2

The Performance Measure

Having already considered the modeling of systems and the determination of state and control constraints, we are now ready to discuss performance measures used in control problems. Our objective is to provide physical motivation for the selection of a performance measure.

Classical design techniques have been successfully applied to *linear, time-invariant, single-input single-output systems with zero initial conditions.* Typical performance criteria are system response to a step or ramp input—characterized by rise time, settling time, peak overshoot, and steady-state accuracy—and the frequency response of the system—characterized by gain and phase margin, peak amplitude, and bandwidth. Classical techniques have proved to be successful in many applications; however, we wish to consider systems of a more general nature with performance objectives not readily described in classical terms.

2.1 PERFORMANCE MEASURES FOR OPTIMAL CONTROL PROBLEMS

The "optimal control problem" is to find a control $\mathbf{u}^* \in U$ which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{2.1-1}$$

to follow a trajectory $\mathbf{x}^* \in X$ that minimizes the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt. \tag{2.1-2}$$

Let us now discuss some typical control problems to provide some physical motivation for the selection of a performance measure.

Minimum-Time Problems

Problem: To transfer a system from an arbitrary initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to a specified target set S in minimum time.

The performance measure to be minimized is

$$\begin{aligned} J &= t_f - t_0 \\ &= \int_{t_0}^{t_f} dt, \end{aligned} \tag{2.1-3}$$

with t_f the first instant of time when $\mathbf{x}(t)$ and S intersect. The automobile example discussed in Section 1.1 is a minimum-time problem. Other typical examples are the interception of attacking aircraft and missiles, and the slewing mode operation of a radar, or gun system.

Terminal Control Problems

Problem: To minimize the deviation of the final state of a system from its desired value $\mathbf{r}(t_f)$.

A possible performance measure is

$$J = \sum_{i=1}^{n} [x_i(t_f) - r_i(t_f)]^2. \tag{2.1-4}$$

Since positive and negative deviations are equally undesirable, the error is squared. Absolute values could also be used, but the quadratic form in Eq. (2.1-4) is easier to handle mathematically. Using matrix notation, we have

$$J = [\mathbf{x}(t_f) - \mathbf{r}(t_f)]^T[\mathbf{x}(t_f) - \mathbf{r}(t_f)], \qquad \dagger(2.1\text{-}5)$$

or this can be written as

$$J = \|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|^2. \tag{2.1-5a}$$

$\|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|$ is called the *norm* of the vector $[\mathbf{x}(t_f) - \mathbf{r}(t_f)]$.

† T denotes the matrix transpose.

To allow greater generality, we can insert a real symmetric positive semi-definite $n \times n$ weighting matrix $\mathbf{H}$† to obtain

$$J = [\mathbf{x}(t_f) - \mathbf{r}(t_f)]^T\mathbf{H}[\mathbf{x}(t_f) - \mathbf{r}(t_f)]. \tag{2.1-6}$$

This quadratic form is also written

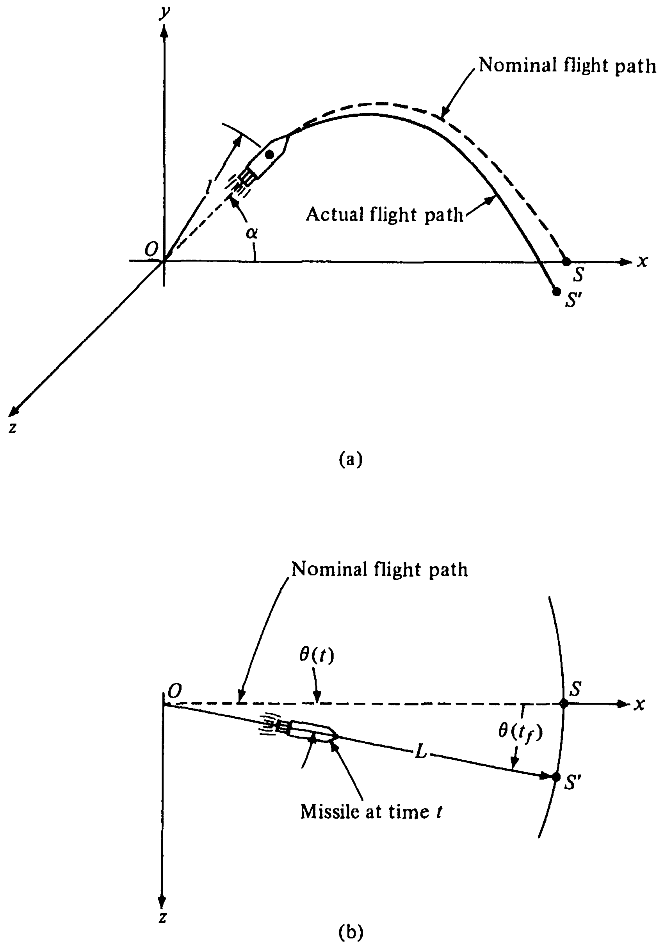

Figure 2-1 A ballistic missile aimed toward the target S

† A real symmetric matrix $\mathbf{H}$ is *positive semi-definite* (*or nonnegative definite*) if for all vectors $\mathbf{z}$, $\mathbf{z}^T\mathbf{H}\mathbf{z} \geq 0$. In other words, there are some vectors for which $\mathbf{H}\mathbf{z} = \mathbf{0}$ in which case $\mathbf{z}^T\mathbf{H}\mathbf{z} = 0$, and for all other $\mathbf{z}$, $\mathbf{z}^T\mathbf{H}\mathbf{z} > 0$.

$$J = \| \mathbf{x}(t_f) - \mathbf{r}(t_f) \|^2_{\mathbf{H}}. \tag{2.1-6a}$$

If **H** is the identity matrix,† (2.1-6) and (2.1-5) are identical.

Suppose that **H** is a diagonal matrix. The assumption that **H** is positive semi-definite implies that all of the diagonal elements are nonnegative. By adjusting the element values we can weight the relative importance of the deviation of each of the states from their desired values. Thus, by increasing h_{ii}‡ we attach more significance to deviation of $x_i(t_f)$ from its desired value; by making h_{jj} zero we indicate that the final value of x_j is of no concern whatsoever.

The elements of **H** should also be adjusted to normalize the numerical values encountered. For example, consider the ballistic missile shown in Fig. 2-1. The position of the missile at time t is specified by the spherical coordinates $l(t)$, $\alpha(t)$, and $\theta(t)$. l is the distance from the origin of the coordinate system, and α and θ are the elevation and azimuth angles. If $L = 5000$ miles and $l(t_f) = L$, an azimuth error at impact of 0.01 rad results in missing the target S by 50 miles! If the performance measure is

$$J = h_{11}[l(t_f) - 5000]^2 + h_{22}[\theta(t_f)]^2, \tag{2.1-7}$$

then we would select $h_{22} = [50/0.01]^2 \cdot h_{11}$ to weight equally deviations in range and azimuth. Alternatively, the variables θ and l could be normalized, in which case $h_{11} = h_{22}$.

Minimum-Control-Effort Problems

Problem: To transfer a system from an arbitrary initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ to a specified target set S, with a minimum expenditure of control effort.

The meaning of the term "minimum control effort" depends upon the particular physical application; therefore, the performance measure may assume various forms. For example, consider a spacecraft on an interplanetary exploration—let $u(t)$ be the thrust of the rocket engine, and assume that the magnitude of thrust is proportional to the rate of fuel consumption. In order to minimize the total expenditure of fuel, the performance measure

† The identity matrix is

$$\mathbf{I} \triangleq \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

‡ h_{ii} denotes the iith element of **H**.

$$J = \int_{t_0}^{t_f} |u(t)|\,dt \tag{2.1-8}$$

would be selected. If there are several controls, and the rate of expenditure of control effort of the ith control is $c_i|u_i(t)|$, $i = 1, \ldots, m$ (c_i is a constant of proportionality), then minimizing

$$J = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} \beta_i |u_i(t)|\right] dt \tag{2.1-8a}$$

would minimize the control effort expended. The β_i's are nonnegative weighting factors.

As another example, consider a voltage source driving a network containing no energy storage elements. Let $u(t)$ be the source voltage, and suppose that the network is to be controlled with minimum source energy dissipation. The source current is directly proportional to $u(t)$ in this case, so to minimize the energy dissipated, minimize the performance measure

$$J = \int_{t_0}^{t_f} u^2(t)\,dt. \tag{2.1-9}$$

For several control inputs the general form of performance measure corresponding to (2.1-9) is

$$\begin{aligned} J &= \int_{t_0}^{t_f} [\mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)]\,dt \\ &= \int_{t_0}^{t_f} \|\mathbf{u}(t)\|_{\mathbf{R}}^2\,dt, \end{aligned} \tag{2.1-9a}$$

where $\mathbf{R}$ is a real symmetric positive definite† weighting matrix. The elements of $\mathbf{R}$ may be functions of time if it is desired to vary the weighting on control-effort expenditure during the interval $[t_0, t_f]$.

Tracking Problems

Problem: To maintain the system state $\mathbf{x}(t)$ as close as possible to the desired state $\mathbf{r}(t)$ in the interval $[t_0, t_f]$.

As a performance measure we select

$$J = \int_{t_0}^{t_f} \|\mathbf{x}(t) - \mathbf{r}(t)\|_{\mathbf{Q}(t)}^2\,dt, \tag{2.1-10}$$

† A real symmetric matrix $\mathbf{R}$ is positive definite if

$$\mathbf{z}^T\mathbf{R}\mathbf{z} > 0$$

for all $\mathbf{z} \neq \mathbf{0}$.

where $\mathbf{Q}(t)$ is a real symmetric $n \times n$ matrix that is positive semi-definite for all $t \in [t_0, t_f]$. The elements of the matrix $\mathbf{Q}$ are selected to weight the relative importance of the different components of the state vector and to normalize the numerical values of the deviations. For example, if $\mathbf{Q}$ is a constant diagonal matrix and q_{ii} is zero, this indicates that deviations of x_i are of no concern.

If the set of admissible controls is bounded, e.g., $|u_i(t)| \leq 1$, $i = 1, 2, \ldots, m$, then (2.1-10) is a reasonable performance measure; however, if the controls are not bounded, minimizing (2.1-10) results in controls with impulses and their derivatives. To avoid placing bounds on the admissible controls, or if control energy is to be conserved, we use the modified performance measure

$$J = \int_{t_0}^{t_f} [\|\mathbf{x}(t) - \mathbf{r}(t)\|^2_{\mathbf{Q}(t)} + \|\mathbf{u}(t)\|^2_{\mathbf{R}(t)}]\, dt. \qquad (2.1\text{-}11)$$

$\mathbf{R}(t)$ is a real symmetric *positive definite* $m \times m$ matrix for all $t \in [t_0, t_f]$. We shall see in Section 5.2 that if the plant is linear this performance measure leads to an easily implemented optimal controller.

It may be especially important that the states be close to their desired values at the final time. In this case, the performance measure

$$J = \|\mathbf{x}(t_f) - \mathbf{r}(t_f)\|^2_{\mathbf{H}} + \int_{t_0}^{t_f} [\|\mathbf{x}(t) - \mathbf{r}(t)\|^2_{\mathbf{Q}(t)} + \|\mathbf{u}(t)\|^2_{\mathbf{R}(t)}]\, dt \qquad (2.1\text{-}12)$$

could be used. $\mathbf{H}$ is a real symmetric positive semi-definite $n \times n$ matrix.

Regulator Problems

A regulator problem is the special case of a tracking problem which results when the desired state values are zero ($\mathbf{r}(t) = \mathbf{0}$ for all $t \in [t_0, t_f]$).

2.2 SELECTING A PERFORMANCE MEASURE

In selecting a performance measure the designer attempts to define a mathematical expression which when minimized indicates that the system is performing in the most desirable manner. Thus, choosing a performance measure is a translation of the system's physical requirements into mathematical terms. In particular, suppose that two admissible control histories which cause admissible state trajectories are specified and we are to select the better one. To evaluate these controls, perform the test shown in Fig.

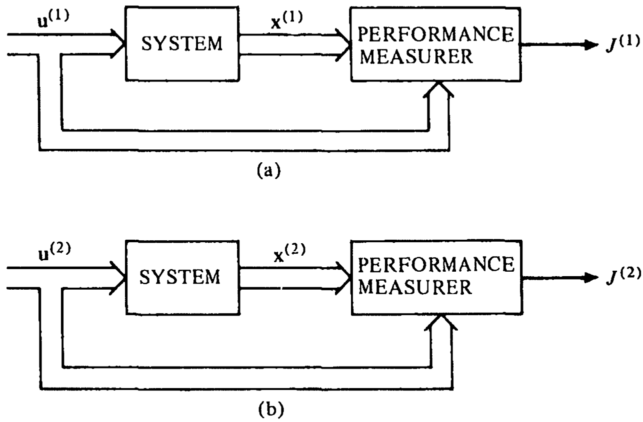

Figure 2-2 Evaluating two specified control histories

2-2. First, apply the control $\mathbf{u}^{(1)}$ to the system and determine the value of the performance measure $J^{(1)}$; then repeat this procedure with $\mathbf{u}^{(2)}$ applied to obtain $J^{(2)}$. If $J^{(1)} < J^{(2)}$, then we designate $\mathbf{u}^{(1)}$ as the better control; if $J^{(2)} < J^{(1)}$, $\mathbf{u}^{(2)}$ is better; if $J^{(1)} = J^{(2)}$ the two controls are equally desirable. An alternative test is to apply each control, record the state trajectories, and then subjectively decide which trajectory is better.

If the performance measure truly reflects desired system performance, the trajectory selected by the designer as being "more to his liking" should yield the smaller value of J. If this is not the case, the performance measure or the constraints should be modified.

Example 2.2-1. Figure 2-3 shows a manned spacecraft whose attitude is to be controlled by a gas expulsion system. As a simplification, we shall

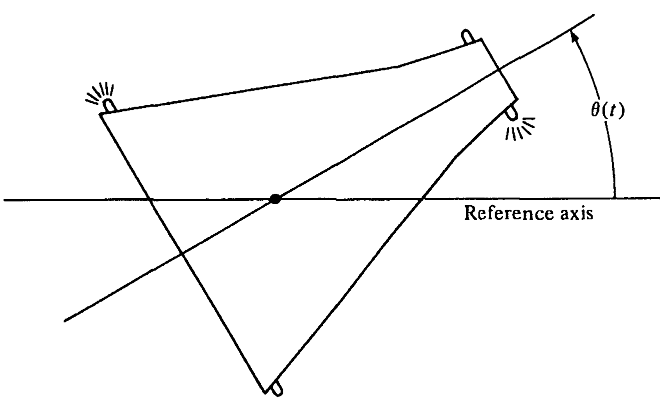

Figure 2-3 Attitude control of a spacecraft

consider only the control of the pitch angle $\theta(t)$. The differential equation that describes the motion is

$$I\frac{d^2}{dt^2}[\theta(t)] = \lambda(t), \tag{2.2-1}$$

where I is the angular moment of inertia and $\lambda(t)$ is the torque produced by the gas jets. Selecting $x_1(t) \triangleq \theta(t)$ and $x_2(t) \triangleq \dot{\theta}(t)$ as state variables, and $u(t) \triangleq \lambda(t)/I$ as the control gives the state equations

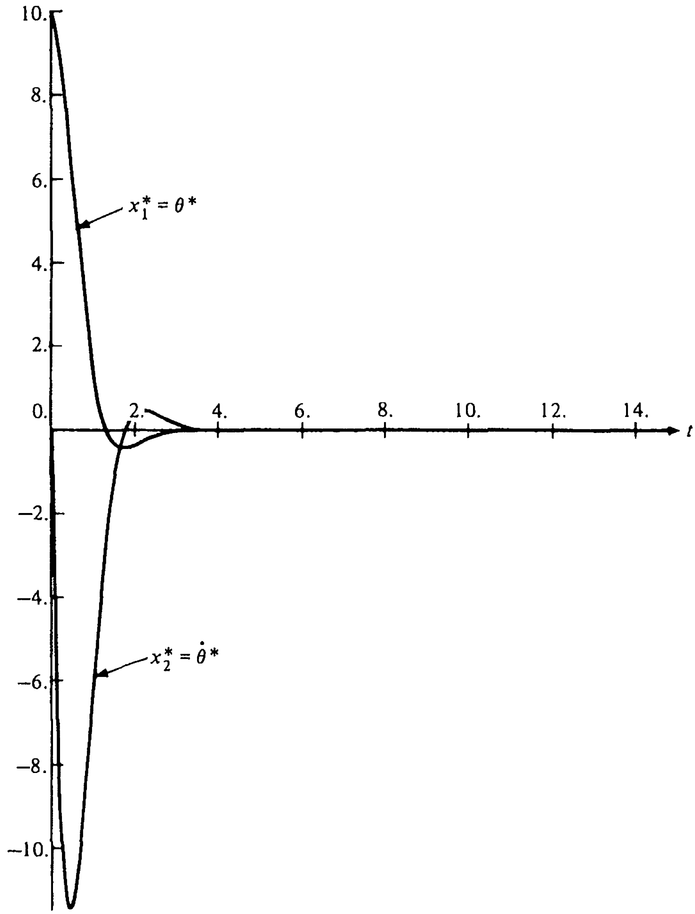

Figure 2-4(a) Position and velocity as functions of time

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}; R = .1; \mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t). \tag{2.2-2}$$

The primary objective of the control system is to maintain the angular position near zero. This is to be accomplished with small acceleration.

As a performance measure we select

$$J = \int_0^\infty [q_{11}x_1^2(t) + q_{22}x_2^2(t) + Ru^2(t)]\,dt, \tag{2.2-3}$$

where $q_{11}, q_{22} \geq 0$, and $R > 0$ are weighting factors. In Figs. 2-4, 2-5, 2-6, and 2-7 the optimal trajectories for $q_{11} = 4.0$, $q_{22} = 0$, and several

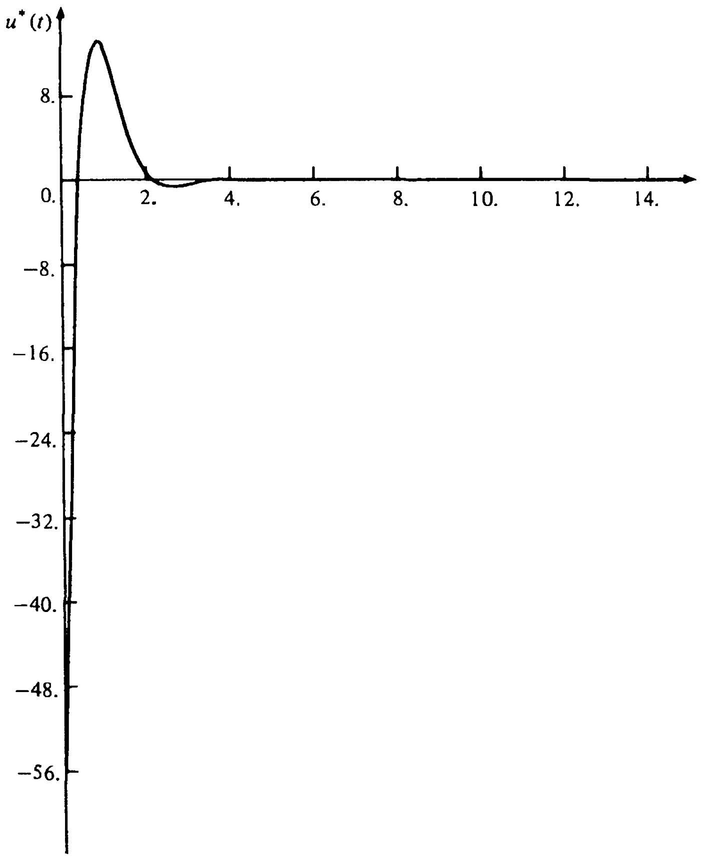

Figure 2-4(b) Acceleration as a function of time

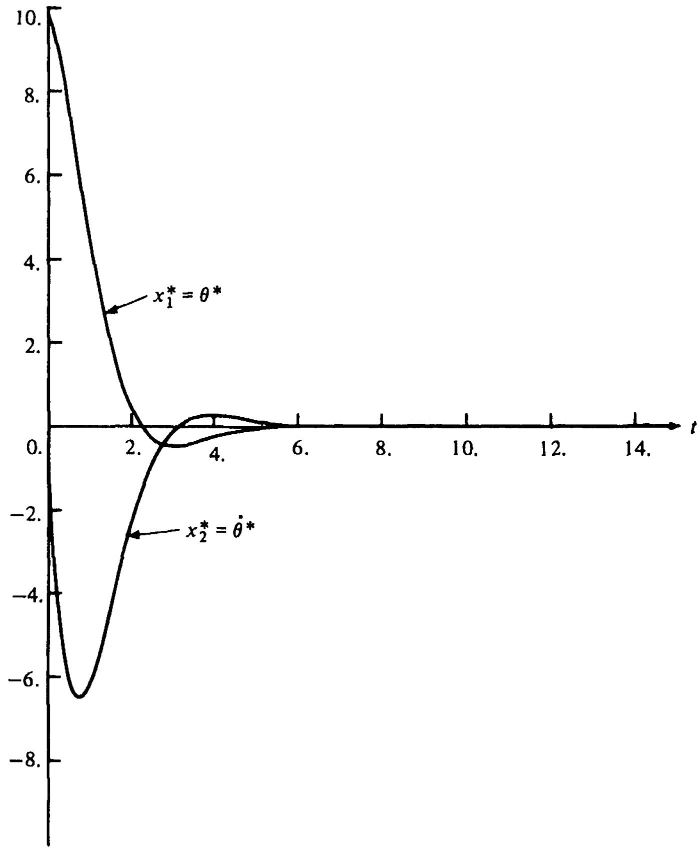

Figure 2-5(a) Position and velocity as functions of time

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}; R = 1.; \mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

values of R are shown.† Increasing R places a heavier penalty on acceleration and control energy expenditure. All of these trajectories are optimal, each for a different performance measure. If we are most concerned about reducing the angular displacement to zero quickly, then the trajectory in Fig. 2-4 would be our choice. The astronauts, however, would probably prefer the trajectory shown in Fig. 2-7 because of the much smaller accelerations.

We must be very careful when interpreting the numerical value of the

† These trajectories were obtained by using the techniques discussed in Sections 3.12 and 5.2.

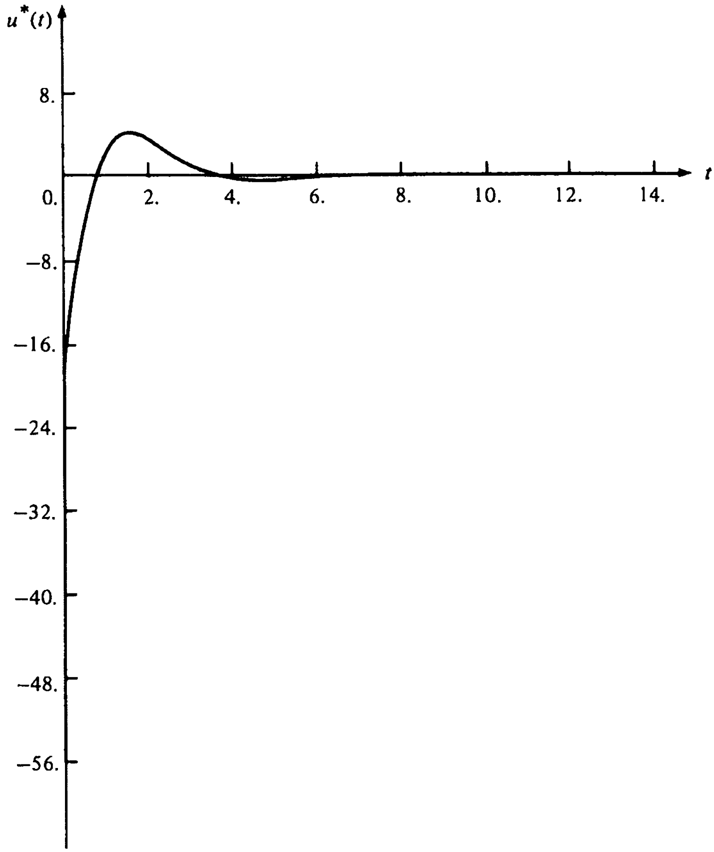

Figure 2-5(b) Acceleration as a function of time

minimum performance measure. By multiplying every weighting factor in the performance measure by a positive constant k, the value of the measure would be k times its original value, but the optimal control and trajectory would remain exactly the same. In fact, it may be possible to adjust the weighting factors by different amounts and still retain the same optimal control and trajectory.†

The physical interpretation of the value of the performance measure is also a factor to be considered. The minimum value of a performance measure such as elapsed time or consumed fuel has a definite physical significance; however, for problems in which the performance measure is a weighted

† See Chapter 8 of reference [S-2].

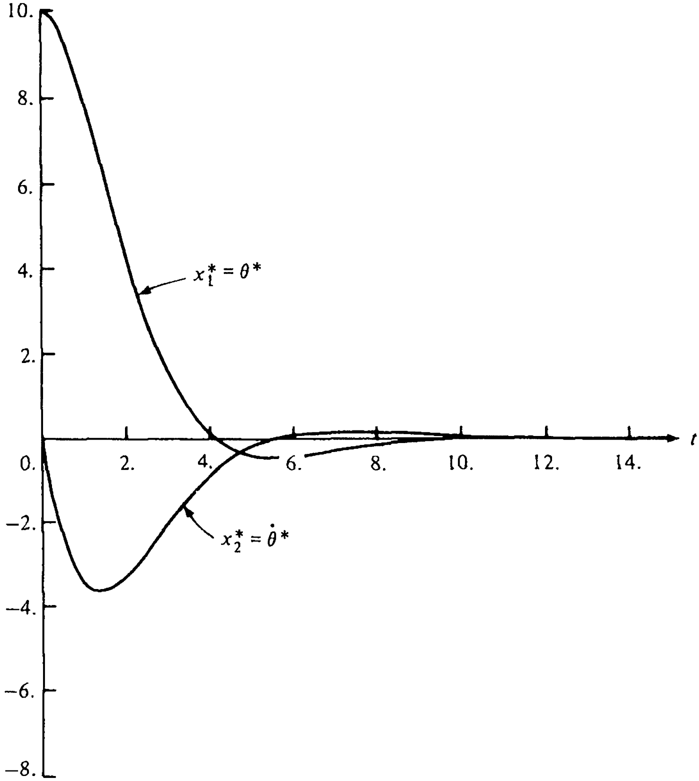

Figure 2-6(a) Position and velocity as functions of time

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}; R = 10; \mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

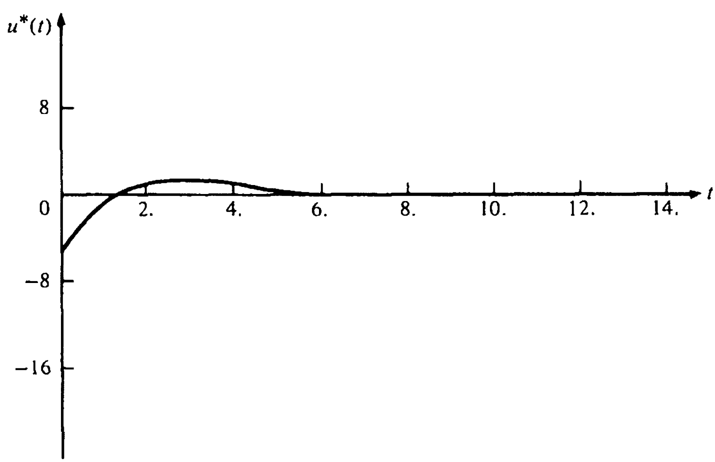

Figure 2-6(b) Acceleration as a function of time

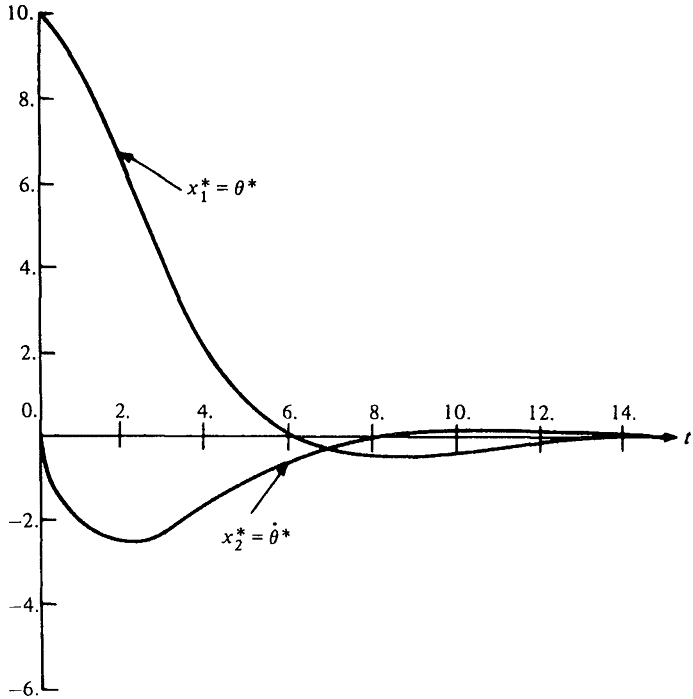

Figure 2-7(a) Position and velocity as functions of time

$$\mathbf{Q} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}; R = 50; \mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

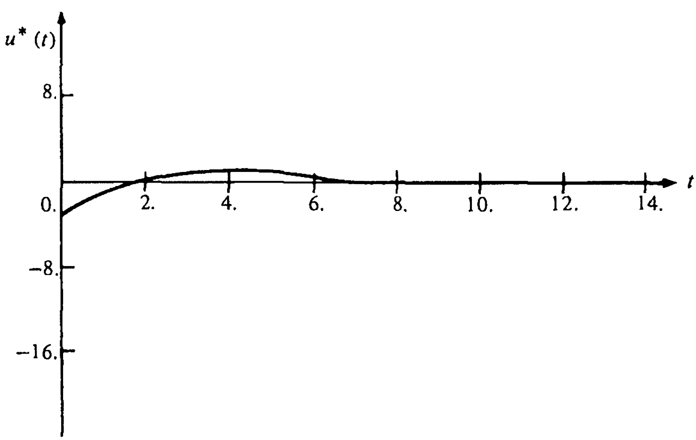

Figure 2-7(b) Acceleration as a function of time

combination of different physical quantities—as in the preceding spacecraft example—the numerical value of the performance measure does not represent a physically meaningful quantity.

2.3 SELECTION OF A PERFORMANCE MEASURE: THE CARRIER LANDING OF A JET AIRCRAFT

The following example, which is similar to a problem considered by Merriam and Ellert [M-1], illustrates the selection of a performance measure. The problem is to design an automatic control system for landing a high-speed jet airplane on the deck of an aircraft carrier.

The jet aircraft is shown in Fig. 2-8. The x direction is along the velocity vector of the aircraft, and the y and z directions are as shown. α is the angle of attack, θ is the pitch angle, and γ is the glide path angle.

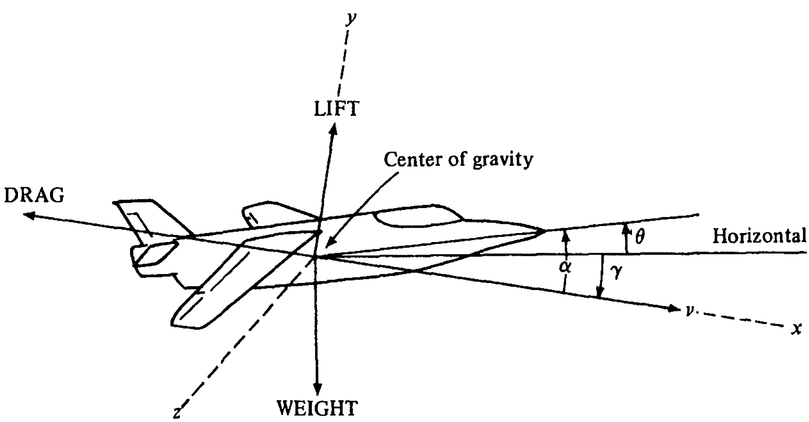

Figure 2-8 Aircraft coordinates and angles

We shall make the following simplifying assumptions:

1. Lateral motion is ignored; only motion in the x-y plane is considered.
2. Random disturbances, such as wind gusts and carrier deck motion, are neglected.
3. The nominal glide path angle γ is small, so that $\cos\gamma \approx 1$ and $\sin\gamma \approx \gamma$ in radians (it will be shown that the nominal γ is -0.0636 rad).
4. The velocity of the aircraft with respect to the nominal landing point is maintained at a constant value of 160 mph (235 ft/sec) by an automatic throttle control device.

5. The longitudinal motion of the aircraft is controlled entirely by the elevator deflection angle [$\delta_e(t)$, shown in Fig. 2-9], which has been trimmed to a nominal setting of 0° at the start of the automatic landing phase.

Figure 2-9 Elevator deflection angle

6. The aircraft dynamics are described by a set of differential equations that have been linearized about the equilibrium flight condition.

Since we desire to have readily available information concerning the system states to generate the required control, the altitude above the flight deck h, altitude rate $\dot{h}$, pitch angle θ, and pitch rate $\dot{\theta}$ are selected as the state variables. h is measured by a radar altimeter, $\dot{h}$ by a barometric rate meter, θ and $\dot{\theta}$ by gyros. If we define $x_1 \triangleq h$, $x_2 \triangleq \dot{h}$, $x_3 \triangleq \theta$, $x_4 \triangleq \dot{\theta}$, and $u \triangleq \delta_e$, the state equations that result from the linearization of the aircraft motion about the equilibrium flight condition are†

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= a_{22}x_2(t) + a_{23}x_3(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= a_{42}x_2(t) + a_{43}x_3(t) + a_{44}x_4(t) + b_4u(t),
\end{aligned}
\tag{2.3-1}
$$

where the a's and b_4 are known constants. In matrix form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t); \tag{2.3-2}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a_{42} & a_{43} & a_{44} \end{bmatrix}; \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_4 \end{bmatrix}.$$

Next, the desired behavior for the aircraft must be defined. The nominal flight path is selected as a straight line which begins at an altitude of 450 ft and at a range of 7,050 ft measured from the landing point on the carrier deck. This results in 30 seconds' being the nominal time required for the terminal phase of the landing. The desired altitude trajectory h_d is shown in Fig. 2-10. This selection for h_d implies that the desired altitude rate

† See [M-1].

$\dot{h}_d$ is as shown in Fig. 2-11. It is desired to maintain the attitude of the aircraft at 5°. This is most important at touchdown because it is required that the main landing gear touch before the nose or tail gear. Since $\theta_d(t) = 5°$ for $t \in [0, 30]$, $\dot{\theta}_d(t) = 0$ during this time interval, and the desired attitude and attitude rate profiles are shown in Figs. 2-12 and 2-13. The desired atti-

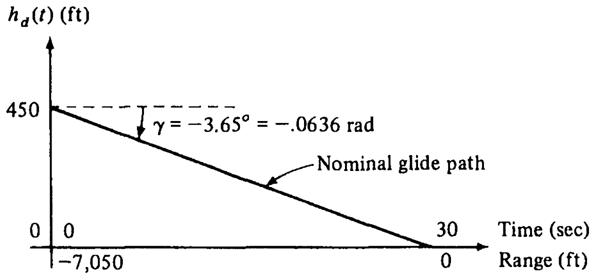

Figure 2-10 Desired altitude history

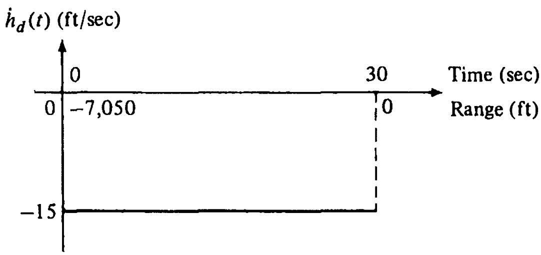

Figure 2-11 Desired rate of ascent history

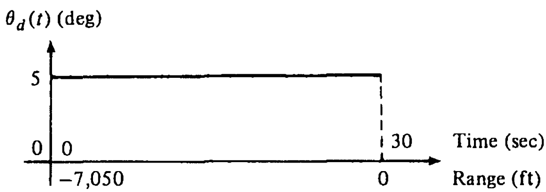

Figure 2-12 Desired attitude profile

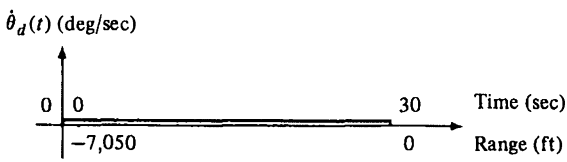

Figure 2-13 Desired attitude rate profile

tude and glide path angle profiles imply that the desired angle of attack α_d is 8.65° during the entire 30 sec interval.

It is assumed that large deviations of δ_e from the nominal 0° setting are indicative of a suboptimal landing and should be avoided; therefore, the desired value of δ_e is 0° throughout the terminal phase of landing.

The assumption is also made that there are limits on allowable departure from nominal values during descent. If any of these limits are exceeded, a wave-off is given, and the pilot takes control.

The translation of the performance requirements into a quantitative measure is the next task. The performance measure is selected as the integral of a sum of quadratic terms in the state and control variables and some additional terms to account for quantities which are crucial at touchdown. The index selected is

$$\begin{aligned} J = {} & k_h[h(30) - h_d(30)]^2 + k_{\dot{h}}[\dot{h}(30) - \dot{h}_d(30)]^2 + k_\theta[\theta(30) - \theta_d(30)]^2 \\ & + \int_0^{30} \{q_h(\tau)[h(\tau) - h_d(\tau)]^2 + q_{\dot{h}}(\tau)[\dot{h}(\tau) - \dot{h}_d(\tau)]^2 \\ & + q_\theta(\tau)[\theta(\tau) - \theta_d(\tau)]^2 + q_{\dot{\theta}}(\tau)[\dot{\theta}(\tau) - \dot{\theta}_d(\tau)]^2 \\ & + r_{\delta_e}(\tau)[\delta_e(\tau) - \delta_{e_d}(\tau)]^2\}\, d\tau, \end{aligned} \tag{2.3-3}$$

where τ is a dummy variable of integration. The k's, q's, and r_{δ_e} are weighting factors that are specified to assign relative importance to each of the terms in the performance measure and to account for differences in the relative size of numerical values encountered. The q's and r_{δ_e} are written as time-varying functions because deviations of some of the variables from nominal values may be more critical during certain periods of time than others. For example, rate of ascent errors are more critical over the flight deck than at earlier instants, so $q_{\dot{h}}(t)$ should increase as t approaches 30 sec. The terms outside of the integral are there to help ensure that the attitude, rate of ascent, and altitude are close to nominal at $t = 30$ sec. Notice that the term containing $h(30)$ penalizes a landing that occurs too soon or too late.

There is no term in the measure containing the angle of attack α explicitly; however, if the values for θ and $\dot{\theta}$ are maintained "close" to their desired values, then it is reasonable to expect that α will be satisfactory. Certainly a term could be added containing the deviation of angle of attack from its nominal value, but this would necessitate the selection of an additional weighting factor, and it is generally desirable to keep the problem simple for the initial solution. The desired, or nominal, aircraft trajectory is specified by Figs. 2-10 through 2-13. Figure 2-10 gives $h_d(t) = 450 - 15t$ ft as the desired altitude history, and the desired altitude at $t = 30$ (the nominal time touchdown occurs) is $h_d(30) = 0$ ft. From Fig. 2-11 the desired altitude rate history is -15 ft/sec throughout the interval [0, 30]; thus, $\dot{h}_d(t) = -15$ ft/sec and $\dot{h}_d(30) = -15$ ft/sec. The desired aircraft attitude is $+5°$ in the entire landing interval; therefore, $\theta_d(t) = 0.0873$ rad, and $\theta_d(30) = 0.0873$ rad. From Fig. 2-13 we have $\dot{\theta}_d(t) = 0$ rad/sec as the nominal attitude rate, and $\dot{\theta}_d(30) = 0$ rad/sec. Substituting the desired values in (2.3-3) gives

$$\begin{aligned} J = {} & k_h[h(30)]^2 + k_{\dot{h}}[\dot{h}(30) + 15]^2 + k_\theta[\theta(30) - 0.0873]^2 \\ & + \int_0^{30} \{q_h(\tau)[h(\tau) - 450 + 15\tau]^2 + q_{\dot{h}}(\tau)[\dot{h}(\tau) + 15]^2 \\ & + q_\theta(\tau)[\theta(\tau) - 0.0873]^2 + q_{\dot{\theta}}(\tau)[\dot{\theta}(\tau)]^2 \\ & + r_{\delta_e}(\tau)[\delta_e(\tau)]^2\}\, d\tau, \end{aligned} \tag{2.3-4}$$

where θ is in radians, $\dot{\theta}$ in radians per second, h in feet, and $\dot{h}$ in feet per second. In matrix form

$$\begin{aligned} J = {} & [\mathbf{x}(30) - \mathbf{r}(30)]^T\mathbf{H}[\mathbf{x}(30) - \mathbf{r}(30)] \\ & + \int_0^{30} \{[\mathbf{x}(\tau) - \mathbf{r}(\tau)]^T\mathbf{Q}(\tau)[\mathbf{x}(\tau) - \mathbf{r}(\tau)] + r_{\delta_e}(\tau)u^2(\tau)\}\, d\tau, \end{aligned} \tag{2.3-5}$$

where $\mathbf{x}(t)$ is the state at time t, $\mathbf{r}(t)$ is the desired or nominal value of the state at time t, $u(t)$ is the control, r_{δ_e} is a positive function of time,

$$\mathbf{H} \triangleq \begin{bmatrix} k_h & 0 & 0 & 0 \\ 0 & k_{\dot{h}} & 0 & 0 \\ 0 & 0 & k_\theta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\mathbf{Q}(t) \triangleq \begin{bmatrix} q_h(t) & 0 & 0 & 0 \\ 0 & q_{\dot{h}}(t) & 0 & 0 \\ 0 & 0 & q_\theta(t) & 0 \\ 0 & 0 & 0 & q_{\dot{\theta}}(t) \end{bmatrix}.$$

The designer must select functional relationships for q_h, $q_{\dot{h}}$, q_θ, $q_{\dot{\theta}}$, and r_{δ_e}, and numerical values for k_h, $k_{\dot{h}}$, and k_θ. In this example the deviation from the desired trajectory is to be minimized; therefore, the q's and k's would assume only nonnegative values, and r_{δ_e} would be positive for all $t \in [0, 30]$. This performance measure allows sufficient flexibility to satisfy system requirements, and also leads to an optimal control law that is relatively easy to implement. Reference [M-1] discusses implementation in more detail and also shows trajectories that illustrate the effects of varying weighting parameters in a performance measure.

REFERENCES

M-1 Merriam, C. W., III, and F. J. Ellert, "Synthesis of Feedback Controls Using Optimization Theory—An Example," *IEEE Trans. Automatic Control* (1963), 89–103.

S-2 Schultz, D. G., and J. L. Melsa, *State Functions and Linear Control Systems.* New York: McGraw-Hill, Inc., 1968.

PROBLEMS

2-1. Refer to the chemical mixing process of Problem 1-6. The amount of dye in tank 2, $v_2(t)$, is to be maintained as closely as possible to M ft³ during a one-day interval.

(a) What would you suggest as a performance measure to be minimized?

(b) Determine a set of physically realistic state and control constraints.

2-2. Repeat Problem 2-1 if the objective is to maximize the amount of dye in tank 2 at the end of one day. It is specified that the total volume of dye that enters tank 1 in the one-day period cannot be more than N ft³.

2-3. An unmanned roving vehicle has been proposed as part of the Mariner Mars exploration series of space missions. The roving vehicle is designed to navigate on the Martian surface and transmit television pictures and other scientific data to earth. Suppose that the rover is to be driven by a dc motor supplied from rechargeable storage batteries; a simplified model is shown in Fig. 2-P3.

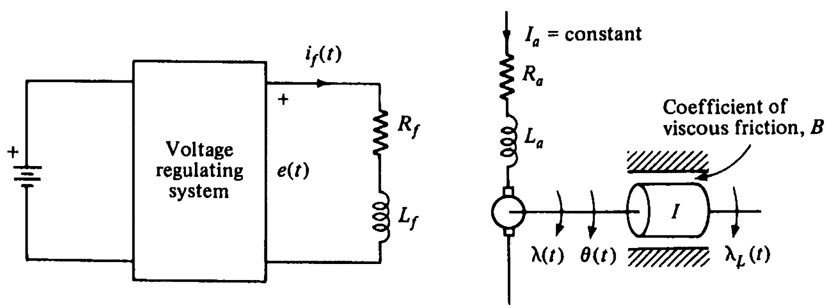

Figure 2-P3

The output of the voltage regulating system is the control signal $e(t)$. The developed torque is $\lambda(t) = K_t i_f(t)$, where K_t is a known constant; $\lambda_L(t)$ is the load torque caused by hills on the Martian surface. The vehicle's speed is to deviate as little as possible from 5 mph without requiring excessive energy output from the voltage regulating system (to prolong the life of the batteries). Let $i_f(t)$ and $\dot{\theta}(t)$ be state variables.

(a) Write state equations for the motor-load combination.
(b) Determine a physically reasonable set of state and control constraints.
(c) Suggest a performance measure if:
  (i) $L_f = 0$.
  (ii) $L_f \neq 0$.

2-4. Refer to the simplified spacecraft model used in Example 2.2-1. Suppose that the objective is to change the spacecraft attitude from an arbitrary initial value to an angle of $+15° \pm 0.1°$ with respect to the reference axis shown in Fig. 2-3. This maneuver is to be accomplished in 30 sec with minimum fuel expenditure.

(a) Determine the state and control constraints.
(b) Suggest an appropriate performance measure.

2-5. Repeat Problem 2-4 if the maneuver is to be accomplished in minimum time.

2-6. Figure 2-P6 shows a rocket that is to be approximated by a particle of instantaneous mass $m(t)$. The instantaneous velocity is $v(t)$, $T(t)$ is the thrust, and

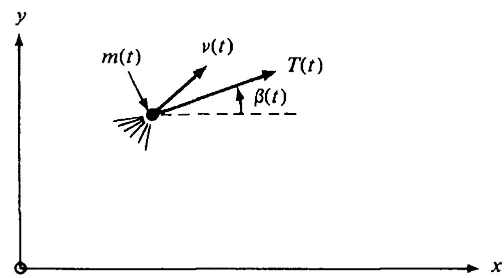

Figure 2-P6

$\beta(t)$ is the thrust angle. If we assume no aerodynamic or gravitational forces, and if we select $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$, $x_3 \triangleq y$, $x_4 \triangleq \dot{y}$, $x_5 \triangleq m$, $u_1 \triangleq T$, $u_2 \triangleq \beta$, the state equations are

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{[u_1(t) \cos u_2(t)]}{x_5(t)}$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{[u_1(t) \sin u_2(t)]}{x_5(t)}$$

$$\dot{x}_5(t) = -\frac{1}{c} u_1(t),$$

where c is a constant of proportionality. The rocket starts from rest at the point $x = 0$, $y = 0$.

(a) Determine a set of physically reasonable state and control constraints.

(b) Suggest a performance measure, and any additional constraints imposed, if the objective is to make $y(t_f) = 3$ mi and maximize $x(t_f)$; t_f is specified.

(c) Suggest a performance measure, and any additional constraints imposed, if it is desired to reach the point $x = 500$ mi, $y = 3$ mi in 2.5 min with maximum possible vehicle mass.