Difference Equations

For any integer $k \geq 1$, an equation of the form

$$u_{n+k} + \alpha_{k-1} u_{n+k-1} + \ldots + \alpha_0 u_n = \varphi_{n+k}, \quad n = 0, 1, \ldots$$

is called a *linear difference equation* of order k.

The coefficients α_0 # 0, α_1, . . . , α_{k-1} may or may not depend on n.
If, for any n, the right side φ_{n+k} is equal to zero, the equation is said *homogeneous,* while if the α_j s are independent of n it is called *linear difference equation with constant coefficients.*

Applications:

1. Difference equations arise for instance in the discretization of ordinary differential equations. All the numerical methods examined so far.

2. Quantities are defined through linear recursive relations.

3. Another relevant application is concerned with the discretization of boundary value problems

Solution of difference equations

Any sequence $\{u_n, n = 0, 1, \ldots\}$ of values that satisfy (1) is called a *solution* to the equation (1).
Given *k initial values* $u_0, \ldots, u_{k-1}$, it is always possible to construct a solution of (1) by computing sequentially.

$$u_{n+k} = [\varphi_{n+k} - (\alpha_{k-1}u_{n+k-1} + \ldots + \alpha_0 u_n)], \quad n = 0, 1, \ldots$$

However, our interest is to find an expression of the solution u_{n+k} which depends only on the coefficients and on the initial values.

Homogeneous case:

We start by considering the *homogeneous case with constant coefficients*,

$$u_{n+k} + \alpha_{k-1}u_{n+k-1} + \ldots + \alpha_0 u_n = 0, \quad n = 0, 1, \ldots$$

We associate with (2) the *characteristic polynomial* $\Pi(r)$

$$\Pi(r) = r^k + \alpha_{k-1}r^{k-1} + \ldots + \alpha_1 r + \alpha_0.$$

Denoting its roots by r_j, $j = 0, \ldots, k-1$, any sequence of the form

$$\{r_j^n, n = 0, 1, \ldots\}, \qquad \text{for } j = 0, \ldots, k-1$$

is a solution of (2).

k sequences defined in (3) are the *fundamental solutions* of the homogeneous equation (2). Any sequence of the form

$$u_n = \gamma_0 r_0^n + \gamma_1 r_1^n + \ldots + \gamma_{k-1} r_{k-1}^n, \qquad n = 0, 1, \ldots \quad (4)$$

is still a solution to (2), since it is a linear equation. The coefficients $\gamma_0, \ldots, \gamma_{k-1}$ can be determined by imposing the k initial conditions $u_0, \ldots, u_{k-1}$. Moreover, it can be proved that if all the roots of Π are simple, then *all* the solutions of (2) can be cast in the form (4).

This last statement no longer holds if there are roots of Π with multiplicity greater than 1.

If, for a certain *j*, the root r_j has multiplicity $m \geq 2$, in order to obtain a system of fundamental solutions that generate all the solutions of (2), it suffices to replace the corresponding fundamental solution

$$\{r_j^n, n = 0, 1, \ldots\}$$

with the *m* sequences

$$\{r_j^n, n = 0, 1, \ldots\}, \{nr_j^n, n = 0, 1, \ldots\}, \ldots, \{n^{m-1}r_j^n, n = 0, 1, \ldots\}.$$

More generally, assuming that $r_0, \ldots, r_k$ are distinct roots of Π, with multiplicities equal to $m_0, \ldots, m_k$, respectively, we can write the solution of (2) as

$$u_n = \sum_{j=0}^{k'} \left(\sum_{s=0}^{m_j-1} \gamma_{sj} n^s \right) r_j^n, \qquad n = 0, 1, \ldots .$$

Notice that even in presence of complex conjugate roots one can still obtain a real solution (see Exercise 3).

Example 1:

For the difference equation $u_{n+2} - u_n = 0$, we have $\Pi(r) = r^2 - 1$, then $r_0 = -1$ and $r_1 = 1$, therefore the solution is given by $u_n = \gamma_{00}(-1)^n + \gamma_{01}$. In particular, enforcing the initial conditions $u0$ and $u1$ gives $\gamma_{00} = (u_0 - u_1)/2$, $\gamma_{01} = (u_0 + u_1)/2$. •

Example 2:

For the difference equation $u_{n+3} - 2u_{n+2} - 7u_n+1 - 4u_n = 0$, $\Pi(r) = r^3 - 2r^2 - 7r - 4$. Its roots are $r_0 = -1$ (with multiplicity 2), $r1 = 4$ and the solution is $u_n = (\gamma_{00} + n\gamma_{10})(-1)^n + \gamma_{01}4^n$. Enforcing the initial conditions we can compute the unknown coefficients as the solution of the following linear system

$$\begin{cases} \gamma_{00} + \gamma_{01} & = u_0, \\ -\gamma_{00} - \gamma_{10} + 4\gamma_{01} & = u_1, \\ \gamma_{00} + 2\gamma_{10} + 16\gamma_{01} & = u_2 \end{cases}$$

that yields

$\gamma_{00} = (24u_0 - 2u_1 - u_2)/25$, $\gamma_{10} = (u_2 - 3u_1 - 4u_0)/5$ and $\gamma_{01} = (2u_1 + u_0 + u_2)/25$.

The above expression is of little practical use since it does not outline the dependence of u_n on the k initial conditions. A more convenient representation is obtained

$$\psi_j^{(i)} = \delta_{ij}, \quad i, j = 0, 1, \dots, k-1. \qquad (11.34)$$

Then, the solution of (11.29) subject to the initial conditions $u_0, \ldots, u_{k-1}$

is given by

$$u_n = \sum_{j=0}^{k-1} u_j \psi_j^{(n)}, \qquad n = 0, 1, \ldots . \tag{5}$$

The new fundamental solutions $\left\{ \psi_j^{(n)},\ n = 0, 1, \ldots \right\}$ can be represented in terms of those in (5) as follows

$$\psi_j^{(n)} = \sum_{m=0}^{k-1} \beta_{j,m} r_m^n \quad \text{for } j = 0, \ldots, k-1,\ n = 0, 1, \ldots \tag{6}$$

By requiring (6), we obtain the *k* linear systems

$$\sum_{m=0}^{k-1} \beta_{j,m} r_m^i = \delta_{ij}, \qquad i, j = 0, \ldots, k-1,$$

whose matrix form is

$$\mathrm{R}\mathbf{b}_j = \mathbf{e}_j, \qquad j = 0, \ldots, k-1.$$

Here $\mathbf{e}_j$ denotes the unit vector $\mathbb{R}^k$, $\mathrm{R} = (r_{im}) = (r^i_m)$ and $\mathbf{b}_j$ $(\beta_{j,0}, \ldots, \beta_{j,k-1})^T$ If all r'_js are simple roots of $\Pi$the matrix $\overline{\overline{\mathrm{R}}}$ is nonsingular

The general case where Π has $k'+1$ distinct roots $r_0, \ldots, r_k$ with multiplicities

$m_0, \ldots, m_{k'}$ respectively, can be dealt with by replacing in (6)

$\{r_j^n, \ \ n = 0, 1, \ldots\}$with$\{r_j^n n^s, \ \ n = 0, 1, \ldots\}$, where $j = 0, \ldots, k'$ and

$s = 0, \ldots, m_j - 1.$

Example 3: We consider again the difference equation of (r+1)^2(r−4) . H

we have $\{r_0^n, \, nr_0^n, \, r_1^n, \ n = 0, 1, \ldots\}$ so that the matrix R becomes

$$\mathrm{R} = \begin{bmatrix} r_0^0 & 0 & r_2^0 \\ r_0^1 & r_0^1 & r_2^1 \\ r_0^2 & 2r_0^2 & r_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & 4 \\ 1 & 2 & 16 \end{bmatrix}.$$

Solving the three systems yields

$$\psi_0^{(n)} = \frac{24}{25}(-1)^n - \frac{4}{5}n(-1)^n + \frac{1}{25}4^n,$$
$$\psi_1^{(n)} = -\frac{2}{25}(-1)^n - \frac{3}{5}n(-1)^n + \frac{2}{25}4^n,$$
$$\psi_2^{(n)} = -\frac{1}{25}(-1)^n + \frac{1}{5}n(-1)^n + \frac{1}{25}4^n,$$

from which it can be checked that the solut $u_n = \sum_{j=0}^{2} u_j \psi_j^{(n)}$

coincides with the one already found .

Now we return to the case of *nonconstant coefficients* and consider the following homogeneous equation

$$u_{n+k} + \sum_{j=1}^{k} \alpha_{k-j}(n) u_{n+k-j} = 0, \qquad n = 0, 1, \ldots . \tag{7}$$

The goal is to transform it into an ODE by means of a function F, called the *generating function* of the equation (7). F depends on the real variable t and is derived as follows. We require that the n–th coefficient of the Taylor series of F around $t = 0$ can be written $\gamma_n u_n$, as $\{\gamma_n\}$ for some unknown constant

$$F(t) = \sum_{n=0}^{\infty} \gamma_n u_n t^n. \qquad (8)$$

The coefficients γ_n, are unknown and must be determined in such a way

$$\sum_{j=0}^{k} c_j F^{(k-j)}(t) = \sum_{n=0}^{\infty} \left[u_{n+k} + \sum_{j=1}^{k} \alpha_{k-j}(n) u_{n+k-j} \right] t^n,$$

where c_j are suitable unknown constants not depending on n. Note that owing to (2) we obtain the ODE

$$\sum_{j=0}^{k} c_j F^{(k-j)}(t) = 0$$

to which we must add the initial conditions

$$F^{(j)}(0) = \gamma_j u_j \text{ for } j = 0, \dots, k-1$$

Once F is available, it is simple to recover u_n through the definition of F

Example 4 Consider the difference equation

$$(n+2)(n+1)u_{n+2} - 2(n+1)u_{n+1} - 3u_n = 0, \quad n = 0, 1, \dots \tag{9}$$

with the initial condit $u_0 = u_1 = 2$. We look for a generating function of t form (8). By term-to-term derivation of the series we get

$$F'(t) = \sum_{n=0}^{\infty} \gamma_n n u_n t^{n-1}, \quad F''(t) = \sum_{n=0}^{\infty} \gamma_n n(n-1) u_n t^{n-2},$$

and, after some algebra, we find

$$F'(t) = \sum_{n=0}^{\infty} \gamma_n n u_n t^{n-1} = \sum_{n=0}^{\infty} \gamma_{n+1}(n+1) u_{n+1} t^n,$$

$$F''(t) = \sum_{n=0}^{\infty} \gamma_n n(n-1) u_n t^{n-2} = \sum_{n=0}^{\infty} \gamma_{n+2}(n+2)(n+1) u_{n+2} t^n.$$

As a consequence, (11.40) becomes

$$\sum_{n=0}^{\infty}(n+1)(n+2)u_{n+2}t^n - 2\sum_{n=0}^{\infty}(n+1)u_{n+1}t^n - 3\sum_{n=0}^{\infty} u_n t^n$$

$$= c_0 \sum_{n=0}^{\infty} \gamma_{n+2}(n+2)(n+1)u_{n+2}t^n + c_1 \sum_{n=0}^{\infty} \gamma_{n+1}(n+1)u_{n+1}t^n + c_2 \sum_{n=0}^{\infty} \gamma_n u_n t^n,$$

so that, equating both sides, we find

$$\gamma_n = 1 \ \forall n \geq 0, \quad c_0 = 1, \ c_1 = -2, \ c_2 = -3.$$

We have thus associated with the difference equation the following ODE with constant coefficients

$$F''(t) - 2F'(t) - 3F(t) = 0,$$

with the initial condition $F(0) = F'(0) = 2$.
The n–th coefficient of the solution $F(t) = e^{3t} + e^{-t}$ is

$$\frac{1}{n!} F^{(n)}(0) = \frac{1}{n!}\left[(-1)^n + 3^n\right],$$

so that $u_n = (1/n!)\left[(-1)^n + 3^n\right]$

is the solution of (9).

Nonhomogeneous Equation

The *nonhomogeneous case* can be tacked by searching for solutions of the form

$$u_n = u_n^{(0)} + u_n^{(\varphi)},$$

where $u_n^{(0)}$ is the solution of homogeneous equation $u_n^{(\varphi)}$ is a particular solution of the nonhomogeneous equation. Once the solution of the homogeneous equation is available, a general technique to obtain the solution of the non homogeneous equation is based on the method of variation of parameters, combined with a reduction of the order of the difference equation (see [Boice]).In the special case of difference equations with constant coefficients, withwright hand side of the form $c^n Q(n),$

where c is a constant and Q is a polynomial of degree p with respect to the variable n, a possible approach is *undetermined coefficients*, where one looks for a particular solution that depends on some undetermined constants and has a known form for some claφ_ns of right Sides . It suffices to look for a particular solution of the form

$$u_n^{(\varphi)} = c^n(b_p n^p + b_{p-1} n^{p-1} + \ldots + b_0),$$

where $b_p, \ldots, b_0$ are constants to be determined in such a way that $u_n^{(\varphi)}$ is actually a solution of (1).

Example 5: Consider the difference equa $u_{n+3}-u_{n+2}+u_{n+1}-u_n = 2^n n^2.$

The particular solution is of the form $u_n = 2^n(b_2n^2 + b_1n + b_0).$ Substituting this solution into the equation, we $5b_2n^2+(36b_2+5b_1)n+(58b_2+18b_1+5b_0) = n^2,$

from which, recalling the principle of identity for polynomials, one gets

$$b_2 = 1/5, \quad b_1 = -36/25 \text{ and } b_0 = 358/125.$$