Advanced Numerical Methods

Problem Statement

Assume that the potential function ϕ is defined as

$$\phi(x, y) = x^2 + y^2 + x^3y^2 \tag{1}$$

This function satisfies the following differential equation:

$$\nabla^2\phi = 4 + 2x^3 + 6xy^2 \tag{2}$$

Now consider a unit square as shown in Figure 1. The exact solution of equation (2) with proper boundary conditions (BCs) is described by equation (1); hence, if we numerically solve equation (2) we should obtain the same results.

Boundary Conditions

We can define as many boundary condition as we like. For the present work, consider the following two cases:

Case I: Dirichlet BCs for all sides. In this case, BCs are obtained from equation (1); i.e.

$$\begin{cases} \phi(0, y) = y^2 & \text{for the left side} \\ \phi(1, y) = 1 + 2y^2 & \text{for the right side} \\ \phi(x, 0) = x^2 & \text{for the lower side} \\ \phi(x, 1) = 1 + x^2 + x^3 & \text{for the upper side} \end{cases} \tag{3}$$

Case II: Dirichlet BCs for the left and lower walls and Neumann BCs for the other walls. Agian in this case, BCs are obtained from equation (1) ; i.e.

$$\begin{cases} \phi(0, y) = y^2 & \text{for the left side} \\ \phi(1, y) = 2 + 3y^2 & \text{for the right side} \\ \phi(x, 0) = x^2 & \text{for the lower side} \\ \phi(x, 1) = 2 + 2x^3 & \text{for the upper side} \end{cases} \tag{4}$$

Requirements

1. Derive the finite difference form of equation (2) with central differencing for inner nodes and proper backward/forward differencing for boundaries.

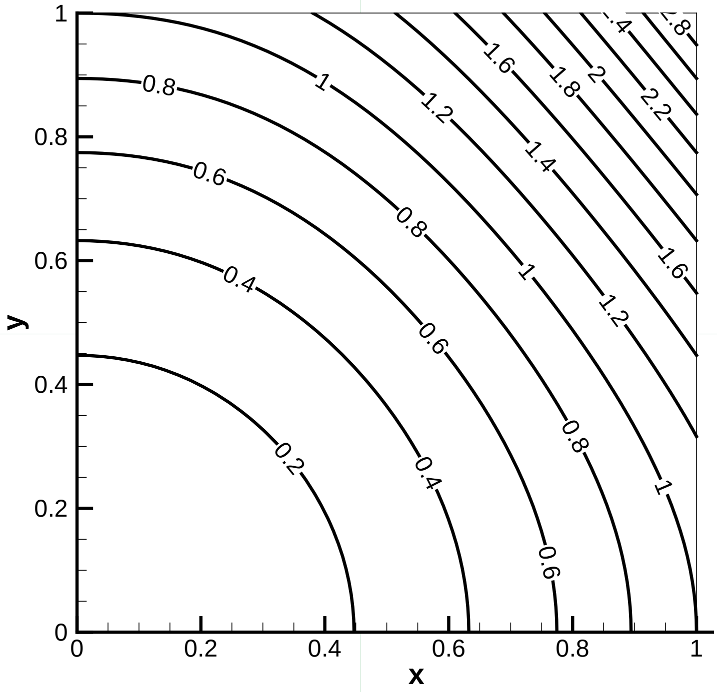

Figure 1: Domain of the problem and the exact solution contour lines.

2. Write a computer program to solve equation (2) with BCs (3) and (4) using the following explicit schemes:
   (a) Point Jaccobi iterative method.
   (b) Point Gauss–Seidel method
3. Write a computer program to solve equation (2) with BCs (3) and (4) using the Jaccobi implicit scheme.

Notice

1. The relative error for convergence should be $\mathrm{Err}_{\max} = 10^{-8}$.
2. In all cases, compare your results with the exact solution; i.e. equation (1).
3. Compare the speed of convergence for different schemes and discuss the results.