground is clamped and between beam and column is rigid at both sides

The conditions of this case are much similar to those of the first one except that the supporting reactions of M_{x_1}, M_{y_1} and M_{z_1} are zero. Therefore, this case is explained using Eqs. (56)–(58) and considering as zero the terms with M_{x_1}, M_{y_1} and M_{z_1} parameters.

4. Discussing and assessment of the results obtained from the least work analysis

As mentioned earlier, the stability and stiffness analyses are both crucial and necessary for designing the structures. So far, no perfect analysis with mathematical basis has been performed on certain cases in which the beam connecting two columns is oblique in the plan. In the previous section the equations are presented and solved for each considered case (1-Columns to ground connections are clamped and columns to beam connections are rigid, 2-Columns to ground connections are clamped and column to beam connection is hinged near the load and other column to beam connection is rigid, 3-Columns to ground connections are clamped and column to beam connection is hinged far from the load and other column to beam connection is rigid, 4-Columns to ground connections are clamped and both columns to beam connections are hinged, 5-Column to ground connection is hinged near the load and other column to ground connection is clamped and both columns to beam connections are rigid, 6-Column to ground is hinged far from the load and other column to ground connection is clamped and both columns to beam connections are rigid). Then, the results obtained from the least work method are verified by OpenSees software. Fig. 12 shows the verification results of the first case (columns to ground and columns to beam connections are clamped).

For plotting Fig. 12, the sections presented in Fig. 13 have been used for beam and columns of steel frame respectively.

The sections used for design the one-story one-span steel frame without beam deviation has been obtained according to the loading code of ASCE07 [9]. The lengths of columns and beam are considered as 3 m and 5 m, respectively, in the studied steel frame. The changing trend of lateral stiffness as per deviation angle is presented in Fig. 12 for all cases investigated in

Section 4, excluding the fifth case (the column to ground is hinged near the load). This trend for the former case is shown in Fig. 14.

In this case the stiffness of steel frame is reduced abruptly due to the application of lateral load to the column with hinged connection to the ground. Any small changes in the deviation angle result in the severe reduction of lateral stiffness of the steel frame. The considerable and important result obtained in this research is that the lateral stiffness values of the steel frame are the same for the case where the column to beam connections are hinged near and far from the load, Fig. 15.

According to the curves presented in Figs. 12 and 15, the stiffness alteration has slight slope at the beginning and increases as the deviation angle increases. It is observed that the highest primary slope occurs in the case with the highest column to beam stiffness ratio. It is assumed that the lowest length ratio and highest moment of inertia ratio of column to beam are 0.6 and 5, respectively. In order to find the point after which the stiffness change increases severely, a one- story one-span steel frame has been investigated in all considered cases excluding the fifth one. In this assessment the column to beam length ratio is considered as 0.6 and column to beam moment of inertia ratio as 5. The obtained results are presented in Table 2. Based on this table, the deviation angles lower than 5° are ignorable in all cases.

In the following, the variation of lateral stiffness is presented through figures for all six cases except the fifth one.

According to Fig. 16, the stiffness values of the second, third, fourth and sixth cases are equal with an acceptable approximation from 60° deviation angle and its beyond. Moreover, forming any kind of hinge in the frame causes the reduction of lateral stiffness of the frame. Another notable point is the cooperation rate of different internal efforts for creating lateral stiffness. The effects of these efforts on the lateral stiffness cannot be calculated easily; therefore, the influences are examined herein on the lateral flexibility. The considered six cases are assessed one by one in the following.

Based on Fig. 17 captured for the first case, 5 parameters have the highest effects on the lateral flexibility. It should be noted that the vertical axis of this figure shows the ratio of the flexibility of each parameter to the flexibility of the whole system.

According to Fig. 18, in the second case, 4 parameters have the highest effects on the lateral flexibility. The torsional parameter of beam is ineffective because one end of the beam is hinged in this case and therefore the value of torsion is zero along the element.

As shown in Fig. 19 for the third case, 4 parameters have the highest effects on the lateral flexibility of the steel frame. This is because the torsion is zero along the beam due to the presence of hinge at one end of the beam.

Based on Fig. 20, in the fourth case, 3 parameters have the highest effects on the lateral flexibility of the steel frame. The effects of torsion and bending about the strong axis of the beam element are removed due to the presence of hinge at both ends of the beam.

Regarding Fig. 21, in the fifth case, 5 parameters have the highest effects on the lateral flexibility. In this case, the frame uses all its potential in providing lateral stiffness due to removing a main parameter responsible for creating lateral stiffness (bending the right near column about the strong axis). Despite this removal, still 5 parameters are effective. Based on this figure, the torsion of the beam element has the highest effect in the case where the applied load is normal to the frame. Moreover, the notable matter here is the high effect of bending about the weak axis of beam in this case.

As presented in Fig. 22, in the sixth case, 4 parameters are influential in the lateral flexibility. In this case, the considerable point is the reduction of bending effect of the left far column about its strong axis due to the presence of the hinge at the column base far from the load.

5. The relation between lateral stiffness and geometrical parameters

In this section, response surface method [10,11,12] has been used to present a relation for each six considered cases except the fifth one as per geometrical parameters, through which lateral stiffness of the steel frame can be calculated.

In this section, the considered structure is analyzed in different statuses using the relations developed in the previous section. Then, the obtained results are represented as the input information of the response surface method. This method needs certain numbers of points in order to fit the best curve for the considered range. The used method is Central Composite Design (CCD) [13,14,15] one and needs 20 points with respect to the number of independent variables.

The most effective parameters are determined after performing sensitivity analysis. In this regard, the relations are expressed based on different parameters in order to find the best fitting curve. Eventually, 4 effective geometrical parameters are presented as the most effective ones as follows:

1. The ratio of moment of inertia of column to the moment of inertia of beam;
2. The ratio of column length to beam length;
3. The ratio of polar moment of inertia of column to the polar moment of inertia of beam;
4. Deviation angle of beam.

The first and second parameters include the effects of bending and the third one those of torsion. The ranges presented for the above mentioned four parameters are: 1–5 for the first; 0.6–1 for the second; 1–5 for the third; and 0–90 for the fourth. As the change amplitude of the fourth factor is much higher than others, it has been divided into 3 parts, with the range of 30 degrees, for accessing appropriate relations.

It should be mentioned that the presented relations are obtained from analyzing the symmetric sections such as Box and H for the columns; and no analysis has been performed for the non-symmetrical sections.

5.1. Explaining the equations for the first case

$$\begin{aligned}\text{For}\quad 0 \le \theta < 30 \quad K = {} & 153863 - 2740 \times \theta + 184676 \times I + 831 \times J \\ & + 24810 \times L - 28.9 \times \theta^2 - 2458 \times I^2 - 139 \times J^2 \\ & - 201678 \times L^2 - 306 \times \theta \times I + 4373 \times \theta \times L \\ & + 3 \times I \times J - 117650 \times I \times L - 4 \times J \times L\end{aligned} \tag{133}$$

Table 2
The results obtained for explaining the deviation angle after which stiffness variation increases severely.

Deviation angle	First case	Third and fourth cases	Fifth case	Seventh case	Error of the first case against initial point (%)	Error of the third and fourth cases against initial point (%)	Error of the third case against initial point (%)	Error of the seventh case against initial point (%)
0	58.3	46.8	42.5	34.1	–	–	–	–
5	56.6	45.6	41.5	33.8	2.9	2.6	2.3	0.8
10	52.3	42.6	39	32.9	10.2	9	8.2	3.5
15	46.9	38.8	35.9	31.6	1.6	17.1	15.5	7.3
20	41.7	35.3	32.9	30.2	28.5	24.6	22.6	11.4

$$\text{For}\quad 30 \le \theta < 60\quad K = 211935 - 6305 \times \theta + 153878 \times I - 2260 \times J + 152154 \times L - 17.5 \times \theta^2 - 2092 \times I^2 + 355 \times J^2 - 452991 \times L^2 - 603 \times \theta \times I - 2 \times \theta \times J + 9067 \times \theta \times L + 16 \times I \times J - 71376 \times I \times L + 32 \times J \times L \tag{134}$$

$$\text{For}\quad 60 \le \theta < 90\quad K = 466153 - 12104 \times \theta + 98156 \times I - 3 \times J + 59291 \times L + 46.7 \times \theta^2 - 853 \times I^2 - 104 \times J^2 - 369789 \times L^2 - 315.4 \times \theta \times I + 6.9 \times \theta \times J + 5198 \times \theta \times L + 29 \times I \times J - 33668 \times I \times L + 278 \times J \times L \tag{135}$$

The parameters used in the above equations are as follows:

θ: deviation angle of beam (as per degree)
I: moment of inertia ratio of column to the beam
J: polar moment of inertia ratio of column to beam
L: ratio of column length to beam length.

In Fig. 23, the curve obtained from Eqs. (133)–(135) are explained and the main real curve is shown as well. The plotted curve is corresponded to the case with $L = 0.6$, $J = 2$ and $I = 2$ ratio values of moment of inertias, torsional moments and lengths.

According to the Fig. 23, the obtained equation error is very low and its worst value (3.4%) is corresponded to the deviation angle of 14°.

5.2. Explaining the equations for the second case

$$\text{For}\quad 0 \le \theta < 30\quad K = 868098 - 852 \times \theta + 184566 \times I + 718 \times J - 2150641 \times L - 15.0 \times \theta^2 - 770 \times I^2 - 120 \times J^2 + 1287760 \times L^2 - 174.9 \times \theta \times I + 1612 \times \theta \times L - 160325 \times I \times L \tag{136}$$

$$\text{For}\quad 30 \le \theta < 60\quad K = 871648 - 2547 \times \theta + 171864 \times I + 820 \times J - 2026218 \times L - 13.1 \times \theta^2 - 1050 \times I^2 - 138 \times J^2 + 1099887 \times L^2 - 463 \times \theta \times I + 4555 \times \theta \times L + 1 \times I \times J - 131250 \times I \times L + 5 \times J \times L \tag{137}$$

$$\text{For}\quad 60 \le \theta < 90\quad K = 869602 - 7482 \times \theta + 128539 \times I - 222 \times J - 1523586 \times L + 21.8 \times \theta^2 - 275 \times I^2 + 30 \times J^2 + 721936 \times L^2 - 286 \times \theta \times I + 0.5 \times \theta \times J + 4562 \times \theta \times L + 1 \times I \times J - 96020 \times I \times L + 17 \times J \times L \tag{138}$$

The parameters used in the Eqs. (136)–(138) are those of the Eqs. (133)–(135).

Fig. 24 presents the values obtained from Eqs. (136)–(138) as well as the real values. The plotted curve is corresponded to the case with $L = 0.6$, $J = 2$ and $I = 2$ ratio values of moment of inertias, torsional moments and lengths. According to this figure, the obtained error is very low and its worst value is 3.7% corresponded to the deviation angle of 65°.

5.3. Explaining the equations for the third case

$$\text{For}\quad 0 \le \theta < 30\quad K = -126409 - 1148 \times \theta + 147514 \times I + 436 \times J + 501669 \times L - 13.3 \times \theta^2 - 666 \times I^2 - 72 \times J^2 - 388077 \times L^2 - 161.8 \times \theta \times I - 0.2 \times \theta \times J + 1948 \times \theta \times L - 1 \times I \times J - 95373 \times I \times L \tag{139}$$

$$\text{For}\quad 30 \le \theta < 60\quad K = -60107 - 2217 \times \theta + 137007 \times I - 1171 \times J + 430030 \times L - 20.0 \times \theta^2 - 1343 \times I^2 + 210 \times J^2 - 456367 \times L^2 - 474.0 \times \theta \times I - 1.4 \times \theta \times J + 5007 \times \theta \times L - 11 \times I \times J - 65587 \times I \times L - 24 \times J \times L \tag{140}$$

$$\text{For}\quad 60 \le \theta < 90\quad K = 233204 - 7485 \times \theta + 97422 \times I - 161 \times J + 245198 \times L + 24.1 \times \theta^2 - 465 \times I^2 + 124 \times J^2 - 359678 \times L^2 - 331.1 \times \theta \times I - 5.8 \times \theta \times J + 4031 \times \theta \times L - 18 \times I \times J - 34369 \times I \times L - 211 \times J \times L \tag{141}$$

The parameters used in the Eqs. (139)–(141) are those of the Eqs. (133)–(135).

Fig. 25 presents the values obtained from Eqs. (139)–(141) as well as the real values.

The plotted curve corresponds to the case with $L = 0.6$, $J = 2$ and $I = 2$ ratio values of moment of inertias, torsional moments and lengths.

According to this figure, the equations error is very low and its worst value (1.7%) is corresponded to the deviation angle of 15°.

5.4. Explaining the equations for the fourth case

$$\text{For}\quad 0 \le \theta < 30\quad K = 709584 - 389 \times \theta + 177013 \times I + 540 \times J - 1860899 \times L - 6.5 \times \theta^2 - 205 \times I^2 - 90 \times J^2 + 1150613 \times L^2 - 124.0 \times \theta \times I + 876 \times \theta \times L - 158009 \times I \times L \tag{142}$$

$$\text{For}\quad 30 \le \theta < 60\quad K = 713854 - 712 \times \theta + 171055 \times I - 1135 \times J - 1818700 \times L - 15.4 \times \theta^2 - 945 \times I^2 - 166 \times J^2 + 1034809 \times L^2 - 440 \times \theta \times I + 14 \times \theta \times J + 3063 \times \theta \times L + 108 \times I \times J - 132636 \times I \times L + 1217 \times J \times L \tag{143}$$

$$\text{For}\quad 60 \le \theta < 90\quad K = 850081.3 - 6500.2 \times \theta + 136432.8 \times I - 4080.96 \times J - 1552106 \times L + 16.41 \times \theta^2 - 225.13 \times I^2 + 90.98 \times J^2 + 741789.4 \times L^2 - 340.93 \times \theta \times I + 40.85 \times \theta \times J + 4750.64 \times \theta \times L - 478.6 \times I \times J - 99763.4 \times I \times L + 3064.21 \times J \times L \tag{144}$$

The parameters used in the Eqs. (142)–(144) are those of the Eqs. (133)–(135).

Fig. 26 presents the values obtained from Eqs. (142)–(144) as well as the real values.

The plotted curve corresponds to the case with $L = 0.6$, $J = 2$ and $I = 2$ ratio values of moment of inertias, torsional moments and lengths.

According to this figure, the equations error is very low and its worst value (2.63%) is related to the deviation angle of 40°.

5.5. Explaining the equations of the sixth case

$$\text{For}\quad 0 \le \theta < 30\quad K = -11117 - 1153 \times \theta + 91838 \times I - 673 \times J + 288417 \times L - 11.8 \times \theta^2 - 2041 \times I^2 + 112 \times J^2 - 288677 \times L^2 - 96.9 \times \theta \times I - 0.2 \times \theta \times J + 1713 \times \theta \times L - 41035 \times I \times L + 3 \times J \times L \tag{145}$$

$$\text{For}\quad 30 \le \theta < 60\quad K = 30495 - 3070 \times \theta + 79117 \times I - 1250 \times J + 304155 \times L - 4.7 \times \theta^2 - 1102 \times I^2 + 207 \times J^2 - 353666 \times L^2 - 156.4 \times \theta \times I - 0.3 \times \theta \times J + 3685 \times \theta \times L - 30333 \times I \times L + 16 \times J \times L \tag{146}$$

$$\text{For}\quad 60 \le \theta < 90\quad K = 124404 - 5246 \times \theta + 61063 \times I + 335 \times J + 290776 \times L + 20.92 \times \theta^2 - 147 \times I^2 - 62 \times J^2 - 290163 \times L^2 - 35.9 \times \theta \times I + 1955 \times \theta \times L - 24300 \times I \times L + 33 \times J \times L \tag{147}$$

The parameters used in the Eqs. (145)–(147) are those of the Eqs. (133)–(135).

Fig. 27 presents the values obtained from Eqs. (145)–(147) as well as the real values.

The plotted curve corresponds to the case with $L = 0.6$, $J = 2$ and $I = 2$ ratio values of moment of inertias, torsional moments and lengths.

According to this figure, the equations error is very low and its worst value (4.27%) is related to the deviation angle of 10°.

6. Conclusion

Both issues of effective length actor and lateral stiffness investigated in this article are very important matters for engineers. Regarding the first one, there isn't any practical relation for calculating effective length factor (which is one of the primary parameters in design of columns) for the columns with oblique restraint. Regarding the second issue, this investigation shows the contribution of elements in the lateral stiffness of moment resisting system with oblique beam.

The main results obtained in this research are briefly summarized as follows:

1) For obtaining the effective length factor in the columns with connected oblique beams, it is sufficient to modify G_T and G_B parameters. After modifying these parameters, the graphs presented by Julian and Lawrence can be used for calculating the column effective length factor:

$$G_T = \frac{\sum\left(\frac{I_C}{L_C}\right)}{\sum\left(\frac{I_{BT}}{L_{BT}}\cos^2\phi_T\right)} \tag{26-repeating}$$

$$G_B = \frac{\sum\left(\frac{I_C}{L_C}\right)}{\sum\left(\frac{I_{BB}}{L_{BB}}\cos^2\phi_B\right)} \tag{27-repeating}$$

2) Lateral stiffness values of all six cases (1-Columns to ground connections are clamped and columns to beam connections are rigid, 2-Columns to ground connections are clamped and column to beam connection is hinged near the load and other column to beam connection is rigid, 3-Columns to ground connections are clamped and column to beam connection is hinged far from the load and other column to beam connection is rigid, 4-Columns to ground connections are clamped and both columns to beam connections are hinged, 5-Column to ground connection is hinged near the load and other column to ground connection is clamped and both columns to beam connections are rigid, 6-Column to ground is hinged far from the load and other column to ground connection is clamped and both columns to beam connections are rigid) in the deviation angle of 90° are the same with an acceptable approximation and equal to the stiffness of a cantilever column, excluding the case in which the base column connection is hinged near the load. However, the stiffness changing paths are different from the deviation angles of 0° to 90°.
3) Lateral stiffness of the steel frame is approximately zero in the deviation angle of 90° in the case where the connection between column base and ground is hinged near the load. The column subjected to the load is connected to the ground in the hinged form having no stiffness. Moreover, the investigated structural frame is normal to the load direction and has approximately no lateral stiffness.
4) In the case with constant effective lateral stiffness factors, excluding deviation angle, the changing of stiffness is nonlinear. It is initially occurred slightly and increases after a certain deviation angle. This deviation angle has been calculated for the studied cases.
5) Lateral stiffness values are the same for the third and fourth cases. It means that if only one end of the beam has hinged connection, then the hinge location has no effect on the lateral stiffness of the structure.
6) The most effective factors for creating lateral stiffness are:

a. bending of the left far column about the strong axis
b. bending of the right near column about the strong axis
c. bending of the beam element about the strong axis
d. bending of the right near column about the weak axis
e. torsion of the beam element

The rates of these effective factors are high or low with respect to the connections of column to beam and column to ground.

In all studied cases, torsion has lower effect comparing to other effective factors. This fact is derived from the factors obtained by Eqs. (133)–(147).

7. Torsion has the highest effect in the deviation angles of 30°–60° in all studied cases. This fact is derived from the factors obtained from Eqs. (133)–(147).

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