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Large deformation equations of the high-order numerical manifold method using triangular element

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Abstract

Numerical finite element method, as a basic method for analyzing engineering problems have shortcomings and defects, such as computational and time high costs in precise modeling and problems with large deformation, requiring the mesh refinement in complex geometries, the sensitivity of elements to boundary conditions and loading, and low ability in analyzing and modeling problems with a discontinuity. A method that can solve these problems is needed. The Numerical Manifold Method (NMM) is an appropriate option. Using two independent mathematical and physical covers, there is no need for the mathematical cover to fit into problem geometry. Given this feature, the meshing of continuous and discontinuous problems is easily possible, which reduces computational costs. Besides, the NMM, using high-order local approximation (LA) functions, makes it possible to obtain accurate answers without increasing the problem's elements. The NMM has been developed using various elements (mathematical covers) in small deformation problems. But so far, the performance of this method has not been studied in large deformation problems.

In this study, we investigated the NMM for problems with large deformations in the elastic range (nonlinear elastic behavior). In the NMM the triangle element was used. To prevent occurring the linear dependence problem in the high-order NMM, the proposed methods in previous studies are used. In this thesis, the behavior of hyperelastic materials which are a typical example of materials with nonlinear elastic behavior was investigated. To analyze these materials, the Mooney-Rivlin material model was used. The required equations for nonlinear analysis of these materials by NMM in large deformation problems were extracted using the Newton-Raphson method and Total Lagrangian formulation. Then, the performance of this method was evaluated by analyzing conventional tests of hyperelastic materials such as uniaxial tensile and compression tests and shear test. The results showed that the conventional (1-order) and high-order NMM in the analysis of uniaxial tensile and compression problems had very good responses and the results were consistent with the results of the simulation by Abaqus. However, the conventional manifold method with non-conforming mathematical covers on the boundaries of the body had a convergence problem in the shear test, and this issue was corrected by increasing the order of NMM and using the high-order manifold method of orders 2 and 3.

Keywords: High-order Numerical Manifold Method, large deformations, hyperelastic materials, total Lagrangian formulation.

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Chapter 1

1 Preface

1.1 Introduction

For most geotechnical engineering problems, deformations are of interest to the extent that it permits the application of the small deformation hypothesis so that analysis can be performed with respect to the original and unformed geometry. However, there is a need for the analysis of problems whose deformations are clearly out of range that can be called small. These problems include landslides, installation of foundations using penetration (such as piles and suction caissons) and the interaction of oil and gas substructures (such as pipelines) with the seabed.

Numerical methods are used to solve engineering problems due to the complexity of analysis of the abovementioned large deformation problems and their large volume of input data. The proposed numerical method for modeling problems must be acceptable and justified in terms of accuracy and speed, and must also be capable of responding to different problems or a problem under different situations.

Numerous numerical methods have been developed for modeling continuous bodies. Among numerical methods that are capable of modeling continuous bodies are finite element method (FEM), finite difference method (FDM), finite volume method (FVM), and numerical manifold method (NMM).

The NMM has been invented to model continuous and discontinuous bodies in a single system. This method uses two independent mathematical and physical meshes to model continuous parts and discontinuities. As these two meshes are independent, the need to adapt the mesh to the fractured and complex boundary surfaces has been eliminated. One of the advantage of this feature is that there is no need to re-mesh when changing the geometry during problem analysis. Moreover, to improve the accuracy and reduce the computational cost, high-order NMM has been developed [1].

Unlike the FEM and the conventional NMM (1^{st}-order NMM), in the high-order NMM, to obtain accurate results, there is no need to refine mesh and increase the number of elements, and using high-order LA functions, more accurate results can be obtained with a lower computational cost.

The NMM, despite these advantages, has not been studied in large deformations. Today, analyzing problems in large deformations is one of the exciting challenges for engineers. Despite the existence of many numerical methods, these methods have many limitations in their application to large deformation. The NMM, like other numerical methods, needs to be evaluated in analyzing these problems.

1.2 Problem design

Nowadays, solving a problem with higher accuracy and lower computational cost is of particular importance. In most numerical methods, such as FEM, to achieve higher accuracy, the elements need to be increased. By increasing the elements, the number of nodes in the problem increases, resulting in a high computational cost. Furthermore, in cases where the geometry of the problem has complexities such as the presence of a cut or a hole, it is necessary to increase the element at areas of complexity. Due to the independence of the mathematical and physical meshes in the NMM, the accuracy of the results can be increased, the computational cost of the problem can also be significantly reduced without increasing the number of elements and using high-order approximation functions. In this research, we develop the relations of the NMM for large deformation problems using the triangular element in the elastic range and investigate the performance, accuracy and analysis time of this method. In this process, the total Lagrangian formulation is implemented that uses the undeformed geometry as reference frame and the Newton-Raphson method is used to linearize the nonlinear equations.

1.3 Research Questions

In developing the NMM for large deformations to analyze continuous bodies, the following questions are raised:

1. Are NMM equations attainable for large deformations?

2. How the simulation of large deformation problems can be performed by the NMM?

3. Is it possible to use the NMM for hyperelastic materials with large deformations?

4. Is the NMM more effective than other common methods in these problems?

5. What is the impact of using different covers (conforming or nonconforming) on the accuracy of the obtained results and the speed of the analysis?

1.4 Research innovations

1. The NMM is used in this study to solve large deformation problems.

2. To derive NMM equations for problems with large deformations, the total Lagrangian formulation is exploited which uses undeformed geometry as the reference frame.

3. The obtained large deformation NMM equations used for the analysis of the hyperelastic materials, and in this process the Mooney-Rivlin material model used to describe the hyperelastic materials.

1.5 Research Method

In this thesis, first, the existing relations of the NMM in small deformations using the triangular element are described. Then the Newton-Raphson method is briefly introduced and after that the total Lagrangian formulation is discussed. Following is the introduction of the hyperelastic materials and the Mooney-Rivlin material model. The governing equations of the NMM in large deformations for hyperelastic materials have been obtained using the methods discussed. Finally, a computer program for the analysis of engineering problems in large elastic deformations specific to hyperelastic materials is defined and the results are compared with those obtained with Abaqus.

1.6 Research structure

After giving an overview in this chapter, the second chapter deals with the recent studies on the NMM, large deformations, and hyperelastic materials. In Chapter 3, the relations of the NMM are first presented using triangular element, and then the Newton-Raphson method, the total Lagrangian formulation, and the

Mooney-Rivlin model for describing the hyperelastic materials are discussed. In the following, the governing equations for the analysis of problems with large elastic deformations are developed. In chapter 4, various examples are solved for hyperelastic materials and the results are compared with the results of the Abaqus simulation or other numerical methods. Finally, Chapter 5 concludes and makes suggestions for future research.

Chapter 2

2 Literature review

2.1 Introduction

In this section, we look at the history of studies on hyperelastic materials, large deformations, the numerical manifold method, and other developed methods with the help of NMM and the problem of linear dependence.

2.2 Hyperelastic materials

Different constitutive relations can reflect different behaviors of material. For example, the Hooke's law for elastic materials is in the linear range or the model of St. Venant-Kirchhoff material which presents a linear relationship between stress and strain (the second Piola-Kirchhoff stress and Green strain). Models of linear elastic systems give meaningful and realistic results when the strains are infinitesimal, while most materials in large deformations have nonlinear behavior [2].

When the status of material can be fully describable using a given total strain, the constitutive relation is called hyperelasticity [2]. In hyperelastic materials, a strain energy density exists as a function of strain and the stress can be obtained by differentiating the strain energy density with respect to the strain. This model of material is independent of the deformation history, meaning that the same deformation is expected when the ultimate force is the same for two different states. Rubber-like materials or human tissues fall into this category. Hypoelasticity on the other hand is when the constitutive relation is given using terms of stress and strain rates, which is often used to describe plastic behavior. In such materials, the deformation history should be taken into account for stress calculation, since two states having the same strain depending on the loading history may have different stresses [2].

To illustrate the importance of precise analysis of hyperelastic materials elastomers can be named, elastomeric materials can be used in various parts of the vehicle such as tires, motors, etc. Nowadays, the design of these extremely vital sections demonstrates the necessity of using software based on numerical methods. Good numerical calculations require choosing a suitable constitutive model. The general theory of nonlinear hyperelasticity is classically applied to predict the behavior of components under static loading conditions as well as developing more sophisticated viscoelasticity models. Many models have been proposed to describe the response of elastomers, but only a few are capable of describing the complete behavior of the material. Attractive models are those that can describe the complete behavior with the least number of material parameters (material constants) that are obtained approximately [3]. Few studies evaluate and compare the ability of hyperelastic models in simulating the complete behavior of elastomers. Some researchers describe the performance of their model specifically for large strains [4] or compare one model with another model to establish the equivalence of formulations [5].

In this study, total Lagrangian formulation-based static analysis of hyperelastic materials is investigated using NMM. There are several models for hyperelastic materials, but the present study employs the Mooney-Rivlin material model, which is one of the most famous hyperelastic material models. In general, hyperelastic materials display a property of being incompressible in small deformations (volume of the material remains constant). This is a frequent behavior of hyperelastic materials in small deformations. Numerically, a constraint such as $J = 1$ (reduced invariant) needs to be imposed to establish the incompressibility of the material. However, this causes problems in the numerical analysis process known as volumetric locking. Because of the incompressibility, the hydrostatic part of the stress cannot be obtained from the volumetric strain. The selective reduced integration method, mixed formulation method, and penalty method have been successfully applied to incompressible materials [2].

If there is a strain energy density, which stress can be differentiated from this strain density with respect to the strain, this system is called path independent. Therefore, solving the nonlinear equation for the total applied force is theoretically possible. The problems analyzed in this thesis include two types of nonlinearities: material nonlinearity due to constitutive relations and geometric nonlinearity caused by kinematic.

2.3 Large deformations

In the analysis of large deformations, nonlinearities such as geometric nonlinearity must be considered. Many formulations can be used to solve such problems. The Lagrangian formulation, which includes the total Lagrangian formulation and the updated Lagrangian formulation, and the Eulerian formulation, are two types of classical formulations. Unlike the Lagrangian formulation that uses the material description (the mesh is fixed on the body of material and changes with the deformation of the material), the Eulerian formulation uses spatial description, in which the mesh is fixed in space. This formulation is widely used in computational fluid dynamics. Some problems cannot be analyzed using these two methods because they can cause a great deal of error or computational disruption. For example, it is difficult to analyze moving boundary problems using Eulerian meshes, and Lagrangian methods are not suitable for analyzing fluid mechanic problems at high flow rates. Therefore, hybrid techniques that combine the advantages of Lagrangian and Eulerian methods have been developed. These methods are called arbitrary Eulerian-Lagrangian methods [6].

Different numerical methods have been used to model and analyze problems in large strains. To remove the meshing or meshing constraints, the Element Free Gherkin method (EFG) [7] and the Reproducing Kernel Particle Method (RKPM) [8] have been used to analyze large deformations [9]. In addition, the cracked particle method is very suitable for simulating complex crack propagation in two or three dimensions [10]. The Material Point Method (MPM) [11] can also be used [12]. Alternative methods include the Natural Element Method (NEM) [13], the rotational formulation, and the Nearest-Nodes Finite Element Method (NN-FEM) method [14]. Also, the Lattice Spring Model (LSM) [15] and the Distinct Lattice Spring Model (DLSM) [16] have been used to analyze large deformations. The Arbitrary Eulerian-Lagrangian (ALE) formulation is based on the Extended Finite Element Method (XFEM) [17] and is used for instance to model thixoforming processes [18].

2.4 Numerical Manifold Method

The NMM was first developed by Shi Gen-Hua [19] based on the theory of Partition of Unity (PU). The fundamental feature of this method is the use of two independent mathematical and physical meshes for modeling engineering problems. The mathematical mesh is used to define mathematical functions. In contrast, the physical mesh is used to define boundaries and locate discontinuities.

The mathematical mesh is formed by several mathematical covers (MCs). Shape of the MCs is arbitrary. Mathematical mesh independence from the physical mesh makes meshing easy and there is no need to change the meshing if the geometry of the problem is changed during the solution [20]. The properties of this method can be described as follows:

1. The independence of the mathematical mesh from the geometry of the problem makes the meshing easier [21].

2. The independence of the mathematical and the physical meshes makes the boundaries and discontinuities of the problem easy to model [22].

3. The unknowns in the NMM, are assigned to MCs in contrast to the FEM, which are assigned to nodes, and this feature increases the simulation capabilities [23].

4. Continuous and discontinuous problems modeling takes place in a single system [24].

5. High-order simulations can be easily defined using high-order polynomial functions as local functions of MCs and PU functions, and there is no need to add cover or node to the problem [20].

In the NMM, the principle of minimum potential energy is often used to obtain equations, but if the problem exceeds the elastic strain, we will need the principle of virtual work [2]. Using these methods (principle of minimum potential energy and principle of virtual work), governing equations of problems such as static and dynamic analysis of continuous and discontinuous bodies in large or small, elastic or elastoplastic strains can be obtained. One of the other methods for formulating these equations is the Weighted Residual Method (WRM). Li et al. Obtained the NMM equations by applying the WRM [25] and confirmed the results obtained by the energy method. The WRM is a mathematical method for solving differential equations and has been used extensively in numerical methods such as FEM, Boundary Element Method, and Element Free Galerkin method [7]. Ghasemzadeh et al. also proposed a dynamic NMM based on the WRM [26]. In the present study, since strains are in the elastic range, the principle of minimum potential energy is used to obtain the governing equations of the continuous bodies.

Chen developed the high-order NMM for increasing precision and reducing computational cost to solve static problems in a continuous body [27]. Although the high-order NMM has several advantages, researchers refuse to use it because of the linear dependence problem [23]. Resolving the problem of linear dependence in static and dynamic analysis of the high orders of the NMM depends on how mathematical covers are selected. Ghasemzadeh et al. analyzed the linear dependence problem on the NMM by applying hexagonal covers which results in the production of triangular manifold elements and proposed a method to solve it [26].

The formulations of high-order approximations are given in detail in [27]. As other methods based on the partition of unity, linear dependence exists on the high orders of the NMM, which causes the singularity of the total matrices. An algorithm for predicting the rank deficiency of the total stiffness matrix is presented in [28] and extended in [29]. An effective and generalized method for generating covers was also proposed in [30], and a strategy for multilayer covers was introduced in [31] using NMM.

The main role of the NMM is to analyze the crack propagation [32]. Several new strategies for simulating crack propagation using the NMM have recently been proposed [33]. NMM-based Moving Least Squares was used to analyze the cracked bodies under dynamic loading [34]. In addition, a method for analyzing three-dimensional crack propagation by the NMM is presented in [35]. It is stated in this paper that the Heaviside functions are not required to analyze the crack by the NMM and the three-dimensional crack propagation can be analyzed similar to the two-dimensional states by NMM. Furthermore, in [36] simulating hydraulic fracture in rocks using a hydromechanical model coupled with the enriched NMM is discussed. In [37] a two-dimensional Voronoi element based on the NMM was developed as a novel idea to investigate the macro- and micro-mechanical properties of intact rocks. A two-dimensional Voronoi element is developed based on the NMM as a new approach to investigate the characteristics of acoustic emission during the process of intact rocks failure [38]. In [39] a unified pipe-mesh-based NMM is developed to simulate the immiscible two-phase flow in geological media. Also in [40] rock scratches have been studied numerically using a particle-based NMM. The NMM has been used to study fluid flow [41], seepage flow [42], and plate bending [43]. The NMM is also used for vibrational analysis of arbitrarily shaped thin plates [44]. In addition, NMM is used to solve two-dimensional heat transfer problems in Functionally graded materials [45]. In [46], three new boundary conditions are developed to use the NMM for seismic response analysis: 1) the classical viscous boundary condition, 2) the free field boundary condition, and 3) the static-dynamic unified boundary condition.

Given the history mentioned about the NMM, its various applications, and research on this issue in small strain problems, the lack of sufficient studies in the large strain range is evident. This thesis attempted to examine this deficit. For this purpose, the problems with nonlinear large elastic strains, of which hyperelastic material behaviors in large strains are the most common example, are investigated in this study.

2.5 linear dependence

The high-order NMM has advantages such as lower computational cost, various Local Approximation (LA) functions, and convenient construction of high-order Global Approximation (GA) functions, over other common numerical methods [28]. However, the high-order NMM has the linear dependence problem of the unknowns, which is observed in the rank deficiency of the GA function, and it results in the singularity of the total stiffness matrix in the static analysis [47] as well as the singularity of the total mass matrix in dynamic analysis. This problem occurs when the PU functions and LA functions consist of polynomial functions [48]. The linear dependence problem has a significant effect on the convergence of results in dynamic analysis and causes the divergence of solution. The occurrence of the linear dependence problem depends on the number of unknowns, the geometry of the elements, and the method of defining LA functions.

In addition to the high-order NMM, the linear dependence problem has been observed in other PU-based methods. The linear dependence of unknowns has been investigated by Babuska and Melenk in one-dimensional analysis [49], by Oden et al. in two-dimensional analysis [50] and by Duarte et al. in three-dimensional analysis [51].

A lot of effort has been made to manage the linear dependence problem. Babuska and Melenk [49] proposed PU functions that solve the linear dependence problem in one-dimensional simulation. Oden et al. [50] suggested that to eliminate the linear dependence some phrases in the PU functions should be removed from the local functions, while Duarte et al. [51] stated that it is not enough. Strouboulis et al. [52] used mapped quadrilateral element shape functions to prevent the linear dependence problem. However, Tian et al. [53] argued that then the PU functions would not remain explicit polynomials, and to ensure complete mapping of the quadrilateral shape functions, the mesh would have to be completely irregular, which is problematic in practice. Tian et al. [47] evaluated the problem of linear dependence in two-dimensional problems by numerical experiments using the generalized FEM. They explained that (a) having the same phrases in LA and PU polynomial functions does not cause linear dependence problem, (b) constraining the high order degrees of freedom (DOF) has a significant impact on preventing the linear dependence, and (c) the zero eigenvalues in the stiffness matrix for the triangular covers are independent of the geometry and refining covers has no effect on it. Whereas in the quadrilateral covers there is a significant dependence between these two issues and the zero eigenvalues of the stiffness matrix. However, they have not spoken of the definitive elimination of the linear dependence, and implementing these methods does not guarantee the resolving of linear dependence problem. Sai et al. [54] developed a new triangular element based on PU using dual local approximation scheme by treating boundary and interior nodes separately. The linear dependence problem is solved in their method and the boundary conditions can be applied directly like FEM. However, boundary points require a modified least-squares approach to construct local functions that can be computationally expensive. Also, they utilized incomplete quadratic polynomial functions without x^2 and y^2 phrases for the inner covers and constant functions for the boundary covers, which could reduce the accuracy of the results. An et al. [28] discussed the origins of the rank deficiency of the analysis in two-dimensional and three-dimensional space. They presented a method for predicting the rank deficiency of a simple triangular and quadrilateral mesh, acknowledging that the rank deficiency differs from the nullity of the global matrix and only depends on the shape of the element, PU functions, and LA functions. However, they did not provide a way to solve the linear dependence problem. Cho [55] evaluated the kernel of the total stiffness matrix in a singular system by the generalized FEM. This researcher has used common finite element shape functions as PU functions and harmonic polynomial functions as LA functions. Cho showed that the kernel of the stiffness matrix in the triangular mesh does not change, and it is independent of the mesh geometry and mesh parameter, whereas in the rectangular mesh the kernel increases significantly as the mesh is refined. However, when complete polynomial functions are used as local functions, the characteristics of the kernel of the stiffness matrix are not obvious. To manage the linear dependence problem, Babuska et al. [56] proposed the Stable Generalized FEM (SGFEM). This method enhances the ill-conditioning property of the solution so that they improve as much as the standard FEM conditions. SGFEM has convergence problems of the GFEM. Kim and Bathe [57] proposed a FEM enriched by interpolation covers that did not have a linear dependence problem. Their pattern is capable of detecting higher gradients and reduces stress jumps between elements. They managed the linear dependence problem by removing some unknowns from local functions. While removing these unknowns can result in reduced accuracy, and it does not guarantee the removal of linear dependence. Tian [58] proposed a linearly independent enrichment method for the GFEM without the use of additional unknowns. However, this method requires two interpolation steps, which increases the computational cost, and steps include the interpolation on the points of each cover for constructing LA function and the interpolation between covers for constructing GA function. Zheng and Xu [59] used first-order polynomial functions as LA and enrichment functions for modeling cracks in the NMM. They reduced the rank deficiency of the stiffness matrix by suppressing the gradient-dependent DOFs. However, the elimination of the linear dependence is not guaranteed. Hong and Lee [60] combined the FEM and the flat-top PU method, which is free of linear dependence problem. Using the flat-top PU method, An et al. [29] demonstrated that high-order simulation is linearly independent and provided a method to simplify the construction of its approximation functions. Zhang [61] combines local modification of the enrichments based on flat-top PU method of common covers to construct a new base for defining local functions. This method reduces the high-order local functions and consequently reduces the accuracy significantly. Using the hexagonal mathematical mesh in 2D space, Ghasemzadeh et al. [26] presented a unique and executable model to eliminate the problem of linear dependence of unknowns. Characteristics of this method can be (a) not using of new unknowns which increases the computational cost, (b) the correction of local functions for only one element in the whole mesh that makes this method easy to use, and (c) the maximum use of high-order unknowns that increase the accuracy of the results and eliminate dependent unknowns that reduce computational cost. With all the efforts made to understand the problem of the linear dependence of the unknowns, new dimensions are emerging every day, and no comprehensive solution has yet been provided.

There has been no specific study about the NMM in large deformations. The only researchers who worked on an issue related to this subject were Huo Fan et al. In 2016, Huo Fan et al. investigated the theory of strain-rotation decomposition-based NMM using the Generalized-α method by applying the weak form of the momentum conservation law [62]. In this study, we investigated the NMM using triangular element in large deformations for hyperelastic materials to begin the development of the NMM for large deformations using different types of materials with different properties and behaviors.

Chapter 3

3 Theory and Equations

3.1 Introduction

In this chapter, first, conventional and high-order NMM relations using triangular element are presented for static modeling in continuous bodies. Then, hyperelastic materials and their relations and models are discussed. And finally, the theory and equations of the NMM using total Lagrangian formulation for large deformations are presented and developed.

3.2 Manifold numerical method

3.2.1 Basic concepts of the NMM

Numerical methods in order to simulate a field F with an unknown function $f(x)$, divide the physical area of the problem (Ω) into small finite regions and then define a simple function such as $u_i(x)$ in each region such that it approaches $f(x)$. In NMM, these small regions are called Mathematical Covers (MCs), denoted by M_i $(i=1,\ldots,n)$ in which n is the number of MCs. These MCs should cover the entire physical domain of the problem and can theoretically have any shape, overlap or be separate. This is illustrated in Figure 3.1. Boundaries of MCs do not have to match external boundaries Γ_Ω or internal discontinuities Γ_d. The MCs union creates a mathematical mesh which is used for numerical calculations. External boundaries and internal discontinuities may divide each MC into one or several parts. If these parts fall within the physical domain of the problem, they create Physical Covers (PCs) denoted by $P_{i,j}$ $(j=1,\ldots,m_i)$ in which m_i is the number of PCs created from M_i [20].

As can be seen in Figure 3.1, MCs can be divided into four groups according to their position with respect to the internal and external boundaries: (a) MCs that do not cross any internal or external boundaries, such as M_i, (b) those that only intersect with external boundaries, such as M_j that is partially outside the problem domain, so that it has a PC in the problem domain, (c) those that only intersect with internal discontinuities, such as M_k that has two parts and both parts are within the domain of the problem, thus producing two PCs, and (d) those that intersect with both the external boundary and the internal discontinuity, such as the M_n divided into three parts, while only two parts are within the problem domain, producing two PCs, $p_{n,1}$ and $p_{n,2}$.

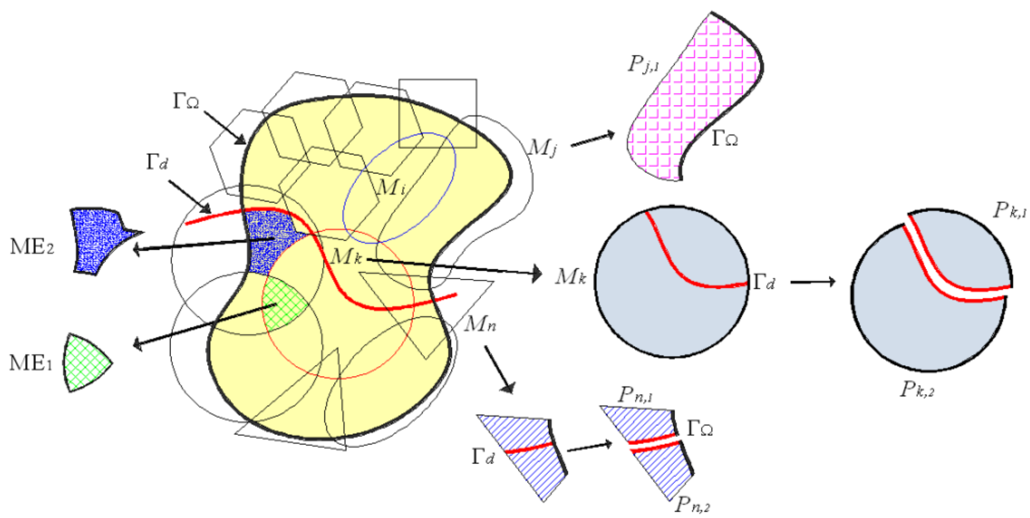

Figure 3.1 An illustration of Mathematical Covers, Physical Covers, and Manifold Elements

The union of all PCs creates a physical mesh which is used to model the problem domain and discontinuities. In the NMM, the Manifold Element (ME) is generated from the area overlapped by several PCs or the area covered by only one PC. For example, in Figure 3.1 the ME_1 is created from the overlap of three PCs and the ME_2 is made of only one PC. It is clear that the MEs do not overlap and their union covers the whole domain of the problem Ω.

To reflect the local properties of the analysis in global calculations in each PC, $p_{i,j}$, the function $u_{i,j}(x)$, is defined as the LA function. The optimal approximation of the result, $u(x)$, can be obtained by a weighted average of local functions $u_{i,j}(x)$ by applying the PU over ME as [63]

$$u(x)=\sum_{i=1}^{n}\sum_{j=1}^{m_i} w_i(x)u_{i,j}(x), \quad x\in ME \;, \tag{3.1}$$

where $w_i(x)$ is the PU function, n is the number of MCs, m_i is the number of PCs obtained from the M_i and the PU functions must satisfy the following conditions:

$$\begin{cases} 0\le w_i \le 1, & \forall x \in M_i \\ w_i = 0, & \forall x \notin M_i \\ \sum_{i=1}^{n} w_i = 1, & \forall x \in \Omega \end{cases} . \tag{3.2}$$

In the simulation by NMM, three steps should be considered which include 1) selection of the MCs' shape, 2) construction of PU functions, and 3) definition of LA functions. Since the NMM does not require the adaptation of the MCs and the internal and external boundaries, and in practice the use of regular MCs makes it easy to operate, regular MCs are employed in this study. In the triangular mesh, each point is placed at the center of an MC so that all the triangles around this point create a hexagonal MC. For example, in Figure 3.2 the M_i (an MC) is distinguished by square hash. The union of MCs forms the mathematical mesh.

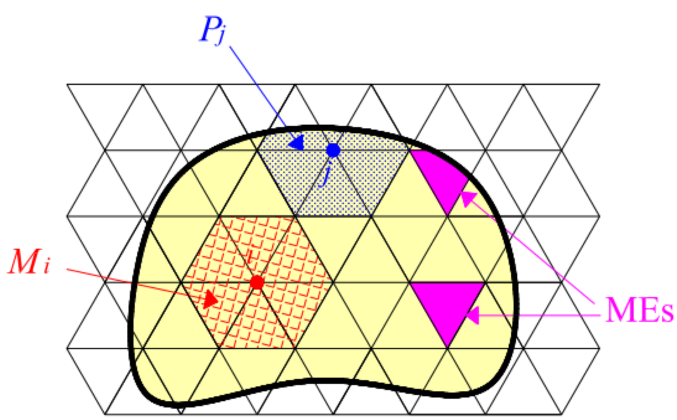

Figure 3.2 Definition of MCs, PCs, and MEs for regularly-patterned triangular mesh

In the NMM, PCs within the problem domain are equal to the MCs. At the problem boundaries, PCs are formed by the common area of the MCs and the problem domain, such as p_j distinguished by point hashes in the triangular mesh in Figure 3.2. In the triangular mesh, each ME is created by the common area of three PCs. In Figure 3.3 the formation of an ME by three MCs, (M_i, M_j, M_k) can be seen. It is assumed that these three MCs are completely within the problem domain, thus, three complete PCs, (p_i, p_j, p_k) are formed by these MCs and common area of PCs form the ME.

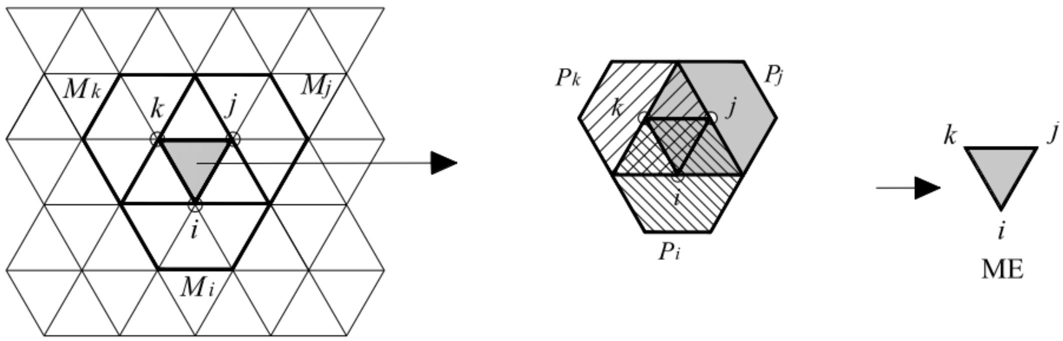

Figure 3.3 Generating MCs and MEs in a regularly patterned triangular mesh

The second step is to construct PU functions that change according to the simulation approach. The NMM often uses mesh-based Lagrange PU functions for mesh-based analysis. In other words, in the NMM, MCs and PCs are generated, then PU functions are constructed based on the process the MEs are formed. For example, in a regular triangular mesh as Figure 3.3, if three MCs, (M_i, M_j, M_k) are within the problem domain, they will produce three complete PCs, (p_i, p_j, p_k) and their common area forms an ME. Centers of the three MCs that create ME are used to define the PU functions. Although internal or external boundaries may divide the MCs into different parts and create different PCs, the centers of the MCs will be used to construct the PU functions. In a triangular mesh, usually, trilateral element shape functions are used to form PUs for MCs.

The third step is to define LA functions. LA functions can be any arbitrary functions such as constant, linear, complete or incomplete polynomials, etc. In this thesis, constant and polynomial functions are used, which are common choices in numerical simulations. The equations of the NMM can be obtained by the principle of minimum potential energy, the principle of virtual work and the weighted residual method, which form a general equation after applying boundary conditions, and the unknowns of the problem can be obtained by solving this equation.

3.2.2 High-order NMM

3.2.2.1 PU functions

PU functions on hexagonal MCs are defined such that the value of the function at the center of the cover is equal to 1 and it declines linearly to zero at the sides of the cover. The triangular common area formed by the three MCs, (M_i, M_j, M_k) in Figure 3.3 creates a ME. The PU functions for these three MCs can be written as

$$w_l(x,y) = a_l + b_l x + c_l y \quad (x,y) \in ME, \;\; l = i, j, k , \tag{3.3}$$

If we show the centers of these MCs as (x_m, y_m), which in that m=i, j, k then these PU functions must be as

$$w_l(x_m, y_m) = \begin{cases} 1 & l = m \\ 0 & l \neq m \end{cases} . \tag{3.4}$$

The above equation shows that the value of the PU function of the MC at the center of the cover is equal to 1 and at the centers of the other two covers is zero. By replacing Eq. (3.4) into Eq. (3.3), a set of equations is obtained that by solving these equations, unknowns a_l , b_l , c_l can be calculated for the triangular element, and it results [1, 20]

$$\begin{aligned} w_i &= \mathbf{w}_i \mathbf{L}(x, y) \\ w_j &= \mathbf{w}_j \mathbf{L}(x, y) . \\ w_k &= \mathbf{w}_k \mathbf{L}(x, y) \end{aligned} \tag{3.5}$$

Therefore, PU functions for the triangular element can be written as

$$\begin{aligned} \mathbf{w}_i &= \frac{1}{\Delta}\left[x_j y_k - x_k y_j \quad y_j - y_k \quad x_k - x_j \right] \\ \mathbf{w}_j &= \frac{1}{\Delta}\left[x_k y_i - x_i y_k \quad y_k - y_i \quad x_i - x_k \right] \\ \mathbf{w}_k &= \frac{1}{\Delta}\left[x_i y_j - x_j y_i \quad y_i - y_j \quad x_j - x_i \right] \\ \Delta &= (x_j - x_i)(y_k - y_i) - (x_k - x_i)(y_j - y_i) \\ \mathbf{L}(x, y) &= \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \end{aligned} \tag{3.6}$$

3.2.2.2 LA functions of the manifold elements

In Chen's research [27], the Displacement LA (DLA) functions in a ME are complete polynomial functions with equal orders. In this thesis, the DLA functions are slightly different such that in a ME it is allowed to use different orders of polynomials either complete or incomplete. In general, the l^{th} DLA (l=i, j, k) function in the form of a complete polynomial can be written as

$$u_l(x,y) = u_{l,0} + u_{l,1}x + u_{l,2}y + u_{l,3}x^2 + u_{l,4}xy + u_{l,5}y^2 + \cdots + u_{l,r}x^{n_l} + \cdots + u_{l,m}y^{n_l}$$
$$v_l(x,y) = v_{l,0} + v_{l,1}x + v_{l,2}y + v_{l,3}x^2 + v_{l,4}xy + v_{l,5}y^2 + \cdots + v_{l,r}x^{n_l} + \cdots + v_{l,m}y^{n_l} \tag{3.7}$$

where $u_{l,k}$ and $v_{l,k}$ ($k=1,\ldots,r,\ldots,m$) are the unknown coefficients of the problem, and n_l is the maximum order in the polynomial function. It is allowed to remove any arbitrary expression from this polynomial function, and thus it can change to any arbitrary polynomial function. The matrix form of Eq. (3.7) can be written as

$$u_l(x,y) = B_l U_l$$
$$v_l(x,y) = B_l V_l \tag{3.8}$$

where

$$B_l = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad \cdots \quad x^{n_l} \quad \cdots \quad y^{n_l}]$$
$$U_l = [u_{l,0} \quad u_{l,1} \quad u_{l,2} \quad u_{l,3} \quad u_{l,4} \quad u_{l,5} \quad \cdots \quad u_{l,r} \quad \cdots \quad u_{l,m}]^T$$
$$V_l = [v_{l,0} \quad v_{l,1} \quad v_{l,2} \quad v_{l,3} \quad v_{l,4} \quad v_{l,5} \quad \cdots \quad v_{l,r} \quad \cdots \quad v_{l,m}]^T \tag{3.9}$$

3.2.2.3 Displacement global approximation (DGA) functions over MEs

Using the PU theory, the DGA function on the ME is defined as [20]:

$$\boldsymbol{u} = w_i u_i + w_j u_j + w_k u_k$$
$$\boldsymbol{v} = w_i v_i + w_j v_j + w_k v_k \,. \tag{3.10}$$

As it is evident in the above equation, the DGA function of the ME is calculated from the multiplication of the PU functions and the LA functions, as a result, the order of the GA function is obtained by summing the orders of PU and LA functions. Since the PU functions of triangular MEs are of order 1, so the least possible order of GA of the NMM using the triangular element is 1. By replacing Eq. (3.5) and Eq. (3.8) into Eq. (3.10) the following equation can be obtained

$$\boldsymbol{u} = \mathbf{w}_i M_i U_i + \mathbf{w}_j M_j U_j + \mathbf{w}_k M_k U_k$$
$$\boldsymbol{v} = \mathbf{w}_i M_i V_i + \mathbf{w}_j M_j V_j + \mathbf{w}_k M_k V_k \tag{3.11}$$

where

$$M_l = L B_l \,. \tag{3.12}$$

Therefore, the DGA function over an element can be obtained in matrix form as

$$\mathbf{u} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} = Td \tag{3.13}$$