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Procrustean Statistical Inference of Deformations

Research Article

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Abstract:
A two step method has been devised for the statistical inference of deformation changes. In the first step of this method and based on Procrustes analysis of deformation tensors, the significance of the change in a time or space series of deformation tensors is statistically analyzed. In the second step significant change(s) in deformations are localized. In other words, they are assigned to certain parameters of deformation tensor. This is done using the Global Model Test. Because of the key role of Procrustes analysis in the proposed method for the inference of deformation changes, it has been given the name of Procrustean Statistical Inference of Deformations. The method has been implemented to synthetic and real deformations. The 3D-deformation tensors of a regional GPS network in the Kenai Peninsula, for analyzing the spatial variation of deformation tensors or the change of deformation within the study area, and a local GPS network in France, for analyzing the temporal variation of deformation tensors or the change of deformation in time at every point of the network in the study area have been used for illustrating the practical application of the proposed method.

Keywords:
Deformation Changes • Procrustes Analysis • Statistical Inference • Robust Estimation

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1. Introduction

From mathematical point of view the problem of analyzing the change in deformations is equivalent to analyzing a time (the analysis of the change in deformations in time) or a space (the analysis of the change in deformations in space) series of deformation tensors. Today, using continuous GPS measurements, deformations of a body can be analyzed in fine resolutions of time. Since even in tectonically active areas the signal to noise ratio for the change of deformations in fine resolutions of time is expected to be low, a sophisticated mathematical technique is required for handling the problem. Mathematical statistics has become a common mathematical tool for this purpose. The development and implementation of statistical methods have been systematically restricted to the analysis of displacement fields (for example: Caspary 1987; Caspary et al., 1999).

Cai (2004) did the first comprehensive study on the statistical inference of the symmetric 2D- and 3D- deformation tensor. His work was based on the statistical inference of the eigenspace components of a random deformation tensor. For this purpose, using the eigenspace synthesis and eigenspace analysis of the symmetric deformation tensor in two- and three-dimensions, Cai formulated the deformation tensor elements as a set of nonlinear functions of the eigenspace components. Assuming that the strain tensor elements are normally distributed and that a time series of deformation tensor is available (either through their direct measurement or through their estimation from other observations) the eigenspace synthesis approach can give a least-squares estimate of the eigenspace elements and their variance-covariance

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information. He has then proposed the test statistics such as the Hotteling's T^2 and the likelihood ratio statistics (e.g. Papoulis and Pillai, 2002) as statistical apparatuses for the inference of estimated eigenspace elements (Cai, 2004).

The eigenspace elements of the deformation tensor are the standard parameters for the interpretation and representation of the accumulation of strain and stress in an area. Nevertheless, a method that can help us analyze deformation changes in a further detail is obviously more desirable. In case of significant change in deformations, such a method would enable us to assign the detected variations to the normal and/or shear strains. This goal is not attainable through the statistical inference of the eigenspace elements. This is because a significant change in an eigenspace element of the deformation tensor is the cumulative result of changes in various parameters of deformation.

A new two-step method has been forwarded that can fulfill the abovementioned requirements. The method can identify significant changes in deformations between the stations of a network (change of deformations in space) and similar stations in a time series of deformation tensor (change of deformations in time). Instead of the standard multivariate test statistics such as the Hotteling's T^2 test, the method is based on (1) the Procrustes analysis (Mosier, 1939; Green, 1952; Cliff, 1966; Schönemann, 1966; Schönemann and Carroll, 1970; Gower, 1975; Lissitz and Schönemann, 1976; Ten Berg, 1977; Goodall, 1991; Dryden and Mardia, 2002) of the deformation tensors and (2) global test of the mathematical model in Procrustes analysis, hence here is given the name: Procrustean Statistical Inference of Deformation. The basic assumptions in procrustean statistical inference are that the strain components are all normally distributed and no gross error is present in the deformation tensors to be analyzed. The two-steps involved in this method should therefore not be confused with the two steps involved in any hypothesis test.

Procrustes analysis has been already used in Geodesy and Photogrammetry for the direct solution of different transformation problems (ex. Crosilla, 1999; Crosilla, 2004; Crosilla and Beinat, 2007). This paper contributes in the application of this method for size and shape comparison of deformation tensors.

To introduce the method, different solutions to the least-squares problem of Procrustes analysis are briefly introduced. It shall be shown that available methods for inculcating the stochastic properties of observations in the solution of weighted Procrustes problem are not appropriate for this specific application of Procrustes analysis. For this reason, the problem will be formulated and solved using the standard least-squares algorithms for solving nonlinear constrained minimization problems. The method has been implemented to synthetic deformations as well as the 3D-deformation tensors of two test areas: the regional GPS network of the Kenai Peninsula and a local GPS network in France.

2. Procrustes Analysis

Procrustes analysis is a mathematical technique for superimposing one or more configurations (shapes) onto another. This is done by the transformation of desired configuration(s) onto the target one under the choice of a rotation, a translation and a central dilation. In statistical shape analysis, a configuration or shape is known as a set of landmark coordinates in an arbitrary coordinate system. A configuration or shape refers to a realization of an object using a discrete set of points termed as landmarks and are normally selected using certain criteria (Dryden and Mardia, 2002). In the geometric approach to the analysis of deformation, a geodetic network is the shape or configuration of interest and the network station coordinates are the corresponding landmark coordinates. Since the characteristic tensor of deformation quadratic in three and two-dimensions:

$$f(dX, dY, dZ) = \begin{bmatrix} dX & dY & dZ \end{bmatrix} \begin{bmatrix} e_{XX} & \frac{1}{2}e_{XY} & \frac{1}{2}e_{XZ} \\ \frac{1}{2}e_{YX} & e_{YY} & \frac{1}{2}e_{YZ} \\ \frac{1}{2}e_{ZX} & \frac{1}{2}e_{ZY} & e_{ZZ} \end{bmatrix} \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} \quad (1)$$

$$f(dX, dY) = \begin{bmatrix} dX & dY \end{bmatrix} \begin{bmatrix} e_{XX} & \frac{1}{2}e_{XY} \\ \frac{1}{2}e_{YX} & e_{YY} \end{bmatrix} \begin{bmatrix} dX \\ dY \end{bmatrix} \quad (2)$$

governs the geometry of deformation, it is also one possible representation for the shapes or configurations of these forms. Depending on the sign of the corresponding eigenvalues of deformation tensor, the strain quadratic (1) can geometrically represent an ellipsoid, a hyperboloid of one sheet or a hyperboloid of two sheets. Similarly, the characteristic tensor of the strain quadratic (2) represents an ellipse or a parabola depending on the sign of its eigenvalues. Therefore the application of Procrustes analysis to the analysis of deformation is justified.

Depending on the number of involved configurations, the number of involved parameters in the transformation and the incorporation of observational errors in the formulation of the problem, Procrustes problem is termed, formulated and solved differently. When one configuration is transformed onto another by an orthogonal transformation in such a way that the sum of the squares of residuals is minimal, the Procrustes problem is termed as Orthogonal Procrustes Analysis (OPA) (Mosier, 1939; Green, 1952; Cliff, 1966; Schönemann, 1966). Schönemann (1966) proposed the general solution to the least-squares problem of orthogonal Procrustes analysis. The problem and its solution can be mathematically formulated as in the following theorem:

Theorem 1: If $\mathbf{A} \in R^{n\times m}$ and $\mathbf{B} \in R^{n\times m}$ $(m \geq n)$ are two arbitrary real matrices of the same dimension, the necessary and sufficient condition to have a unique orthogonal transformation matrix $\mathbf{T}$ to satisfy the least-squares problem:

$$\mathbf{B} + \mathbf{E} = \mathbf{AT} \quad (3a)$$

$$TT^T = T^TT = I \quad (3b)$$

$$\text{trace}(E^TE) = \min \quad (3c)$$

is that the matrix SS^T, where $S = A^TB$, has not multiple zero eigenvalues and all the singular values are nonnegative. The unique solution T is then given by the equation:

$$T = WV^T \quad (4a)$$

where W, V and $D_s$are the eigenvectors and diagonal matrix of eigenvalues in orthogonal decomposition of matrices $SS^T$and S^TSrespectively, that is (Schönemann, 1966):

$$SS^T = WD_sW^T \quad (4b)$$

$$S^TS = VD_sV^T \quad (4c)$$

In functional relations (3a) and (3b), E and $I_{n\times m}$ are the residual and identity matrices respectively. In the particular application of Procrustes analysis for the statistical inference of deformation parameters matrices A and B are the full column rank strain tensors that are directly observed or obtained from the 2D- or 3D-analysis of deformation. Matrices $SS^T$and$S^TS$are therefore, necessarily positive definite. In other words, the uniqueness of the Procrustes solution for the orthogonal transformation T is already assured and needs not to be verified.

Extended Orthogonal Procrustes analysis (EOP) is an extension of OPA in which the transformation involves a rotation T, translation vector γ and a central dilation c for matching two configurations (Schönemann and Carroll, 1970). The problem and its solution are mathematically formulated in the following theorem:

Theorem 2: If $A \in R^{n\times m}$ and $B \in R^{n\times m}$ $(m \geq n)$ are two arbitrary real matrices of the same dimension, the necessary and sufficient condition to have a unique orthogonal transformation T to satisfy the least-squares problem:

$$B + E = cAT + J\gamma^T \quad (5a)$$

$$TT^T = T^TT = I \quad (5b)$$

$$\text{trace}(E^TE) = \min \quad (5c)$$

is that the matrix SS^T, where $S = A^T\left(I - \frac{1}{m}JJ^T\right)B$, has non-negative eigenvalues. The unique solution for the transformation parameters is then given by:

$$T = VW^T \quad (6a)$$

$$c = \text{trace}\left[T^TA^T(I - \frac{1}{m}JJ^T)B\right] \Big/ \text{trace}\left[A^T(I - \frac{1}{m}JJ^T)A\right] \quad (6b)$$

$$\gamma = \frac{1}{m}(B - cAT)^TJ \quad (6c)$$

$$E = \left(I - \frac{1}{m}JJ^T\right)(B - cAT) \quad (6d)$$

Scalar c is the scale factor of transformation (dilation parameter), $\gamma_{n\times 1}$ is its translation vector, T is the corresponding rotation matrix of the transformation, $J = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}^T_{1\times m}$ and $m = J^TJ$. W and V are the latent vectors in orthogonal decomposition of matrices $SS^T$and S^TSrespectively, that is (Schönemann and Carroll, 1970):

$$SS^T = WD_sW^T \quad (6e)$$

$$S^TS = VD_sV^T \quad (6f)$$

Corollary 1: The residual tensor in the least-squares problem of Procrustes analysis is independent of the translation between the involved configurations. Since the translation vector γ does not contribute in the matrix of residuals (6d), the residual tensor in the problem of EOP is independent of the translation between the involved configurations. Since OPA is a special case of EOP, the residual tensor in orthogonal Procrustes analysis also is not sensitive to the translation between the involved configurations. This can also be verified by putting $c = 1$ and γ=0 in theorem 2 and following similar derivation steps.

Corollary 1 ensures that in general, Procrustes analysis can also be applied for the analysis of the shape change in space. This is because Procrustean residuals are not sensitive to the location of configurations.

Further generalization to the Procrustes problem involved the development of mathematical models that were necessary for transforming more than one configuration to the target shape by a set of transformations. This problem is known in the literature as Generalized Procrustes Analysis (GPA). Kristof and Wingersky (1971) solved this problem for a set of transformations that include different orthogonal rotations. They also proved that the solution of this problem is the geometrical centroid of involved configurations. They couldn't prove the uniqueness of their proposed solution. Later, generalized Procrustes problem was set up and solved for transformations that include different scaling, translations and rotations (Gower, 1975 and Ten Berg, 1977; Goodall, 1991).

The first attempt to include the stochastic model into the solution of Procrustes problem is due to Lissitz and Schönemann (Lissitz and Schönemann, 1976). They proved that the inclusion of the stochastic model through the least-squares minimization of weighted errors of the form: trace $\left(E^TD_1E\right) = \min$ and of the form: trace $\left(ED_2E^T\right) = \min$are equivalent to weighting rows and columns of the residual matrix respectively and minimizing the sum of the weighted residuals. Here, $D_1$and D_2 are positive definite matrices that reflect information about the relative magnitude of variances. In addition to the corresponding solutions to these weighting approaches, they also proposed the solution of the weighted Procrustes problem in which different matrices are used for weighting rows and columns of the residual matrix simultaneously.

3. Procrustean Statistical Inference of Deformations

In the problem of the analysis of the change in deformations of the Earth's crust, change in shape and change in size are equally of interest. Therefore, among possible formulations of the Procrustes problem, OPA is the most appropriate one. This is because the functional model in orthogonal Procrustes analysis does not involve the size and shape information of deformation tensors. Therefore, any possible changes between deformation tensors in terms of size and shape changes will be reflected as the misfit of the functional model to reality. Similar to any least-squares problem, the adequacy of the functional model can be assessed by screening the residuals. For this purpose, the null hypothesis "The model is correct and complete" is firstly analyzed. This hypothesis test is usually known in literature as Global Test of the Model (Caspary, 1987). The inadequacy of functional model in orthogonal Procrustes analysis of deformations, that is its rejection by the global model test, indicates significant change(s) in the form of size and/or shape in deformations. Further inspection of residuals (see Equation 3a) can help us in localizing the variation(s) that was statistically asserted in previous step. For this purpose, deformation changes are treated as outliers. By definition an outlier is a residual, which according to some testing rule is in contradiction to the assumption (Caspary 1987). Therefore, using a test strategy and a clear statistical concept, outliers can be theoretically localized. According to the classical theory of errors an outlier can refer to a systematic or a gross error. In the theory of least-squares it is normally taken as an indication for the presence of gross errors. This is due to the implicit assumption that the functional model is normally taken to be completely in accord with reality. Since in Procrustean statistical inference of deformations outliers are expected to represent the misfit of the functional model to reality, it has been implicitly assumed that no gross errors are present in the deformation tensors under study.

This method for analyzing the change in deformations is naturally a relative method in the sense that it depends on the selected level of risk, the assumed distribution and the testing procedure. To reduce the sensitivity of the method to possible deviations from statistical concepts, robust estimation has been preferred to traditional outlier detection techniques. For this purpose, a robust method by Wicki (2001) has been used. With regard to the application of robust estimation for data snooping, it is not possible to assign any probability to detected outliers. The robust method by Wicki (2001) has been modified in such a way that the modified estimator can assign a certain probability to the detected outliers.

Since the method of Lissitz and Schönemann doesn't minimize the sum of squares of standardized residuals, it is not an appropriate method for inculcating the stochastic properties of the configurations in Procrustean statistical inference of deformations. The most straightforward solution to this problem is using the standard least-squares algorithm for solving the non-linear mathematical model of the orthogonal Procrustes problem. That is, linearizing the model and minimizing the sum of the squares of weighted residuals. Since the problem of Procrustes analysis is a constrained optimization problem, it can be re-formulated as follows:

$$\mathbf{f}(\mathbf{x}, \mathbf{l}) = \mathbf{0} \tag{7a}$$

$$\mathbf{f}_c(\mathbf{x}) = \mathbf{0} \tag{7b}$$

$$\mathbf{r}^T \mathbf{P} \mathbf{r} \rightarrow \min \tag{7c}$$

$\mathbf{l} = vec\,(\mathrm{A}; B)$ is the vector of observations in which A and B are assumed to represent the deformation tensors to be transformed one onto the other. The positive definite matrix $\mathbf{P}$ is the weight matrix which is normally taken as a diagonal matrix. Vector $\mathbf{r} = vec\,(E)$ is the residual vector and $\mathbf{x}$ is the vector of unknown parameters. In orthogonal Procrustes analysis, this vector includes the elements of rotation matrix T. Equations (7a) and (7b) are the implicit representation of Equation (3a) and (3b) above. In this problem, the non-linearity resides in the constraints. The constraints ensure the orthogonality of the rotation tensor. The initial values for the rotation tensor components are computed from the direct solution to the orthogonal Procrustes problem, which was given in the theorem 1 above. Linearizing the nonlinear model (7) above leads to the following system of simultaneous equations:

$$\begin{matrix} \mathbf{A}_1\delta + \mathbf{B}_1\mathbf{r} + \mathbf{w} = \mathbf{0} \\ \mathbf{D}\delta + \mathbf{w}_c = \mathbf{0} \end{matrix} \tag{8a}$$

$$\mathbf{A}_1 = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0, \mathbf{l}=\mathbf{l}_0} \tag{8b}$$

$$\mathbf{B}_1 = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{l}}\right|_{\mathbf{l}=\mathbf{l}_0, \mathbf{x}=\mathbf{x}_0} \tag{8c}$$

$$\mathbf{D} = \left.\frac{\partial \mathbf{f}_c}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0, \mathbf{l}=\mathbf{l}_0} \tag{8d}$$

$$\mathbf{w} = \mathbf{f}(\mathbf{x}^0, \mathbf{l}^0) \tag{8e}$$

$$\mathbf{w}_c = \mathbf{f}_c(\mathbf{x}^0) \tag{8f}$$

In these equations $\mathbf{w}$ and $\mathbf{w}_c$ are the misclosure vectors for observation equations and constraints. The vector of observations $\mathbf{l}$ is given by:

$$\mathbf{l} = vec\,(\mathrm{A}; B) \tag{8g}$$

where A and B are assumed to represent the deformation tensors to be transformed one onto the other respectively. By definition, the operator $vec(.)$ in Equation (8g) changes an m-by-n matrix A to a column vector of length $m \times n$by stacking it columns to each other.

The solution of the linear constrained implicit model (8a) and (8b) is given by (Vanicek and Krakiwsky, 1986):

$$\hat{\delta} = \mathbf{x} - \mathbf{x}_0 = -\mathbf{N}^{-1}\mathbf{u} - \mathbf{N}^{-1}\mathbf{D}^T\left(\mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T\right)^{-1}\left(\mathbf{w}_c + \mathbf{D}\delta^{(1)}\right) \tag{9a}$$

$$\mathbf{N} = \mathbf{A}_1^T\left(\mathbf{B}_1\mathbf{Q}_l\mathbf{B}_1^T\right)^{-1}\mathbf{A}_1 \tag{9b}$$

$$\mathbf{u} = \mathbf{A}_1^T\left(\mathbf{B}_1\mathbf{Q}_l\mathbf{B}_1^T\right)^{-1}\mathbf{w} \tag{9c}$$

$$\delta^{(1)} = -\mathbf{N}^{-1}\mathbf{u} \tag{9d}$$

$$\mathbf{C}_{\hat{\delta}} = \mathbf{N}^{-1} - \mathbf{N}^{-1}\mathbf{D}^T(\mathbf{D}\mathbf{N}^{-1}\mathbf{D}^T)^{-1}\mathbf{D}\mathbf{N}^{-1} \tag{9e}$$

$$\hat{\mathbf{r}} = -\mathbf{Q}_l\mathbf{B}_1^T\left(\mathbf{B}_1\mathbf{Q}_l\mathbf{B}_1^T\right)^{-1}(\mathbf{A}_1\hat{\delta} + \mathbf{w}) \tag{9f}$$

$$\mathbf{Q}_{\hat{\mathbf{r}}} = \mathbf{Q}_l\mathbf{B}_1^T(\mathbf{B}_1\mathbf{Q}_l\mathbf{B}_1^T)^{-1}\mathbf{B}_1\mathbf{Q}_l \tag{9g}$$

In the following parts of this paper the theoretical background of the two steps mentioned above are reviewed. For this purpose the global model test is firstly re-introduced. The stochastic concepts for the Procrustean statistical inference of deformations are also established in this section. The robust method by Wicki (2001) together with implemented modifications to this estimator is then introduced. Finally, the method is applied to simulated and real deformations.

3.1. Global Model Test

Global model test is based on the following theorem from mathematical statistics:

Theorem 3: The sum of the squares of n independent random variables $z_i$that are normally distributed with distribution parameters $\mu = 0$ and$\sigma = 1$, i.e.$x = z_1^2 + z_2^2 + \ldots + z_n^2$, has the probability density function (Papoulis and Pillai, 2002):

$$f_x(x) = \begin{cases} \frac{x^{n/2-1}}{2^n\Gamma(n/2)}e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{10a}$$

where $\Gamma(x)$represents the gamma function defined as:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx \tag{10b}$$

A random variable with the probability density function (10) is said to have chi-square distribution with n degrees of freedom and is normally denoted by$\chi^2(n)$. Assuming that observations are normally distributed, that is $\mathbf{l} \sim N\left(\bar{\mathbf{l}}, \mathbf{\Sigma}\right)$ where $\bar{\mathbf{l}} = E(\mathbf{l})$ and $\mathbf{\Sigma}$is the variance-covariance matrix of observations, it can be easily seen that residuals $\mathbf{r} = \mathbf{l} - \bar{\mathbf{l}}$ are also normally distributed with distribution parameters: $\mathbf{r} \sim N\left(\mathbf{0}, \sigma_0^2\mathbf{Q}_r\right)$ where σ_0^2 and $\mathbf{Q}_r$are the a-priori variance of unit weight and the cofactor matrix of residuals. Therefore, according to theorem 3, when no outliers are present, the loss function in the least-squares estimation characterizes a random variable with stochastic properties:$\mathbf{r}^T\mathbf{P}\mathbf{r} \sim \sigma_0^2\chi^2(df)$. The global model test questions these assumptions by comparing the a-posteriori variance factor $s_0^2$with σ_0^2under the null hypothesis that "The model is correct; the distributional assumptions meet the reality". Following theorem 3, the test statistics to be used is given by (Koch, 1999):

$$T = \mathbf{r}^T\mathbf{P}\mathbf{r}/\sigma_0^2 = df\frac{s_0^2}{\sigma_0^2} \sim \chi^2(df) \tag{11}$$

In statistical inference of the change in deformations, the rejection of global model test is an indication for the inadequacy of functional model in the orthogonal Procrustes analysis of deformations. This may be due to significant change(s) in the form of size and/or shape in deformations as well as inadequacy of stochastic components in the functional model (distributional assumptions), assuming that the a priori precision of observations is reliably known. In the later case, some of strain components may not be perfectly normally distributed. If $0 \leq \varepsilon < 1$ is a known parameter and H is an unknown contaminating distribution, for such a type of deformation parameters the probability density function can be written as (Huber, 1964):

$$F = (1-\varepsilon)N + \varepsilon H \tag{12}$$

3.2. M-Estimator

Robust statistics is the appropriate mathematical tool when the stochastic assumptions are only approximations to reality (Huber, 1964; Hampel et al., 1981). Among different kinds of robust estimators, maximum likelihood estimators (also known as M-estimators) are more in commensurate with the least-squares estimator which in principle is also a maximum likelihood estimator. For example: Huber (1964) proposed the following M-estimator in which, in contrary to the least-squares estimator, residuals (r_i) are bounded by a constant parameter $(\bar{c})$:

$$\Psi_{\bar{c}}(r_i) = \begin{cases} r_i & \text{for } |r_i| < \bar{c} \\ sign(r_i)\,\bar{c} & \text{for } |r_i| \geq \bar{c} \end{cases} \tag{13}$$

When $\bar{c} \to \infty$, the Huber estimator gives identical results to the least-squares estimator. Based on this estimator, Wicki (2001) and Xu (1989, 1993) proposed a maximum likelihood estimator for the adjustment of geodetic networks. In contrary to Huber's estimator in which residuals are treated independently of their quality, the new estimator takes the quality of residuals into account. This was done by including the standard deviations of residuals (σ_{r_i}) in the formulation of this M-estimator:

$$\Psi_c(w_i) = \begin{cases} w_i = \frac{r_i}{\sigma_{r_i}} & \text{for } |w_i| < c \\ sign(w_i)\,c & \text{for } |w_i| \geq c \end{cases} \tag{14a}$$

From experience, a value in the range $2.5 \leq c \leq 4$has been proposed for the constant parameter $c$in Equation (14a) (Wicki 2001).

The application of robust estimation techniques for estimating the unknown parameters of a functional model with stochastic components can not only provide a solution which is less sensitive to possible deviations from stochastic assumptions, but also can inform us on the outliers that may exist in the acquired measurements. This is because outliers are one of the causes for the deviation of stochastic assumptions from reality. In this respect, one should be cautious that it is not possible to assign any probability to the outliers to be identified by robust estimation techniques.

According to Baarda's method of data snooping (Baarda, 1968), for a certain error level of type I ($\alpha = \alpha_0$) and a certain error level of type II ($\beta = \beta_0$) the outlier on an observation l_i can be identified if it reaches the value:

$$\Delta_i = \sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}} \tag{15}$$

In this equation, $\lambda_0 = \lambda(\alpha_0, \beta_0, df)$ is non-centrality parameter in non-central chi-square distribution with dfdegree of freedom and $z_i = [\mathbf{Q}_r\mathbf{P}]_{ii}$is the local redundancy number (Baarda, 1968). For the conventional complementary hypothesis H_a: "One of the measurements is an outlier": $df = 1$ and therefore, the non-centrality parameter is only a function of $\alpha_0$andβ_0. Figure 1 illustrates the ratio $\sqrt{\lambda_0}/\sqrt{z_i}$ for α_0 =0.001, 0.005, 0.01, 0.05, 0.1, 1.0, 2.5 and 5 percent against β_0 =10, 20 and 30 percent when the conventional alternative hypothesis is taken into account. It is seen in this figure that for a risk level of $\alpha_0 \geq 1$ percent and for β_0 =10, 20 and 30 percent, when local redundancy numbers are larger or as large as 0.9 the range of this ratio coincides with the proposed range for the constant parameter $c$in Equation (14a). For a significance level of 95% ($\alpha_0 = 0.05$) the range of redundancy numbers for which a similar property is seen enlarges to$z_i \geq 0.6$. Implementing this ratio instead of the empirical bounding parameter $c$in Equation (14a) not only strengthens the theoretical basis of the BIBER-estimator but also assigns a certain probability to the outliers that are to be detected by using this estimator. Therefore, the following modification is proposed here to the BIBER-estimator:

$$\Psi_{\sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}}}(w_i) = \begin{cases} w_i = \frac{r_i}{\sigma_{r_i}} & \text{for } |w_i| < \sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}} \\ sign(w_i)\sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}} & \text{for } |w_i| \geq \sqrt{\lambda_0}\frac{\sigma_{l_i}}{\sqrt{z_i}} \end{cases} \tag{16}$$

The M-estimator above is called here the modified BIBER-estimator and is proposed for a detailed analysis of the change in deformations.

4. Applications

4.1. Synthetic Deformations

To test the forwarded method of this study, three different sets of synthetic deformations has been produced and analyzed. Procrustes analysis is firstly done on two identical synthetic deformations (Case I in Table1). This special case is a good measure for checking the computer codes that has been developed for practical applications. Deformation tensors that have been taken into account in the second case (Case II in Table1) are the characteristic tensors of the deformation quadratics that represent completely different geometric shapes (ellipsoid and hyperboloid). In the last case, once significant changes are made on dilatational strains of a synthetic deformation tensor (Case III-a in Table1). Then, only the shear components are subjected to significant changes (Case III-b in Table1). Finally, deformation changes have been implemented on both shear and normal strains (Case III-c in Table1). Simulated deformation changes fulfill the criterion of Equation (15). The statistical significance of simulated deformation changes has been analyzed. For this purpose a risk level of $\alpha = 1\%$ and a power of $\beta = 10\%$ has been selected. Table1 provides the obtained numerical results.

4.2. Spatial Variation of Deformation in the Kenai Peninsula

Southern Alaska, including the Aleutian Island chain (extending from city of Fairbanks in the north to the Gulf of Alaska in the south) is one of the world's most active seismic zones. South central StateAlaska was severely affected by the 1964 PWS (Prince William Sound) earthquake. Kanamori (1977) estimated a moment magnitude of $M_w = 9.2$ for this earthquake. Hossainali et al. (2010) analyzed the three dimensional pattern of deformation kinematics for the post-seismic deformation of this area using GPS measurements of three successive campaigns. Estimated velocity fields in this area computed using the Bernese GPS processing software as well as the three-dimensional pattern of crustal deformation obtained for this area suggests spatial variations in the deformations: According to estimated velocity filed the network stations in the western Kenai are moving SSE in contrary to the network stations in the eastern Kenai. This spatial variation in deformation is also confirmed by other studies (Cohen and Freymueller, 1997; Freymueller et al., 2000; Zweck and Freymueller, 2002). Figure 2 illustrates the estimated horizontal velocity vectors and their scaled confidence regions obtained from the analysis of the corresponding GPS measurements together with vertical pattern of deformation in this area obtained from the corresponding deformation tensors in three-dimensions (Hossainali et al., 2010). Table 2 gives the 3D- principal strains, their 95% confidence intervals and computed surface compressions as illustrated in Figure 2(b). The a-priori variance of unit weight is assumed to be one. In other words, the a priori precision of measurements is assumed to be correct. The acceptance of the null hypothesis of the Global

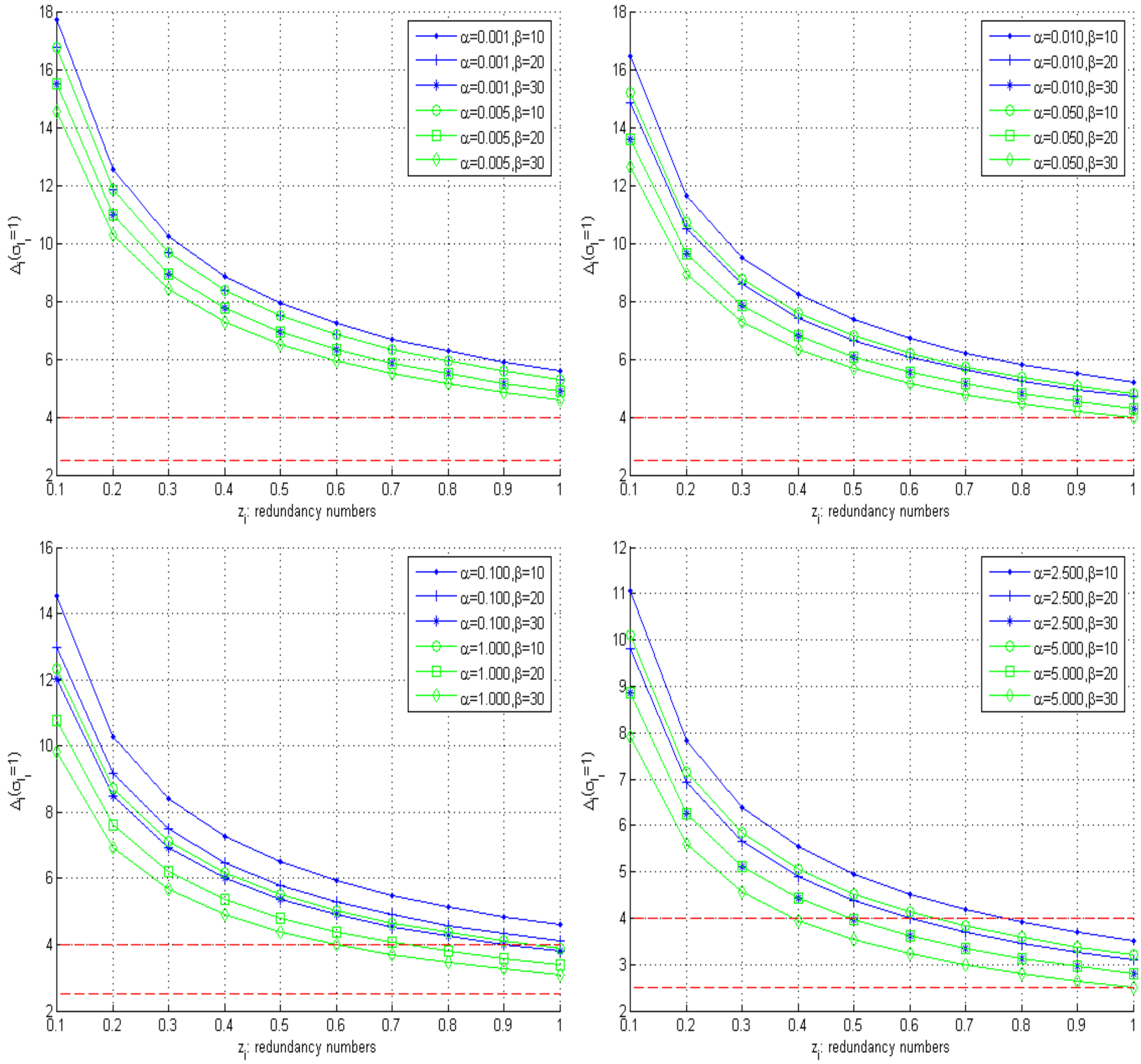

Figure 1. The ratio $\sqrt{\lambda_0}/\sqrt{z_i}$ for α_0 = 0.001, 0.005, 0.01, 0.05, 0.1, 1.0, 2.5 and 5 percent against β_0 =10, 20 and 30 when the conventional null hypothesis is taken into account.

Model Test in processing the GPS data of this area confirms this assumption.

The method of this paper has been implemented to the three-dimensional deformation tensors of this network. The analysis was done at the risk level of $\alpha = 5\%$ and the test power of $\beta = 30\%$. According to this analysis, no significant change between the deformations of two stations GRAV and HOMA can be statistically asserted in deformation tensors of these two stations in the Kenai Peninsula. The change in deformations of the other stations is stochastically significant for the GPS network in this area.

4.3. Temporal variation of Deformation in a Local Network in France

A local GPS network in France has been used to analyze the temporal variation of deformation. This GPS network has been measured in four successive campaigns. The GPS results of this network has been provided to this study for estimating the deformation tensors at the position of the GPS stations in this area using a method similar to one applied to the Kenai Peninsula (Hossainali, 2006). The configuration of the network stations as well as estimated velocity vectors are shown in Figure 3 below. The comparison of inflated confidence regions to computed velocity vectors suggests

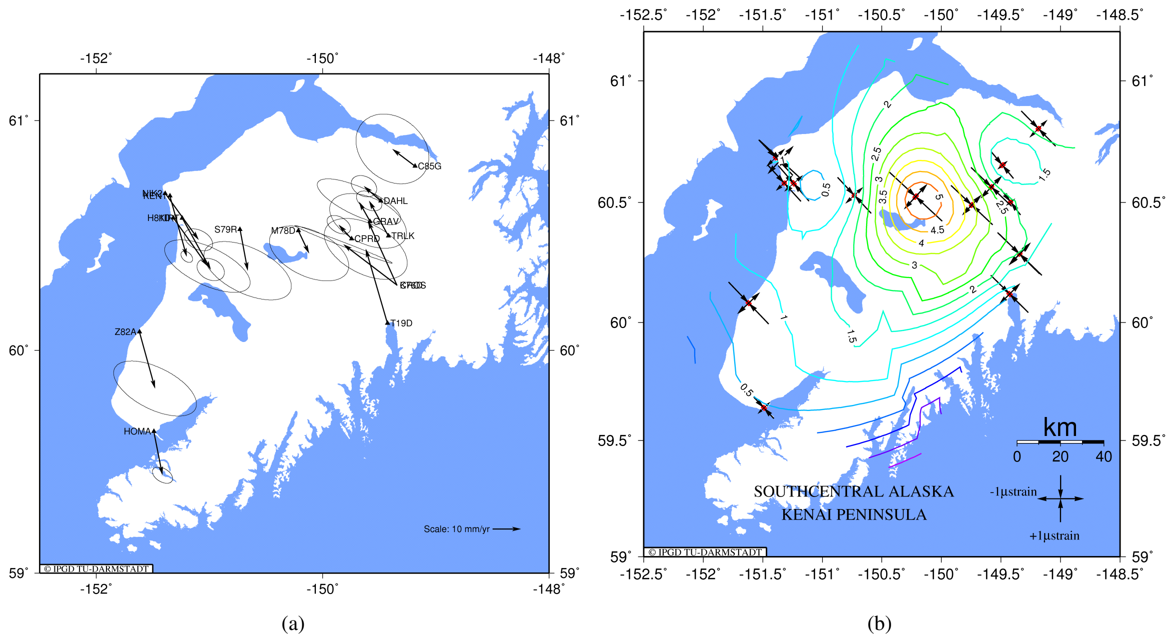

Figure 2. Estimated horizontal velocity vectors and their scale confidence regions (a) and vertical pattern of crustal deformation obtained from estimated deformation tensors in three-dimensions (b)

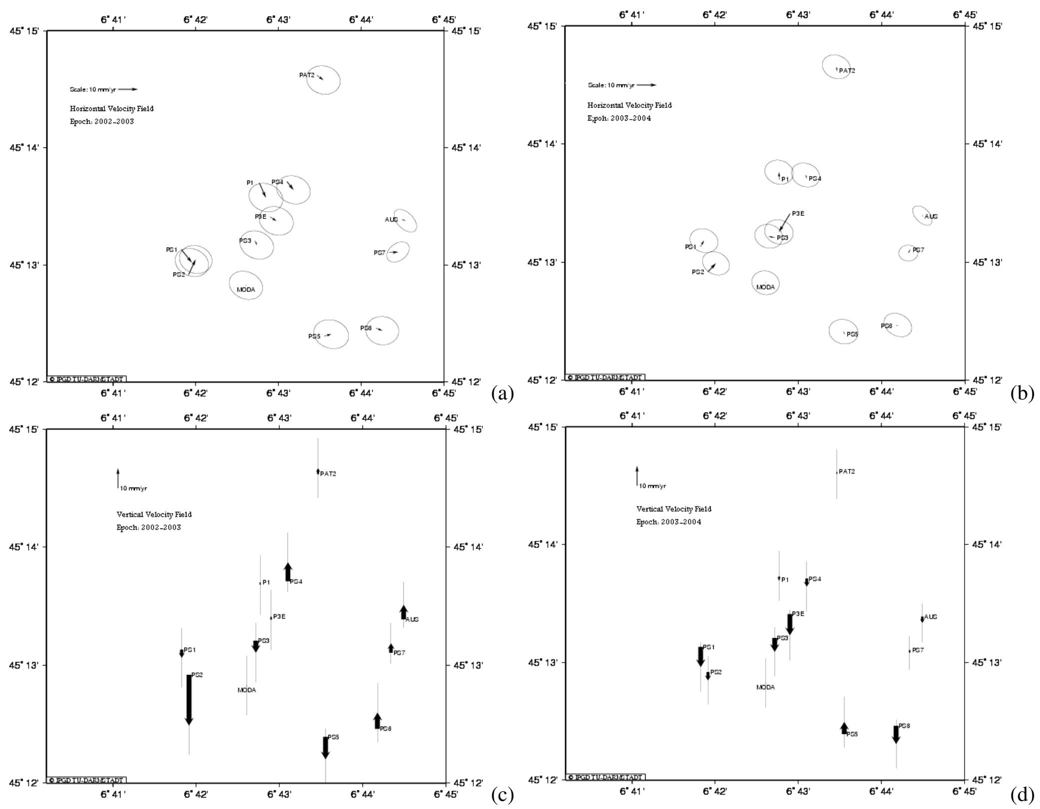

Figure 3. Horizontal (a and b) and vertical (c and d) velocity vectors and their scaled confidence regions for the local network in France.

Table 1. Results of Procrustean statistical inference of simulated deformations.

Case	Kind of deformation	The Procrustean Statistical Inference Results
I	Two Identical Deformation Tensors	The null hypothesis cannot be rejected, no significant change in deformations can be statistically asserted at the probability of 99%
II	Variation in the geometric shapes	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyx, ezx, exy, eyy, ezy, exz, eyz, ezz
III-a	Variation in dilatational parameters	The null hypothesis is rejected, Significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyy, ezz
III-b	Variation in shear parameters	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: eyx, ezx, exy, ezy, exz, eyz, ezz
III-c	Variation in all parameters of strain	The null hypothesis is rejected, significant change in deformations can be statistically asserted at the probability of 99%
		Significant change has occurred on the parameters: exx, eyx, ezx, exy, eyy, ezy, exz, eyz, ezz

Table 2. 3D- principal strains, their 95% confidence intervals and surface compressions as illustrated in Figure 2(b).

Station Code	Station Name	Horizontal Principal Strains ($\mu - strain$)					Δ ($\mu - strain$)	
		e_I	σ_{e_I}	e_{II}	$\sigma_{e_{II}}$	Azimuth	Δ	σ_Δ
3	KEN1	0.815	0.107	-1.249	0.182	132.71	-0.551	0.516
4	C85G	0.555	0.163	-1.086	0.094	135.32	-0.957	0.800
5	CPRD	0.603	0.171	-1.223	0.086	133.34	-1.279	2.250
6	CROS	0.621	0.151	-1.196	0.108	134.66	-0.930	0.878
7	DAHL	0.274	0.167	-0.738	0.091	140.23	-0.660	1.000
8	GRAV	0.666	0.154	-1.198	0.105	133.59	-0.968	0.766
9	H81D	0.483	0.145	-1.053	0.116	137.59	-0.792	0.726
10	HOMA	0.364	0.115	-0.626	0.166	136.55	-0.297	0.122
11	K76D	0.682	0.157	-1.311	0.101	134.09	-1.087	1.042
12	KIRT	0.520	0.109	-1.000	0.178	139.05	-0.439	0.697
13	M78D	0.732	0.179	-1.544	0.081	132.77	-1.829	2.144
14	NIK2	0.540	0.120	-1.048	0.157	136.17	-0.622	0.511
16	S79R	0.513	0.147	-1.049	0.113	137.14	-0.765	2.046
17	T19D	0.643	0.108	-1.124	0.181	134.75	-0.513	0.061
18	TRLK	0.249	0.137	-0.693	0.128	142.46	-0.454	0.033
19	Z82A	0.670	0.113	-1.238	0.170	136.27	-0.641	0.174

that except for station P3E, no other station has had significant movements during the time interval of GPS measurements in this area. In processing the GPS data of this network the a-priori variance of unit weight is also assumed to be one. In other words, the a priori precision of measurements is again assumed to be correct. The acceptance of the null hypothesis of the Global Model Test in processing the GPS data of this area confirms this assumption.

The significance of the temporal variation of deformations in this area has been also analyzed. This analysis approves that at the risk level of $\alpha = 5\%$ and the error level of type II$\beta = 30\%$, except for station P3E, no other deformation changes can be statistically asserted at the locale of the other stations within the involved period of measurements. The detected deformation changes are assigned to vertical parameters of deformation, that is e_{xz}, e_{yz} and e_{zz}.

5. Conclusions

The statistical inference of deformation changes, similar to the other statistical inference techniques, has the following characteristic features:

A) The method is relative in the sense that it depends on the selected level of risk, the assumed distribution and the testing procedure.

B) It is always possible to approve a null hypothesis (assert significant changes in a space or time series of deformation tensors) whereas the null hypothesis is not correct (deformation parameters have not changed). The probability for the commitment of this type of error is equal to β. Therefore, to increase the power of the test, β should be decreased.

Therefore, like any other statistical inference technique, the efficiency of the method should be judged within the framework of the characteristics mentioned above. The proposed method can be considered as a mathematical technique for analyzing time or space series of tensors. This generalization makes the application of the method for the statistical assessment of a time or space series of deformation tensors possible.

References

Baarda W. (1968): A testing method for use in geodetic networks, Netherlands Geodetic Commission, Publications on Geodesy 2.

Cai J. (2004): The statistical Inference of the eigenspace components of a symmetric random deformation tensor, Heft Nr.577, Deutsche Geodätische Kommission, Reihe C. Ph.D.

Cai J., Grafarend E.W., and, Schaffrin B. (2005): Statistical inference of the eigenspace components of a two-dimensional symmetric rank two random tensor, J. of Geodesy, 78: 426-437.

Caspary W. F. (1987): Concepts of Network and Deformation Analysis. Kensington, N.S.W., country-regionAustralia, Monograph 11, School of Surveying, The University of New South Wales.

Caspary W.F., Haen W. (1999): Deformation analysis by statistical methods, Technometrics 32(1): 49-57.

Cliff N. (1966): Orthogonal rotation to congruence, Psychometrika 31: 33-42.

Cohen S.C. and Freymueller J.T. (1997): Deformation on the Kenai Peninsula, Alaska, Joural of Geophys. Research 102: 20,479-20,487.

Crosilla F. (2003): Procrustes Analysis and Geodetic Science, in Geodesy The Challenge of the 3rd Millennium, W. Friedrich, et al., Editors. Springer Verlag. p. 287-292.

Crosilla F. (2004): Procrustes statistics to test for significant OLS similarity transformation parameters, IAG Symposia 127"V Hotime-Marussi Symposium on Mathematical Geodesy", Matera, Italy, 17-21 June 2002, edited by F. Sanso, Springer, 113-119.

Dryden I. L. and Mardia K.V. (2002): Statistical Shape Analysis, John Wiley & Sons.

Goodall C.R. (1991): Procrustes methods in the statistical analysis of shape (with discussion), Journal of the Royal Statistical Society, Series B 53: 285-339.

Gower J.C. (1975): Generalized Procrustes analysis, Psychometrika 40: 33-50.

Green B.F. (1952): The orthogonal approximation of an oblique structure in factor analysis, Psychometrika 17: 429-440.

Grafarend E.W. and Awange J.L. (2003): Nonlinear analysis of the three-dimensional datum transformation, J. of Geodesy, 77: 66-76.

Hampel F.R., Ronchetti E.M., et al. (1981): Robust Statistics, The Approach Based on Influence Functions, Willy Series in Probability and Mathematical Statistics.

Huber P.J. (1964): Robust estimation of a location parameter, Annals of Mathematical Statistics 35: 73-101.

Kristof W. and Wingersky B. (1971): Generalization of the orthogonal Procrustes rotation procedure to more than two matrices, Proceedings of the 79th Annual Convention of the American Psychological Association.

Koch K.R. (1999): Parameter estimation and hypothesis testing in linear models, 2^{nd} edn., Springer, Berlin Heidelberg New York.

Lissitz R.W. and Schönemann P.H. (1976): A solution to the weighted Procrustes problem in which the transformation is in agreement with the loss function, Psychometrika 41: 547-550.

Mosier C.I. (1939): Determining a simple structure when loadings for certain tests are known, Psychometrika 4: 149-162.

Papoulis A. and Pillai S.U. (2002): Probability, Random Variables and Stochastic Processes, Mc Graw Hill Co.

Schönemann P. H. (1966): A generalized solution to the orthogonal Procrustes problem, Psychometrika 31: 1-10.

Schönemann P. H. and Carroll R.M. (1970): Fitting one matrix onto another under choice of central dilation and rigid motion, Psychometrika 35: 245-255.

Ten Berg J.M.F. (1977): Orthogonal Procrustes rotation for two or more matrices, Psychometrika 42: 267-276.

Vaniček P. and Krakiwsky E.J. (1986): Geodesy, The Concepts, Elsevier Science Publishers.

Wicki F. (2001), Robust Estimator for the Adjustment of Geodetic Networks, First International Symposium on Robust Statistics and Fuzzy Techniques in Geodesy and GIS, Swiss Federal Institute of Technology Zurich (ETH), Istitute of Geodesy and Photogrametry.

Xu P.L. (1989), Statistical criteria for robust methods, ITC Jnl 1:37-40.

Xu P.L. (1993): Consequences of constant parameters and confidence intervals for robust estimation. Boll Geod. Sci. Aff. 52: 231-249.

Zweck C. and Freymueller J. (2002): Three-dimensional elastic dislocation modeling of the postseismic response to the 1964 Alaska earthquake, Joural of Geophys. Research 107.