Vibration with Control

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

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Daniel J. Inman
Virginia Tech, USA

John Wiley & Sons, Ltd

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West Sussex PO19 8SQ, England

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Library of Congress Cataloging in Publication Data

Inman, D. J.
Vibration with control / Daniel J. Inman.
p. cm.
Includes bibliographical references and index.
ISBN-13 978-0-470-01051-8 (HB)
ISBN-10 0-470-01051-7 (cloth : alk. paper)
1. Damping (Mechanics). 2. Vibration. I. Title.
TA355.I523 2006
620.3—dc22 2006005568

British Library Cataloguing in Publication Data

A catalogue record for this book is available from the British Library

ISBN-13 978-0-470-01051-8 (HB)
ISBN-10 0-470-01051-7 (HB)

Typeset in 10/12pt Times by Integra Software Services Pvt. Ltd, Pondicherry, India
Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham, Wiltshire
This book is printed on acid-free paper responsibly manufactured from sustainable forestry in which at least two trees are planted for each one used for paper production.

Contents

Preface

Advance-level vibration topics are presented here, including lumped-mass and distributed-mass systems in the context of the appropriate mathematics, along with topics from control that are useful in vibration analysis and design. This text is intended for use in a second course in vibration, or in a combined course in vibration and control. This book is also intended as a reference for the field of structural control and could be used as a text in structural control. Control topics are introduced at beginner level, with no knowledge of controls needed to read the book.

The heart of this manuscript was first developed in the early 1980s and published in 1989 under the title *Vibration with Control, Measurement and Stability*. That book went out of print in 1994. However, the text remained in use at several universities, and all used copies seem to have disappeared from online sources in about 1998. Since then I have had yearly requests for copying rights. Hence, at the suggestions of colleagues, I have revised the older book to produce this text. The manuscript is currently being used in a graduate course at Virginia Tech in the Mechanical Engineering Department. As such, presentation materials for each chapter and a complete solutions manual are available for use by instructors.

The text is an attempt to place vibration and control on a firm mathematical basis and connect the disciplines of vibration, linear algebra, matrix computations, control, and applied functional analysis. Each chapter ends with notes on further references and suggests where more detailed accounts can be found. In this way I hope to capture a 'big picture' approach without producing an overly large book. The first chapter presents a quick introduction using single-degree-of-freedom systems (second-order ordinary differential equations) to the following chapters, which extend these concepts to multiple-degree-of-freedom systems (matrix theory, systems of ordinary differential equations) and distributed-parameter systems (partial differential equations and boundary value problems). Numerical simulations and matrix computations are also presented through the use of MATLAB™. New material has been added on the use of MATLAB, and a brief introduction to nonlinear vibration is given. New problems and examples have been added, as well as a few new topics.

ACKNOWLEDGMENTS

I would like to thank Jamil M. Renno, a PhD student, for reading the final manuscript and sorting out several typos and numerical errors. In addition, Drs T. Michael Seigler,

Kaihong Wang, and Henry H. Sodano are owed special thanks for helping with the figures. I would also like to thank my past PhD students who have used the earlier version of the book, as well as Pablo Tarazaga, Dr Curt Kothera, M. Austin Creasy, and Armaghan Salehian who read the draft and made wonderful corrections and suggestions. Professor Daniel P. Hess of the University of South Florida provided invaluable suggestions and comments for which I am grateful. I would like to thank Ms Vanessa McCoy who retyped the manuscript from the hard copy of the previous version of this book and thus allowed me to finish writing electronically.

Thanks are also owed to Wendy Hunter of Wiley for the opportunity to publish this manuscript and the encouragement to finish it. I would also like to extend my thanks and appreciation to my wife Cathy Little, son Daniel, and daughters Jennifer and Angela (and their families) for putting up with my absence while I worked on this manuscript.

Daniel J. Inman
dinman@vt.edu

1

Single-degree-of-freedom Systems

1.1 INTRODUCTION

In this chapter the vibration of a single-degree-of-freedom system will be analyzed and reviewed. Analysis, measurement, design, and control of a single-degree-of-freedom system (often abbreviated SDOF) is discussed. The concepts developed in this chapter constitute an introductory review of vibrations and serve as an introduction for extending these concepts to more complex systems in later chapters. In addition, basic ideas relating to measurement and control of vibrations are introduced that will later be extended to multiple-degree-of-freedom systems and distributed-parameter systems. This chapter is intended to be a review of vibration basics and an introduction to a more formal and general analysis for more complicated models in the following chapters.

Vibration technology has grown and taken on a more interdisciplinary nature. This has been caused by more demanding performance criteria and design specifications for all types of machines and structures. Hence, in addition to the standard material usually found in introductory chapters of vibration and structural dynamics texts, several topics from control theory and vibration measurement theory are presented. This material is included not to train the reader in control methods (the interested student should study control and system theory texts) but rather to point out some useful connections between vibration and control as related disciplines. In addition, structural control has become an important discipline requiring the coalescence of vibration and control topics. A brief introduction to nonlinear SDOF systems and numerical simulation is also presented.

1.2 SPRING–MASS SYSTEM

Simple harmonic motion, or oscillation, is exhibited by structures that have elastic restoring forces. Such systems can be modeled, in some situations, by a spring–mass schematic, as illustrated in Figure 1.1. This constitutes the most basic vibration model of a structure and can be used successfully to describe a surprising number of devices, machines, and structures. The methods presented here for solving such a simple mathematical model may seem to be

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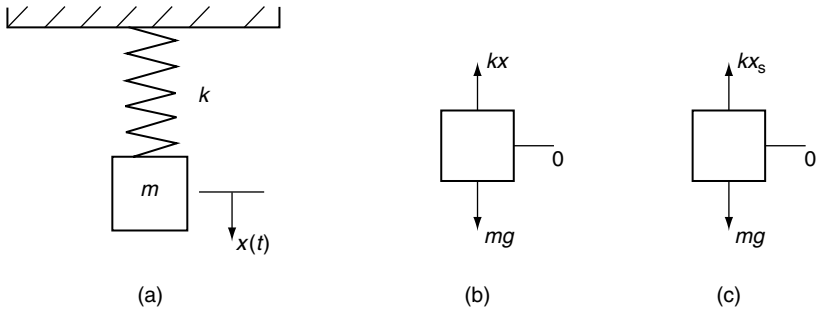

Figure 1.1 (a) Spring–mass schematic, (b) free body diagram, and (c) free body diagram of the static spring–mass system.

more sophisticated than the problem requires. However, the purpose of the analysis is to lay the groundwork for the analysis in the following chapters of more complex systems.

If $x = x(t)$ denotes the displacement (m) of the mass m (kg) from its equilibrium position as a function of time t (s), the equation of motion for this system becomes [upon summing forces in Figure 1.1(b)]

$$m\ddot{x} + k(x + x_s) - mg = 0$$

where k is the stiffness of the spring (N/m), x_s is the static deflection (m) of the spring under gravity load, g is the acceleration due to gravity (m/s^2), and the overdots denote differentiation with respect to time. (A discussion of dimensions appears in Appendix A, and it is assumed here that the reader understands the importance of using consistent units.) From summing forces in the free body diagram for the static deflection of the spring [Figure 1.1(c)], $mg = kx_s$ and the above equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0 \tag{1.1}$$

This last expression is the equation of motion of a single-degree-of-freedom system and is a linear, second-order, ordinary differential equation with constant coefficients.

Figure 1.2 indicates a simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection, x_s. The results of this static experiment can be plotted as force (mass times acceleration) versus x_s, the slope yielding the value of k for the linear portion of the plot. This is illustrated in Figure 1.3.

Once m and k are determined from static experiments, Equation (1.1) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of Equation (1.1) is found from substitution of an assumed periodic motion (from experience watching vibrating systems) of the form

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.2}$$

where $\omega_n = \sqrt{k/m}$ is the *natural frequency* (rad/s). Here, the *amplitude*, A, and the *phase shift*, ϕ, are constants of integration determined by the initial conditions.

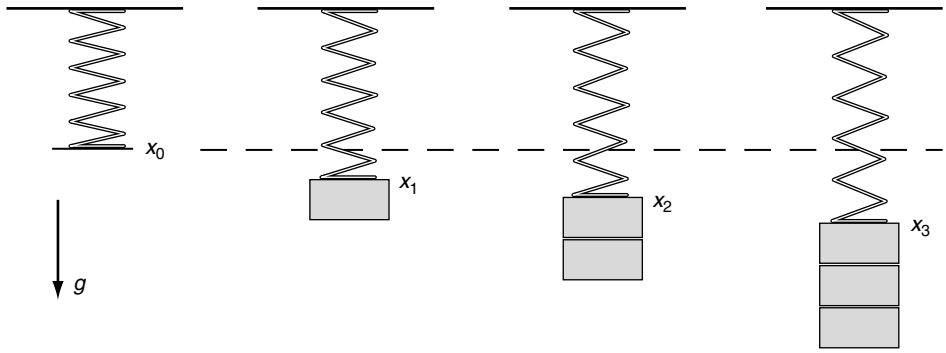

Figure 1.2 Measurement of the spring constant.

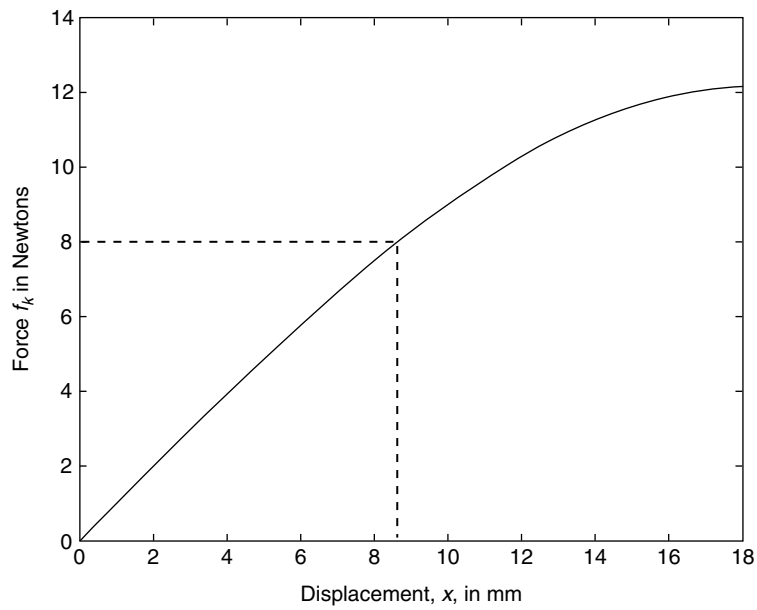

Figure 1.3 Determination of the spring constant.

The existence of a *unique* solution for Equation (1.1) with two specific initial conditions is well known and is given by, for instance, Boyce and DiPrima (2000). Hence, if a solution of the form of Equation (1.2) form is guessed and it works, then it is *the* solution. Fortunately, in this case the mathematics, physics, and observation all agree.

To proceed, if x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants A and ϕ to be evaluated. The unique solution is

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} \sin\left[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)\right] \tag{1.3}$$

Alternatively, $x(t)$ can be written as

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \tag{1.4}$$

by using a simple trigonometric identity.

A purely mathematical approach to the solution of Equation (1.1) is to assume a solution of the form $x(t) = A\,e^{\lambda t}$ and solve for λ, i.e.,

$$m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0$$

This implies that (because $e^{\lambda t} \neq 0$, and $A \neq 0$)

$$\lambda^2 + \left(\frac{k}{m}\right) = 0$$

or that

$$\lambda = \pm j \left(\frac{k}{m}\right)^{1/2} = \pm \omega_n j$$

where $j = (-1)^{1/2}$. Then the general solution becomes

$$x(t) = A_1\, e^{-\omega_n j t} + A_2\, e^{\omega_n j t} \tag{1.5}$$

where A_1 and A_2 are arbitrary complex conjugate constants of integration to be determined by the initial conditions. Use of Euler's formulae then yields Equations (1.2) and (1.4) (see, for instance, Inman, 2001). For more complicated systems, the exponential approach is often more appropriate than first guessing the form (sinusoid) of the solution from watching the motion.

Another mathematical comment is in order. Equation (1.1) and its solution are valid only as long as the spring is linear. If the spring is stretched too far, or too much force is applied to it, the curve in Figure 1.3 will no longer be linear. Then Equation (1.1) will be nonlinear (see Section 1.8). For now, it suffices to point out that initial conditions and springs should always be checked to make sure that they fall in the linear region if linear analysis methods are going to be used.

1.3 SPRING–MASS–DAMPER SYSTEM

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.3). Typically, the periodic motion dies down after some time. The easiest way to treat this mathematically is to introduce a velocity based force term, $c\dot{x}$, into Equation (1.1) and examine the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.6}$$

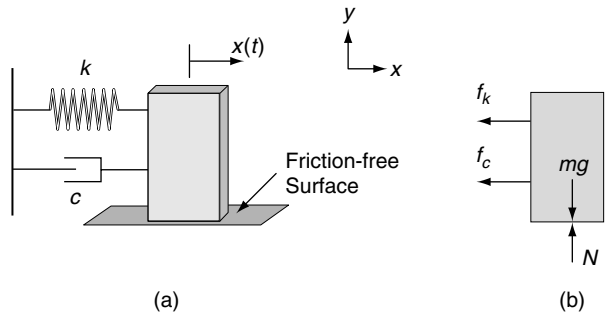

Figure 1.4 (a) Schematic of the spring–mass–damper system and (b) free body diagram of the system in part (a).

This also happens physically with the addition of a *dashpot* or *damper* to dissipate energy, as illustrated in Figure 1.4.

Equation (1.6) agrees with summing forces in Figure 1.4 if the dashpot exerts a dissipative force proportional to velocity on the mass m. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are. In addition, many structures dissipate energy in forms not proportional to velocity. The constant of proportionality c is given in N s/m or kg/s in terms of fundamental units.

Again, the unique solution of Equation (1.6) can be found for specified initial conditions by assuming that $x(t)$ is of the form

$$x(t) = A\,e^{\lambda t}$$

and substituting this into Equation (1.6) to yield

$$A\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right)e^{\lambda t} = 0 \tag{1.7}$$

Since a trivial solution is not desired, $A \neq 0$, and since $e^{\lambda t}$ is never zero, Equation (1.7) yields

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \tag{1.8}$$

Equation (1.8) is called the *characteristic equation* of Equation (1.6). Using simple algebra, the two solutions for λ are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}} \tag{1.9}$$

The quantity under the radical is called the *discriminant* and, together with the sign of m, c, and k, determines whether or not the roots are complex or real. Physically, m, c, and k are all positive in this case, so the value of the discriminant determines the nature of the roots of Equation (1.8).

It is convenient to define the dimensionless *damping ratio*, ζ, as

$$\zeta = \frac{c}{2\sqrt{km}}$$

In addition, let the *damped natural frequency*, ω_d, be defined (for $0 < \zeta < 1$) by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{1.10}$$

Then, Equation (1.6) becomes

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{1.11}$$

and Equation (1.9) becomes

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j, \qquad 0 < \zeta < 1 \tag{1.12}$$

Clearly, the value of the damping ratio, ζ, determines the nature of the solution of Equation (1.6). There are three cases of interest. The derivation of each case is left as a problem and can be found in almost any introductory text on vibrations (see, for instance, Meirovitch, 1986 or Inman, 2001).

Underdamped. This case occurs if the parameters of the system are such that

$$0 < \zeta < 1$$

so that the discriminant in Equation (1.12) is negative and the roots form a complex conjugate pair of values. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\zeta\omega_n t} (A \cos \omega_d t + B \sin \omega_d t) \tag{1.13}$$

or

$$x(t) = Ce^{-\zeta\omega_n t} \sin (\omega_d t + \phi)$$

where A, B, C, and ϕ are constants determined by the specified initial velocity, v_0, and position, x_0:

$$A = x_0, \qquad C = \frac{\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d}$$

$$B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}, \qquad \phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) \tag{1.14}$$

The underdamped response has the form given in Figure 1.5 and consists of a decaying oscillation of frequency ω_d.

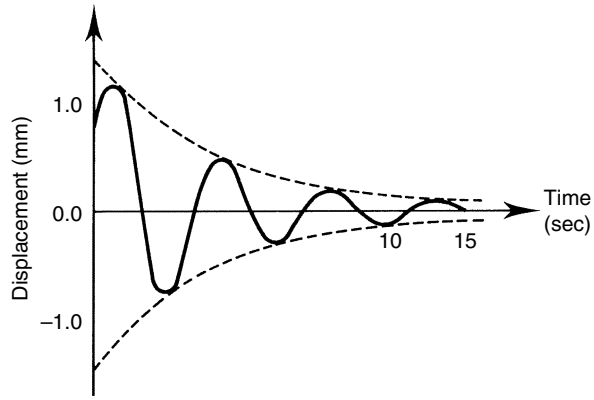

Figure 1.5 Response of an underdamped system.

Overdamped. This case occurs if the parameters of the system are such that

$$\zeta > 1$$

so that the discriminant in Equation (1.12) is positive and the roots are a pair of negative real numbers. The solution of Equation (1.11) then becomes

$$x(t) = A\,e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + B\,e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{1.15}$$

where A and B are again constants determined by v_0 and x_0. They are

$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$B = -\frac{v_0 + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

The overdamped response has the form given in Figure 1.6. An overdamped system does not oscillate, but rather returns to its rest position exponentially.

Critically damped. This case occurs if the parameters of the system are such that

$$\zeta = 1$$

so that the discriminant in Equation (1.12) is zero and the roots are a pair of negative real repeated numbers. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\omega_n t}\left[(v_0 + \omega_n x_0)t + x_0\right] \tag{1.16}$$

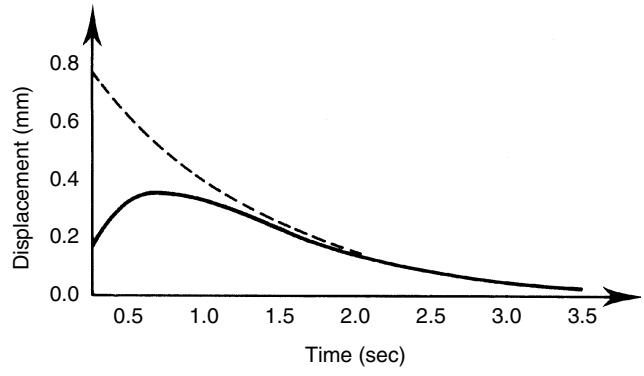

Figure 1.6 Response of an overdamped system.

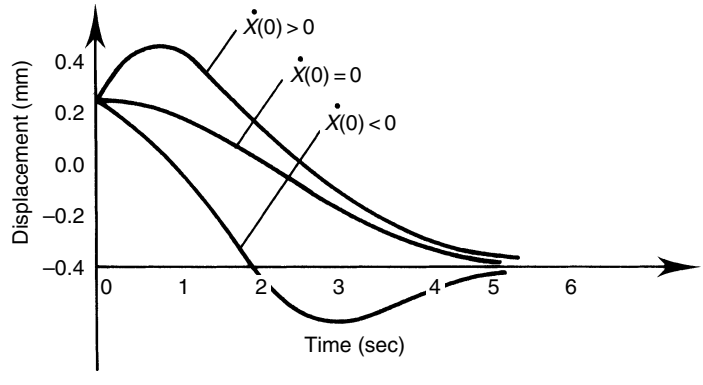

Figure 1.7 Response of a critically damped system.

The critically damped response is plotted in Figure 1.7 for various values of the initial conditions v_0 and x_0.

It should be noted that critically damped systems can be thought of in several ways. First, they represent systems with the minimum value of damping rate that yields a nonoscillating system (Problem 1.5). Critical damping can also be thought of as the case that separates nonoscillation from oscillation.

1.4 FORCED RESPONSE

The preceding analysis considers the vibration of a device or structure as a result of some initial disturbance (i.e., v_0 and x_0). In this section, the vibration of a spring–mass–damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses, and step forcing functions is examined.

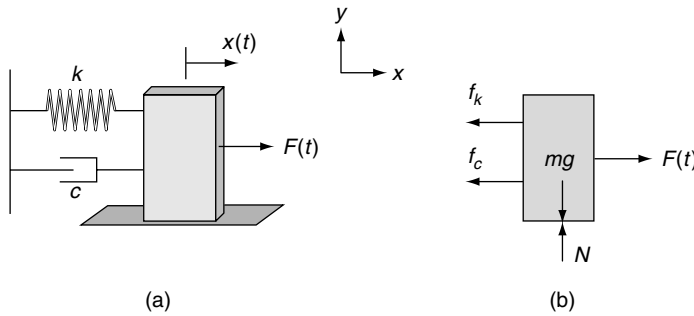

Figure 1.8 (a) Schematic of the forced spring–mass–damper system assuming no friction on the surface and (b) free body diagram of the system of part (a).

In many environments, rotating machinery, motors, and so on, cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. It is common to approximate the driving forces, $F(t)$, as periodic of the form

$$F(t) = F_0 \sin \omega t$$

where F_0 represents the amplitude of the applied force and ω denotes the frequency of the applied force, or the driving frequency (rad/s). On summing the forces, the equation for the forced vibration of the system in Figure 1.8 becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{1.17}$$

Recall from the discipline of differential equations (Boyce and DiPrima, 2000), that the solution of Equation (1.17) consists of the sum of the homogeneous solution in Equation (1.5) and a particular solution. These are usually referred to as the *transient response* and the *steady state response* respectively. Physically, there is motivation to assume that the steady state response will follow the forcing function. Hence, it is tempting to assume that the particular solution has the form

$$x_p(t) = X \sin(\omega t - \theta) \tag{1.18}$$

where X is the steady state amplitude and θ is the phase shift at steady state. Mathematically, the method is referred to as the method of undetermined coefficients. Substitution of Equation (1.18) into Equation (1.17) yields

$$X = \frac{F_0/k}{\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}}$$

or

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{1.19}$$

and

$$\tan\theta = \frac{(c\omega/k)}{1 - m\omega^2/k} = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \tag{1.20}$$

where $\omega_n = \sqrt{k/m}$ as before. Since the system is linear, the sum of two solutions is a solution, and the total time response for the system of Figure 1.8 for the case $0 < \zeta < 1$ becomes

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) + X\sin(\omega t - \theta) \tag{1.21}$$

Here, A and B are constants of integration determined by the initial conditions and the forcing function (and in general will be different from the values of A and B determined for the free response).

Examining Equation (1.21), two features are important and immediately obvious. First, as t becomes larger, the transient response (the first term) becomes very small, and hence the term steady state response is assigned to the particular solution (the second term). The second observation is that the coefficient of the steady state response, or particular solution, becomes large when the excitation frequency is close to the undamped natural frequency, i.e., $\omega \approx \omega_n$. This phenomenon is known as *resonance* and is extremely important in design, vibration analysis, and testing.

Example 1.4.1

Compute the response of the following system (assuming consistent units):

$$\ddot{x}(t) + 0.4\dot{x}(t) + 4x(t) = \frac{1}{\sqrt{2}}\sin 3t, \qquad x(0) = \frac{-3}{\sqrt{2}}, \qquad \dot{x}(0) = 0$$

First, solve for the particular solution by using the more convenient form of

$$x_p(t) = X_1\sin 3t + X_2\cos 3t$$

rather than the magnitude and phase form, where X_1 and X_2 are the constants to be determined. Differentiating x_p yields

$$\dot{x}_p(t) = 3X_1\cos 3t - 3X_2\sin 3t$$
$$\ddot{x}_p(t) = -9X_1\sin 3t - 9X_2\cos 3t$$

Substitution of x_p and its derivatives into the equation of motion and collecting like terms yield

$$\left(-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}}\right)\sin 3t + (-9X_2 + 1.2X_1 + 4X_2)\cos 3t = 0$$

Since the sine and cosine are independent, the two coefficients in parentheses must vanish, resulting in two equations in the two unknowns X_1 and X_2. This solution yields

$$x_p(t) = -0.134\ \sin 3t - 0.032\ \cos 3t$$

Next, consider adding the free response to this. From the problem statement

$$\omega_n = 2\,\text{rad/s}, \qquad \zeta = \frac{0.4}{2\omega_n} = 0.1 < 1, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.99\,\text{rad/s}$$

Thus, the system is underdamped, and the total solution is of the form

$$x(t) = e^{-\zeta\omega_n t}\,(A\sin\omega_d t + B\cos\omega_d t) + X_1\sin\omega t + X_2\cos\omega t$$

Applying the initial conditions requires the following derivative

$$\begin{aligned}\dot{x}(t) = {} & e^{-\zeta\omega_n t}\,(\omega_d A\cos\omega_d t - \omega_d B\sin\omega_d t) + \omega X_1\cos\omega t \\ & - \omega X_2\sin\omega t - \zeta\omega_n e^{-\zeta\omega_n t}\,(A\sin\omega_d t + B\cos\omega_d t)\end{aligned}$$

The initial conditions yield the constants A and B:

$$\begin{aligned}x(0) &= B + X_2 = \frac{-3}{\sqrt{2}} \Rightarrow B = -X_2 - \frac{3}{\sqrt{2}} = -2.089 \\ \dot{x}(0) &= \omega_d A + \omega X_1 - \zeta\omega_n B = 0 \Rightarrow A = \frac{1}{\omega_d}\,(\zeta\omega_n B - \omega X_1) = -0.008\end{aligned}$$

Thus the total solution is

$$x(t) = -e^{-0.2t}\,(0.008\sin 1.99t + 2.089\cos 1.99t) - 0.134\sin 3t - 0.032\cos 3t$$

Resonance is generally to be avoided in designing structures, since it means large-amplitude vibrations, which can cause fatigue failure, discomfort, loud noises, and so on. Occasionally, the effects of resonance are catastrophic. However, the concept of resonance is also very useful in testing structures. In fact, the process of modal testing (see Chapter 8) is based on resonance. Figure 1.9 illustrates how ω_n and ζ affect the amplitude at resonance. The dimensionless quantity Xk/F_0 is called the *magnification factor* and Figure 1.9 is called a *magnification curve* or *magnitude plot*. The maximum value at resonance, called the *peak resonance*, and denoted by M_p, can be shown (see, for instance, Inman, 2001) to be related to the damping ratio by

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{1.22}$$

Also, Figure 1.9 can be used to define the *bandwidth* (BW) of the structure as the value of the driving frequency at which the magnitude drops below 70.7% of its zero frequency value (also said to be the 3 dB down point from the zero frequency point). The bandwidth can be calculated (Kuo and Golnaraghi, 2003, p. 359) in terms of the damping ratio by

$$\text{BW} = \omega_n\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{1.23}$$

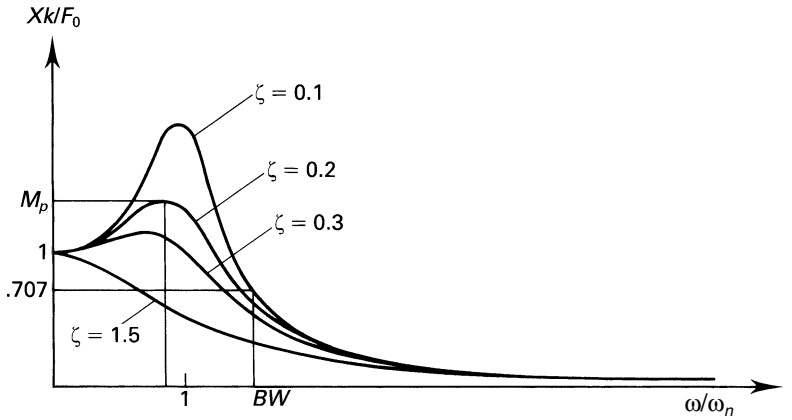

Figure 1.9 Magnification curves for a single-degree-of-freedom system ($r = \omega/\omega_n$).

Two other quantities are used in discussing the vibration of underdamped structures. They are the *loss factor* defined at resonance (only) to be

$$\eta = 2\zeta \tag{1.24}$$

and the *Q value*, or *resonance sharpness* factor, given by

$$Q = \frac{1}{2\zeta} = \frac{1}{\eta} \tag{1.25}$$

Another common situation focuses on the transient nature of the response, namely the response of (1.6) to an impulse, to a step function, or to initial conditions. Many mechanical systems are excited by loads, which act for a very brief time. Such situations are usually modeled by introducing a fictitious function called the *unit impulse function*, or the *Dirac delta function*. This delta function is defined by the two properties

$$\begin{aligned} &\delta(t-a) = 0, \qquad t \neq a \\ &\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \end{aligned} \tag{1.26}$$

where a is the instant of time at which the impulse is applied. Strictly speaking, the quantity $\delta(t)$ is not a function; however, it is very useful in quantifying important physical phenomena of an impulse.

The response of the system of Figure 1.8 for the underdamped case (with $a = x_0 = v_0 = 0$) can be shown to be given by

$$x(t) = \begin{cases} 0, & t < a \\ \dfrac{e^{-\zeta\omega t}\sin\omega_d t}{m\omega_d}, & t > a \end{cases} \tag{1.27}$$

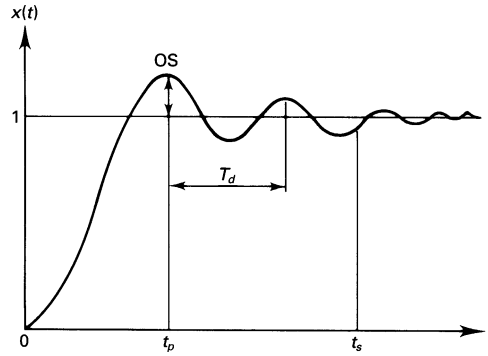

Figure 1.10 Step response of a single-degree-of-freedom system.

Note from Equation (1.13) that this corresponds to the transient response of the system to the initial conditions $x_0 = 0$ and $v_0 = 1/m$. Hence, the impulse response is equivalent to giving a system at rest an initial velocity of $(1/m)$. This makes the impulse response, $x(t)$, important in discussing the transient response of more complicated systems. The impulse is also very useful in making vibration measurements, as described in Chapter 8.

Often, design problems are stated in terms of certain specifications based on the response of the system to step function excitation. The response of the system in Figure 1.8 to a step function (of magnitude $m\omega_n^2$ for convenience), with initial conditions both set to zero, is calculated for underdamped systems from

$$m\ddot{x} + c\dot{x} + kx = m\omega_n^2\mu(t), \qquad \mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1.28}$$

to be

$$x(t) = 1 - \frac{e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{\sqrt{1-\zeta^2}} \tag{1.29}$$

where

$$\phi = \arctan\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] \tag{1.30}$$

A sketch of the response is given in Figure 1.10, along with the labeling of several significant specifications for the case $m = 1$, $\omega_n = 2$, and $\zeta = 0.2$.

In some situations, the steady state response of a structure may be at an acceptable level, but the transient response may exceed acceptable limits. Hence, one important measure is the *overshoot*, labeled OS in Figure 1.10 and defined as the maximum value of the response minus the steady state value of the response. From Equation (1.29) it can be shown that

$$\mathrm{OS} = x_{\max}(t) - 1 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{1.31}$$

This occurs at the *peak time*, t_p, which can be shown to be

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{1.32}$$

In addition, the period of oscillation, T_d, is given by

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 2t_p \tag{1.33}$$

Another useful quantity, which indicates the behavior of the transient response, is the *settling time*, t_s. This is the time it takes the response to get within ±5% of the steady state response and remain within ±5%. One approximation of t_s is given by (Kuo and Golnaraghi, 2003, p. 263):

$$t_s = \frac{3.2}{\omega_n\zeta} \tag{1.34}$$

The preceding definitions allow designers and vibration analysts to specify and classify precisely the nature of the transient response of an underdamped system. They also give some indication of how to adjust the physical parameters of the system so that the response has a desired shape.

The response of a system to an impulse may be used to determine the response of an underdamped system to any input $F(t)$ by defining the *impulse response function* as

$$h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{1.35}$$

Then the solution of

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

can be shown to be

$$x(t) = \int_0^t F(\tau)h(t-\tau)\,d\tau = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t F(\tau)\,e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\,d\tau \tag{1.36}$$

for the case of zero initial conditions. This last expression gives an analytical representation for the response to any driving force that has an integral.

1.5 TRANSFER FUNCTIONS AND FREQUENCY METHODS

The preceding analysis of the response was carried out in the time domain. Current vibration measurement methodology (Ewins, 2000) as well as much control analysis (Kuo and Golnaraghi, 2003) often takes place in the frequency domain. Hence, it is worth the effort to reexamine these calculations using frequency domain methods (a phrase usually associated with linear control theory). The frequency domain approach arises naturally from mathematics (ordinary differential equations) via an alternative method of solving differential equations, such as Equations (1.17) and (1.28), using the Laplace transform (see, for instance, Boyce and DiPrima, 2000, Chapter 6).

Taking the Laplace transform of Equation (1.28), assuming both initial conditions to be zero, yields

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right]\mu(s) \tag{1.37}$$

where $X(s)$ denotes the Laplace transform of $x(t)$, and $\mu(s)$ is the Laplace transform of the right-hand side of Equation (1.28). If the same procedure is applied to Equation (1.17), the result is

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right] F_0(s) \tag{1.38}$$

where $F_0(s)$ denotes the Laplace transform of $F_0 \sin \omega t$. Note that

$$G(s) = \frac{X(s)}{\mu(s)} = \frac{X(s)}{F_0(s)} = \frac{1}{ms^2 + cs + k} \tag{1.39}$$

Thus, it appears that the quantity $G(s) = [1/(ms^2 + cs + k)]$, the ratio of the Laplace transform of the output (response) to the Laplace transform of the input (applied force) to the system, characterizes the system (structure) under consideration. This characterization is independent of the input or driving function. This ratio, $G(s)$, is defined as the *transfer function* of this system in control analysis (or of this structure in vibration analysis). The transfer function can be used to provide analysis of the vibrational properties of the structure as well as to provide a means of measuring the dynamic response of the structure.

In control theory, the transfer function of a system is defined in terms of an output to input ratio, but the use of a transfer function in structural dynamics and vibration testing implies certain physical properties, depending on whether position, velocity, or acceleration is considered as the response (output). It is quite common, for instance, to measure the response of a structure by using an accelerometer. The resultant transfer function is then $s^2X(s)/U(s)$, where $U(s)$ is the Laplace transform of the input and $s^2X(s)$ is the Laplace transform of the acceleration. This transfer function is called the *inertance* and its reciprocal is referred to as the *apparent mass*. Table 1.1 lists the nomenclature of various transfer functions. The physical basis for these names can be seen from their graphical representation.

The transfer function representation of a structure is very useful in control theory as well as in vibration testing. The variable s in the Laplace transform is a complex variable, which can be further denoted by

$$s = \sigma + j\omega_d$$

where the real numbers σ and ω_d denote the real and imaginary parts of s respectively. Thus, the various transfer functions are also complex valued.

In control theory, the values of s where the denominator of the transfer function $G(s)$ vanishes are called the *poles* of the transfer function. A plot of the poles of the compliance (also called receptance) transfer function for Equation (1.38) in the complex s plane is given

Table 1.1 Various transfer functions.

Response measurement	Transfer function	Inverse transfer function
Acceleration	Inertance	Apparent mass
Velocity	Mobility	Impedance
Displacement	Compliance	Dynamic stiffness

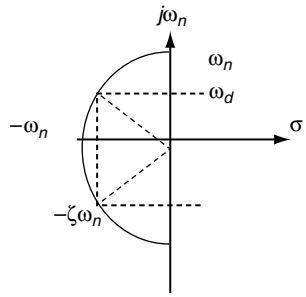

Figure 1.11 Complex s plane of the poles of Equation (1.39).

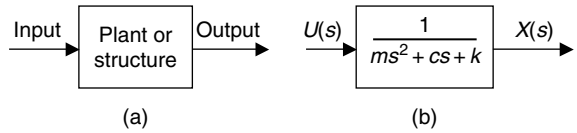

Figure 1.12 Block diagram representation of a single-degree-of-freedom system.

in Figure 1.11. The points on the semicircle occur where the denominator of the transfer function is zero. These values of $s(s=-\zeta\omega_n \pm \omega_d j)$ are exactly the roots of the characteristic equation for the structure. The values of the physical parameters m, c, and k determine the two quantities ζ and ω_n, which in turn determine the position of the poles in Figure 1.11.

Another graphical representation of a transfer function useful in control is the *block diagram* illustrated in Figure 1.12(a). This diagram is an icon for the definition of a transfer function. The control terminology for the physical device represented by the transfer function is the *plant*, whereas in vibration analysis the plant is usually referred to as the structure. The block diagram of Figure 1.12(b) is meant exactly to imply the formula

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k} \tag{1.40}$$

The response of Equation (1.38) to a sinusoidal input (forcing function) motivates a second description of the transfer function of a structure, called the *frequency response function* (often denoted by FRF). The frequency response function is defined as the transfer function evaluated at $s=j\omega$, i.e., $G(j\omega)$. The significance of the frequency response function follows from Equation (1.21), namely that the steady state response of a system driven sinusoidally is a sinusoid of the same frequency with different amplitude and phase. In fact, substitution of $j\omega$ into Equation (1.40) yields exactly Equations (1.19) and (1.20) from

$$\frac{X}{F_0} = |G(j\omega)| = \sqrt{x^2(\omega) + y^2(\omega)} \tag{1.41}$$

where $|G(j\omega)|$ indicates the magnitude of the complex frequency response function,

$$\phi = \tan^{-1} G(j\omega) = \tan^{-1}\left[\frac{y(\omega)}{x(\omega)}\right] \tag{1.42}$$

indicates the phase of the frequency response function, and

$$G(j\omega) = x(\omega) + y(\omega)j \tag{1.43}$$

This mathematically expresses two ways of representing a complex function, as the sum of its real part $[\mathrm{Re}\, G(j\omega) = x(\omega)]$ and its imaginary part $[\mathrm{Im}\, G(j\omega) = y(\omega)]$, or by its magnitude ($|G(j\omega)|$) and phase ($\phi$). In more physical terms, the frequency response function of a structure represents the magnitude and phase shift of its steady state response under sinusoidal excitation. While Equations (1.17), (1.21), (1.41), and (1.42) verify this for a single-degree-of-freedom viscously damped structure, it can be shown in general for any linear time-invariant plant (Melsa and Schultz, 1969, p. 187).

It should also be noted that the frequency response function of a linear system can be obtained from the transfer function of the system, and vice versa. Hence, the frequency response function uniquely determines the time response of the structure to any known input.

Graphical representations of the frequency response function form an extensive part of control analysis and also form the backbone of vibration measurement analysis. Next, three sets of frequency response function plots that are useful in testing vibrating structures are examined. The first set of plots consists simply of plotting the imaginary part of the frequency response function versus the driving frequency and the real part of the frequency response function versus the driving frequency. These are shown for a damped single-degree-of-freedom system in Figure 1.13 (the compliance frequency response function for $\zeta = 0.01$ and $\omega_n = 20\,\mathrm{rad/s}$).

The second representation consists of a single plot of the imaginary part of the frequency response function versus the real part of the frequency response function. This type of plot is called a *Nyquist plot* (also called an *Argand plane plot*) and is used for measuring the natural frequency and damping in testing methods and for stability analysis in control system design. The Nyquist plot of the mobility frequency response function of a structure modeled by Equation (1.37) is given in Figure 1.14.

The last plots considered for representing the frequency response function are called *Bode plots* and consist of a plot of the magnitude of the frequency response function versus the driving frequency and the phase of the frequency response function versus the driving frequency (a complex number requires *two* real numbers to describe it completely). Bode plots

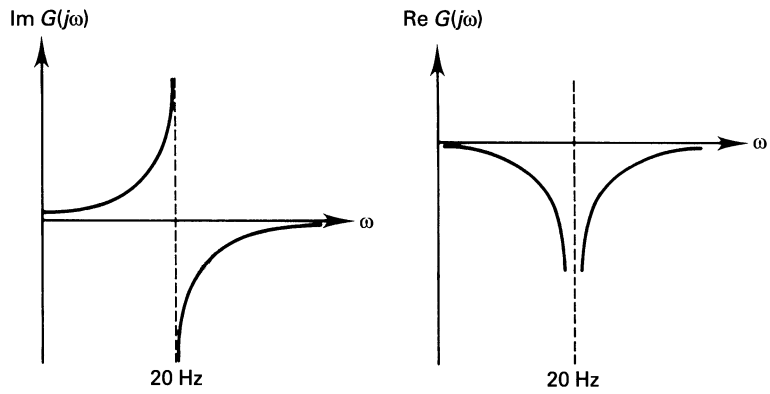

Figure 1.13 Plots of the real part and the imaginary part of the frequency response function.

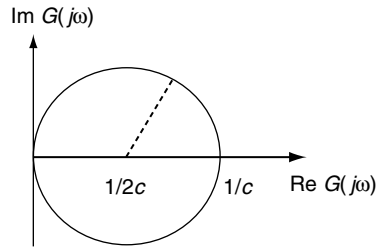

Figure 1.14 Nyquist plot for Equation 1.39.

have long been used in control system design and analysis as well as for determining the plant transfer function of a system. More recently, Bode plots have been used in analyzing vibration test results and in determining the physical parameters of the structure.

In order to represent the complete Bode plots in a reasonable space, $\log_{10}$ scales are often used to plot $|G(j\omega)|$. This has given rise to the use of the decibel and decades in discussing

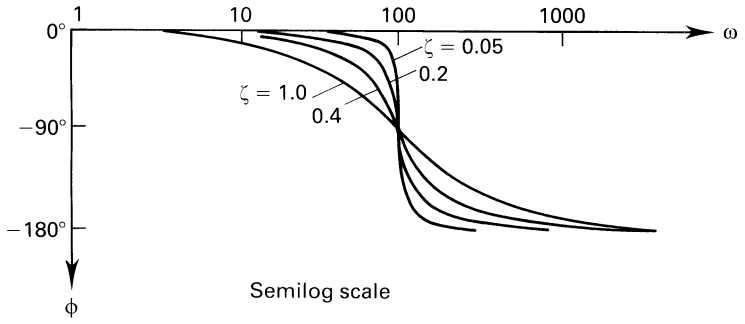

Figure 1.15 Bode phase plot for Equation (1.39).

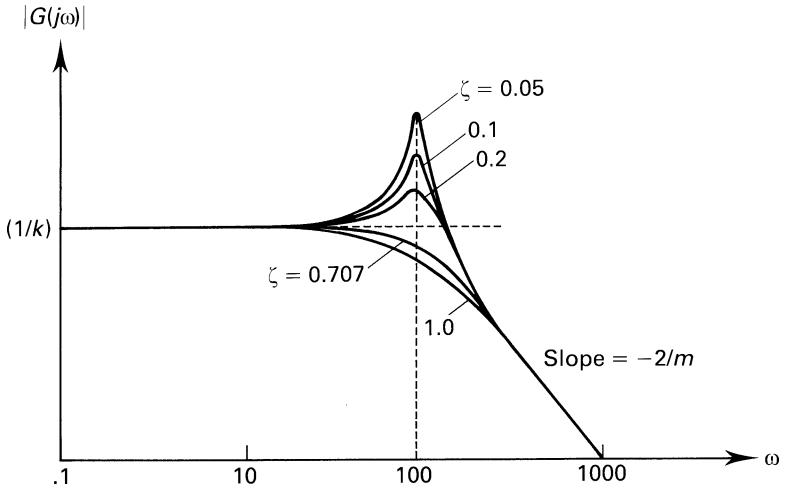

Figure 1.16 Bode magnitude plot for Equation (1.39).

the magnitude response in the frequency domain. The magnitude and phase plots (for the compliance transfer function) for the system of Equation (1.17) are shown in Figures 1.15 and 1.16 for different values of ζ. Note the phase change at resonance (90°), as this is important in interpreting measurement data.

1.6 MEASUREMENT AND TESTING

One can also use the quantities defined in the previous sections to measure the physical properties of a structure. As mentioned before, resonance can be used to determine the natural frequency of a system. Methods based on resonance are referred to as resonance testing (or modal analysis techniques) and are briefly introduced here and discussed in more detail in Chapter 8.

As mentioned earlier, the mass and stiffness of a structure can often be determined by making simple static measurements. However, damping rates require a dynamic measurement and hence are more difficult to determine. For underdamped systems one approach is to realize, from Figure 1.5, that the decay envelope is the function $e^{-\zeta\omega_n t}$. The points on the envelope illustrated in Figure 1.17 can be used to curve-fit the function e^{-at}, where a is the constant determined by the curve fit. The relation $a = \zeta\omega_n$ can next be used to calculate ζ and hence the damping rate c (assuming that m and k or ω_n are known).

A second approach is to use the concept of logarithmic decrement, denoted by δ and defined by

$$\delta = \ln \frac{x(t)}{x(t + T_d)} \tag{1.44}$$

where T_d is the period of oscillation. Using Equation (1.13) in the form

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{1.45}$$

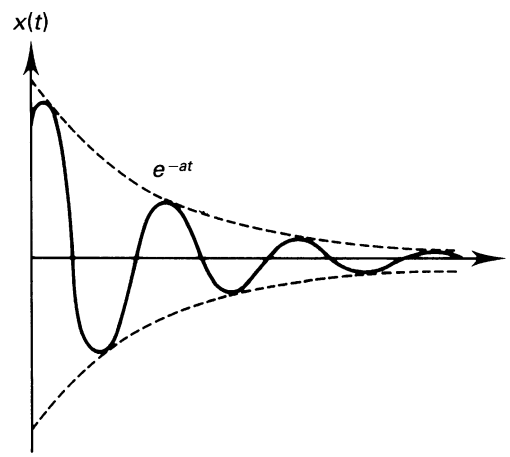

Figure 1.17 Free decay measurement method.

the logarithmic decrement δ becomes

$$\delta = \ln\left[\frac{e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{e^{-\zeta\omega_n(t+T_d)}\sin(\omega_d t + \omega_d T_d + \phi)}\right] = \ln e^{\zeta\omega_n T_d} = \zeta\omega_n T_d \tag{1.46}$$

where the sine functions cancel because $\omega_d T_d$ is a one-period shift by definition. Further evaluation of δ yields

$$\delta = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{1.47}$$

Equation (1.47) can be manipulated to yield the damping ratio in terms of the decrement, i.e.,

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{1.48}$$

Hence, if the decrement is measured, Equation (1.48) yields the damping ratio.

The various plots of the previous section can also be used to measure ω_n, ζ, m, c, and k. For instance, the Bode diagram of Figure 1.16 can be used to determine the natural frequency, stiffness, and damping ratio. The stiffness is determined from the intercept of the frequency response function and the magnitude axis, since the value of the magnitude of the frequency response function for small ω is $\log(1/k)$. This can be seen by examining the function $\log_{10}|G(j\omega)|$ for small ω. Note that

$$\log|G(j\omega)| = \log\frac{1}{k} - \frac{1}{2}\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right] = \log\left(\frac{1}{k}\right) \tag{1.49}$$

for very small values of ω. Also, note that $|G(j\omega)|$ evaluated at ω_n yields

$$k\,|G(j\omega_n)| = \frac{1}{2\zeta} \tag{1.50}$$

which provides a measure of the damping ratio from the magnitude plot of the frequency response function.

Note that Equations (1.48) and (1.22) appear to contradict each other, since

$$\frac{1}{2\zeta\sqrt{1-\zeta^2}} = k\,\max|G(j\omega)| = M_p \neq k\,|G(j\omega_n)| = \frac{1}{2\zeta}$$

except in the case of very small ζ (i.e., the difference between M_p and $|G(j\omega_n)|$ goes to zero as ζ goes to zero). This indicates a subtle difference between using the damping ratio obtained by taking resonance as the value of ω, where $|G(j\omega_n)|$ is a maximum, and using the point where $\omega = \omega_n$, the undamped natural frequency. This point is also illustrated by noting that the damped natural frequency [Equation (1.8)] is $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and the frequency at which $|G(j\omega_n)|$ is maximum is

$$\omega_p = \omega_n\sqrt{1-2\zeta^2} \tag{1.51}$$

Also note that Equation (1.51) is valid only if $0 < \zeta < 0.707$.

Finally, the mass can be related to the slope of the magnitude plot for the inertance transfer function, $G_1(s)$, by noting that

$$G_1(s) = \frac{s^2}{ms^2 + cs + k} \tag{1.52}$$

and for large ω (i.e., $\omega_n << \omega$), the value of $|G_1(j\omega)|$ is

$$|G_1(j\omega)| \approx (1/m) \tag{1.53}$$

Plots of these values are referred to as straight-line approximations to the actual magnitude plot (Bode, 1945).

The preceding formulae relating the physical properties of the structure to the magnitude Bode diagrams suggest an experimental way to determine the parameters of a structure: namely, if the structure can be driven by a sinusoid of varying frequency and if the magnitude and phase (needed to locate resonance) of the resulting response are measured, then the Bode plots and the preceding formulae can be used to obtain the desired physical parameters. This process is referred to as plant identification in the control's literature and can be extended to systems with more degrees of freedom (see, for instance, Melsa and Schultz, 1969, for a more complete account).

There are several other formulae for measuring the damping ratio and natural frequency from the results of such experiments (sine sweeps). For instance, if the Nyquist plot of the mobility transfer function is used, a circle of diameter $1/c$ results (see Figure 1.14). Another approach is to plot the magnitude of the frequency response function on a linear scale near the region of resonance, as shown in Figure 1.18. If the damping is small enough for the peak at resonance to be sharp, the damping ratio can be determined by measuring the frequencies at 0.707 at the maximum value (also called the 3 dB down point or half-power points), denoted by ω_1 and ω_2 respectively, and then using the formula (Ewins, 2000)

$$\zeta = \frac{1}{2}\left[\frac{\omega_2 - \omega_1}{\omega_d}\right] \tag{1.54}$$

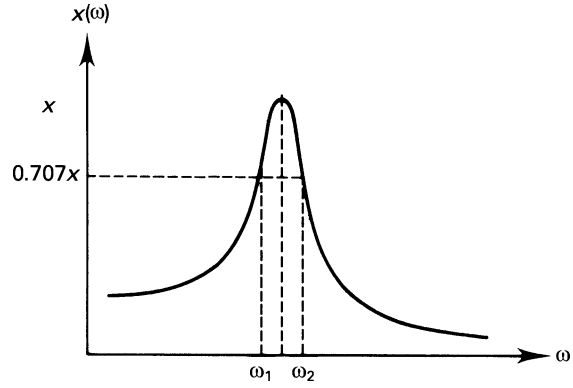

Figure 1.18 Quadrature peak picking method.

to compute the damping ratio. This method is referred to as *quadrature peak picking* and is illustrated in Figure 1.18.

1.7 STABILITY

In all the preceding analysis, the physical parameters m, c, and k are, of course, positive quantities. There are physical situations, however, in which expressions in the form of Equations (1.1) and (1.6) result but have one or more negative coefficients. Such systems are not well behaved and require some additional analysis.

Recalling that the solution to Equation (1.1) is of the form $A\sin(\omega t + \phi)$, where A is a constant, it is easy to see that the response, in this case $x(t)$, is bounded. That is to say,

$$|x(t)| < A \tag{1.55}$$

for all t, where A is some finite constant and $|x(t)|$ denotes the absolute value of $x(t)$. In this case, the system is well behaved or *stable* (called marginally stable in the control's literature). In addition, note that the roots (also called *characteristic values* or eigenvalues) of

$$\lambda^2 m + k = 0$$

are purely complex numbers $\pm j\omega_n$ as long as m and k are positive (or have the same sign). If k happens to be negative and m is positive, the solution becomes

$$x(t) = A\sinh\omega_n t + B\cosh\omega_n t \tag{1.56}$$

which increases without bound as t does. Such solutions are called *divergent* or *unstable*.

If the solution of the damped system of Equation (1.6) with positive coefficients is examined, it is clear that $x(t)$ approaches zero as t becomes large because of the exponential term. Such systems are considered to be *asymptotically stable* (called stable in the controls literature). Again, if one or two of the coefficients are negative, the motion grows without bound and becomes unstable as before. In this case, however, the motion may become unstable in one of two ways. Similar to overdamping and underdamping, the motion may grow without bound and not oscillate, or it may grow without bound and oscillate. The first case is referred to as *divergent instability* and the second case is known as *flutter instability*; together, they fall under the topic of self-excited vibrations.

Apparently, the sign of the coefficient determines the stability behavior of the system. This concept is pursued in Chapter 4, where these stability concepts are formally defined. Figures 1.19 through 1.22 illustrate each of these concepts.

These stability definitions can also be stated in terms of the roots of the characteristic equation [Equation (1.8)] or in terms of the poles of the transfer function of the system. In fact, referring to Figure 1.11, the system is stable if the poles of the structure lie along the imaginary axis (called the $j\omega$ axis), unstable if one or more poles are in the right half-plane, and asymptotically stable if all of the poles lie in the left half-plane. Flutter occurs when the poles are in the right half-plane and not on the real axis (complex conjugate pairs of roots

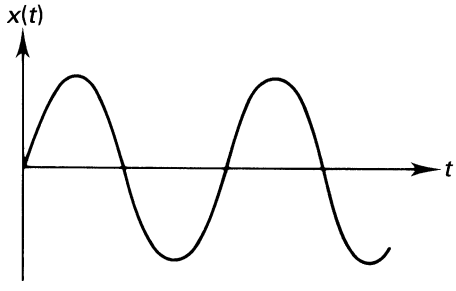

Figure 1.19 Response of a stable system.

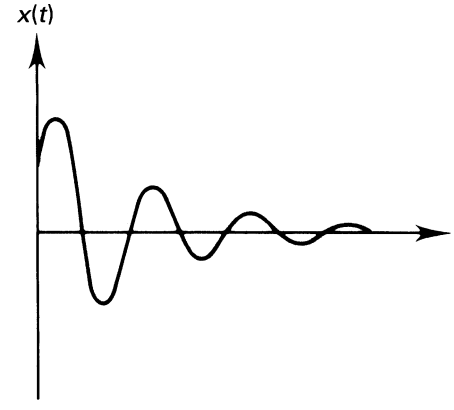

Figure 1.20 Response of an asympotically stable system.

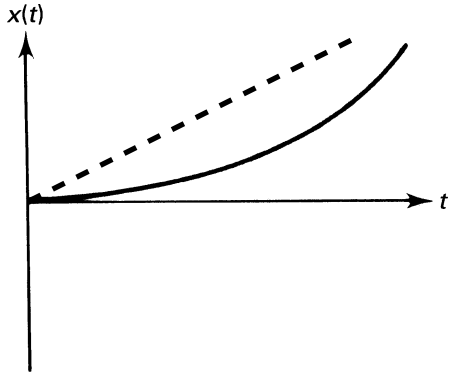

Figure 1.21 Response of a system with a divergent instability.

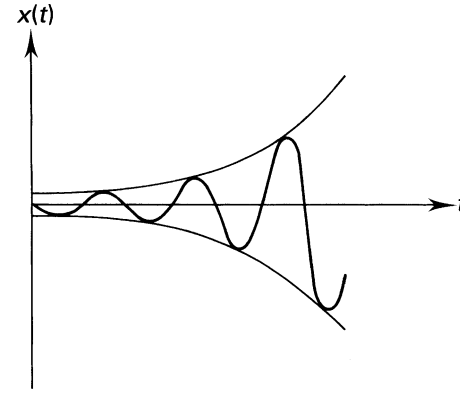

Figure 1.22 Response of a system with flutter instability.

with a positive real part), and divergence occurs when the poles are in the right half-plane along the real axis. In the simple single-degree-of-freedom case considered here, the pole positions are entirely determined by the signs of m, c, and k.

The preceding definitions and ideas about stability are stated for the free response of the system. These concepts of a well-behaved response can also be applied to the forced motion of a vibrating system. The stability of the forced response of a system can be defined by considering the nature of the applied force or input. The system is said to be *bounded-input, bounded-output stable* (or, simply, BIBO stable) if, for *any* bounded input (driving force), the output (response) is bounded for any arbitrary set of initial conditions. Such systems are manageable at resonance.

It can be seen immediately that Equation (1.17) with $c = 0$, the undamped system, is not BIBO stable, since, for $f(t) = \sin(\omega_n t)$, the response $x(t)$ goes to infinity (at resonance) whereas $f(t)$ is certainly bounded. However, the response of Equation (1.17) with $c > 0$ is

bounded whenever $f(t)$ is. In fact, the maximum value of $x(t)$ at resonance M_p is illustrated in Figure 1.9. Thus, the system of Equation (1.17) with damping is said to be BIBO stable.

The fact that the response of an undamped structure is bounded when $f(t)$ is an impulse or step function suggests another, weaker, definition for the stability of the forced response. A system is said to be *bounded*, or *Lagrange stable*, with respect to a given input if the response is bounded for any set of initial conditions. Structures described by Equation (1.1) are Lagrange stable with respect to many inputs. This definition is useful when $f(t)$ is known completely or known to fall in some specified class of functions.

Stability can also be thought of in terms of whether or not the energy of the system is increasing (unstable), constant (stable), or decreasing (asymptotically stable) rather than in terms of the explicit response. Lyapunov stability, defined in Chapter 4, extends this idea. Another important view of stability is based on how sensitive a motion is to small perturbations in the system parameters (m, c, and k) and/or small perturbations in initial conditions. Unfortunately, there does not appear to be a universal definition of stability that fits all situations. The concept of stability becomes further complicated for nonlinear systems. The definitions and concepts mentioned here are extended and clarified in Chapter 4.

1.8 DESIGN AND CONTROL OF VIBRATIONS

One can use the quantities defined in the previous sections to design structures and machines to have a desired transient and steady state response to some extent. For instance, it is a simple matter to choose m, c, and k so that the overshoot is a specified value. However, if one needs to specify the overshoot, the settling time, and the peak time, then there may not be a choice of m, c, and k that will satisfy all three criteria. Hence, the response cannot always be completely shaped, as the formulae in Section 1.4 may seem to indicate.

Another consideration in designing structures is that each of the physical parameters m, c, and k may already have design constraints that have to be satisfied. For instance, the material the structure is made of may fix the damping rate, c. Then, only the parameters m and k can be adjusted. In addition, the mass may have to be within 10% of a specified value, for instance, which further restricts the range of values of overshoot and settling time. The stiffness is often designed on the basis of the static deflection limitation.

For example, consider the system of Figure 1.10 and assume it is desired to choose values of m, c, and k so that ζ and ω_n specify a response with a settling time $t_s = 3.2$ units and a time to peak, t_p, of 1 unit. Then, Equations (1.32) and (1.34) imply that $\omega_n = 1/\zeta$ and that $\zeta = 1/\sqrt{1+\pi^2}$. This, unfortunately, also specifies the overshoot, since

$$\mathrm{OS} = \exp\left(\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}\right)$$

Thus, all three performance criteria cannot be satisfied. This leads the designer to have to make compromises, to reconfigure the structure, or to add additional components.

Hence, in order to meet vibration criteria such as avoiding resonance, it may be necessary in many instances to alter the structure by adding vibration absorbers or isolators (Machinante,

1984, or Rivin, 2003). Another possibility is to use active vibration control and feedback methods. Both of these approaches are discussed in Chapters 6 and 7.

As just mentioned, the choice of the physical parameters m, c, and k determines the shape of the response of the system. In this sense, the choice of these parameters can be considered as the design of the structure. Passive control can also be considered as a redesign process of changing these parameters on an already existing structure to produce a more desirable response. For instance, some mass could be added to a given structure to lower its natural frequency. Although passive control or redesign is generally the most efficient way to control or shape the response of a structure, the constraints on m, c, and k are often such that the desired response cannot be obtained. Then the only alternative, short of starting over, is to try active control.

There are many different types of active control methods, and only a few will be considered to give the reader a feel for the connection between the vibration and control disciplines. As mentioned earlier, the comments made in this text on control should not be considered as a substitute for studying standard control or linear system texts. Output feedback control is briefly introduced here and discussed in more detail in Chapter 7.

First, a clarification of the difference between active and passive control is in order. Basically, an active control system uses some external adjustable or active (for example, electronic) device, called an actuator, to provide a means of shaping or controlling the response. Passive control, on the other hand, depends only on a fixed (passive) change in the physical parameters of the structure. Active control often depends on current measurements of the response of the system, and passive control does not. Active control requires an external energy source, and passive control typically does not.

Feedback control consists of measuring the output, or response, of the structure and using that measurement to determine the force to apply to the structure to obtain a desired response. The device used to measure the response (sensor), the device used to apply the force (actuator), and any electronics required to transfer the sensor signal into an actuator command (control law) make up the *control hardware*. This is illustrated by the block diagram in Figure 1.23. Systems with feedback are referred to as closed-loop systems, while control systems without feedback are called open-loop systems, as illustrated in Figures 1.23 and 1.24 respectively. A major difference between open-loop and closed-loop control is simply that closed-loop control depends on information about the response of the system, and open-loop control does not.

The rule that defines how the measurement from the sensor is used to command the actuator to effect the system is called the *control law*, denoted by $H(s)$ in Figure 1.23. Much

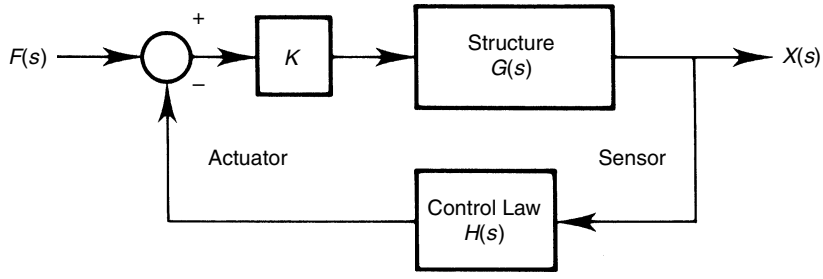

Figure 1.23 Closed-loop system.

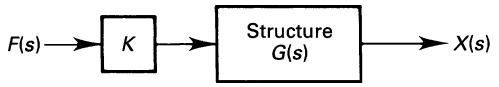

Figure 1.24 Open-loop system.

of control theory focuses on clever ways to choose the control law to achieve a desired response.

A simple open-loop control law is to multiply (or amplify) the response of the system by a constant. This is referred to as *constant gain control*. The magnitude of the frequency response function for the system in Figure 1.23 is multiplied by the constant K, called the *gain*. The frequency domain equivalent of Figure 1.23 is

$$\frac{X(s)}{F(s)} = KG(s) = \frac{K}{ms^2 + cs + k} \tag{1.57}$$

where the plant is taken to be a single-degree-of-freedom model of structure. In the time domain, this becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = Kf(t) \tag{1.58}$$

The effect of this open-loop control is simply to multiply the steady state response by K and to increase the value of the peak response, M_p.

On the other hand, the closed-loop control, illustrated in Figure 1.23, has the equivalent frequency domain representation given by

$$\frac{X(s)}{F(s)} = \frac{KG(s)}{1 + KG(s)H(s)} \tag{1.59}$$

If the feedback control law is taken to be one that measures both the velocity and position, multiplies them by some constant gains g_1 and g_2 respectively, and adds the result, the control law $H(s)$ is given by

$$H(s) = g_1 s + g_2 \tag{1.60}$$

As the velocity and position are the *state variables* for this system, this control law is called *full state feedback*, or PD control (for position and derivative). In this case, Equation (1.56) becomes

$$\frac{X(s)}{F(s)} = \frac{K}{ms^2 + (Kg_1 + c)s + (Kg_2 + k)} \tag{1.61}$$

The time domain equivalent of this equation (obtained by using the inverse Laplace transform) is

$$m\ddot{x}(t) + (c + Kg_1)\dot{x}(t) + (k + Kg_2)x(t) = Kf(t) \tag{1.62}$$

By comparing Equations (1.58) and (1.62), the versatility of closed-loop control versus open-loop, or passive, control is evident. In many cases the choice of values of K, g_1, and

g_2 can be made electronically. By using a closed-loop control, the designer has the choice of three more parameters to adjust than are available in the passive case to meet the desired specifications.

On the negative side, closed-loop control can cause some difficulties. If not carefully designed, a feedback control system can cause an otherwise stable structure to have an unstable response. For instance, suppose the goal of the control law is to reduce the stiffness of the structure so that the natural frequency is lower. From examining Equation (1.62), this would require g_2 to be a negative number. Then, suppose that the value of k was overestimated and g_2 calculated accordingly. This could result in the coefficient of $x(t)$ becoming negative, causing instability. That is, the response of Equation (1.62) would be unstable if $(k + Kg_2) < 0$. This would amount to positive feedback and is not likely to arise by design on purpose, but it can happen if the original parameters are not well known. On physical grounds, instability is possible because the control system is adding energy to the structure. One of the major concerns in designing high-performance control systems is to maintain stability. This introduces another constraint on the choice of the control gains and is discussed in more detail in Chapter 7. Of course, closed-loop control is also expensive because of the sensor, actuator, and electronics required to make a closed-loop system. On the other hand, closed-loop control can always result in better performance provided the appropriate hardware is available.

Feedback control uses the measured response of the system to modify and add back into the input to provide an improved response. Another approach to improving the response consists of producing a second input to the system that effectively cancels the disturbance to the system. This approach, called *feedforward control*, uses knowledge of the response of a system at a point to design a control force that, when subtracted from the uncontrolled response, yields a new response with desired properties, usually a response of zero. Feedforward control is most commonly used for high-frequency applications and in acoustics (for noise cancellation) and is not considered here. An excellent treatment of feedforward controllers is given by Fuller, Elliot, and Nelson (1996).

1.9 NONLINEAR VIBRATIONS

The force versus displacement plot for a spring in Figure 1.3 curves off after the deflections and forces become large enough. Before enough force is applied to deform permanently or break the spring, the force deflection curve becomes nonlinear and curves away from a straight line, as indicated in Figure 1.25. Therefore, rather than the linear spring relationship $f_k = kx$, a model such as $f_k = \alpha x - \beta x^3$, called a *softening spring*, might better fit the curve. This nonlinear spring behavior greatly changes the physical nature of the vibratory response and complicates the mathematical description and analysis to the point where numerical integration usually has to be employed to obtain a solution. Stability analysis of nonlinear systems also becomes more complicated.

In Figure 1.25 the force–displacement curves for three springs are shown. Notice that the linear range for the two nonlinear springs is a good approximation until about 1.8 units of displacement or 2000 units of force. If the spring is to be used beyond that range, then the linear vibration analysis of the preceding sections no longer applies.

Consider, then, the equation of motion of a system with a nonlinear spring of the form

$$m\ddot{x}(t) + \alpha x(t) - \beta x^3(t) = 0 \tag{1.63}$$

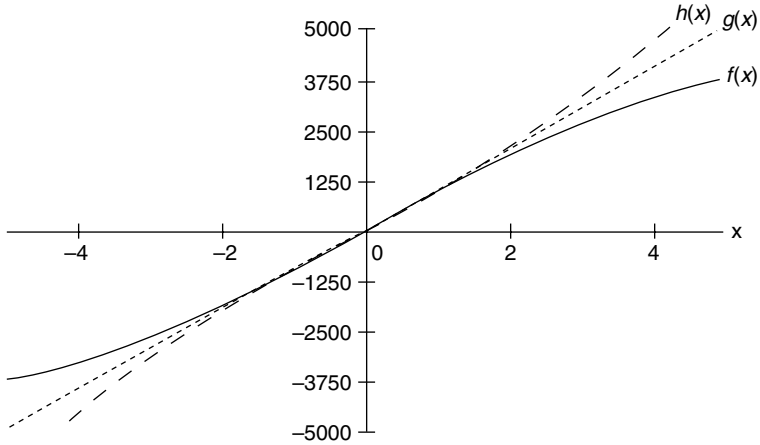

Figure 1.25 Force deflection curves for three different springs, indicating their linear range. The curve $g(x) = kx$ is a linear spring, the curve $f(x) = kx - bx^3$ is called a softening spring, and the curve $h(x) = kx + bx^3$ is called a hardening spring (here $k = 1000$, $b = 10$).

which is subject to two initial conditions. In the linear system there was only one equilibrium point to consider, $v(t) = x(t) = 0$. As will be shown in the following, the nonlinear system of Equation (1.63) has more than one equilibrium position. The equilibrium point of a system, or set of governing equations, may be defined best by first placing the equation of motion into state-space form.

A general single-degree-of-freedom system may be written as

$$\ddot{x}(t) + f(x(t), \dot{x}(t)) = 0 \tag{1.64}$$

where the function f can take on any form, linear or nonlinear. For example, for a linear spring–mass–damper system the function f is just $f(x, \dot{x}) = 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t)$, which is a linear function of the state variables of position and velocity. For a nonlinear system the function f will be some nonlinear function of the state variables. For instance, for the nonlinear spring of Equation (1.63), the function is $f(x, \dot{x}) = \alpha x - \beta x^3$.

The general state-space model of Equation (1.64) is written by defining the two state variables: the position $x_1 = x(t)$, and the velocity $x_2 = \dot{x}(t)$. Then, Equation (1.64) can be written as the first-order pair

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -f(x_1, x_2) \end{aligned} \tag{1.65}$$

This state-space form of the equation of motion is used for numerical integration, in control analysis, and for formally defining an equilibrium position. Define the state vector, $\mathbf{x}$, and a nonlinear vector function $\mathbf{F}$, as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{F} = \begin{bmatrix} x_2(t) \\ -f(x_1, x_2) \end{bmatrix} \tag{1.66}$$

Then, Equations (1.65) may be written in the simple form of a vector equation

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1.67}$$

An *equilibrium point* of this system, denoted by $\mathbf{x}_e$, is defined to be any value of the vector $\mathbf{x}$ for which $\mathbf{F}(\mathbf{x})$ is identically zero (called zero-phase velocity). Thus, the equilibrium point is any vector of constants, $\mathbf{x}_e$, that satisfies the relations

$$\mathbf{F}(\mathbf{x}_e) = \mathbf{0} \tag{1.68}$$

Placing the linear single-degree-of-freedom system into state-space form then yields

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -2\zeta\omega_n x_2 - \omega_n^2 x_1 \end{bmatrix} \tag{1.69}$$

The equilibrium of a linear system is thus the solution of the vector equality

$$\begin{bmatrix} x_2 \\ -2\zeta\omega_n x_2 - \omega_n^2 x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.70}$$

which has the single solution $x_1 = x_2 = 0$. Thus, for any linear system the equilibrium point is a single point consisting of the origin. On the other hand, the equilibrium condition of the soft spring system of Equation (1.63) requires that

$$\begin{aligned} x_2 &= 0 \\ -\alpha x_1 + \beta x_1^3 &= 0 \end{aligned} \tag{1.71}$$

Solving for x_1 and x_2 yields the *three* equilibrium points

$$\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{bmatrix} \tag{1.72}$$

In principle, the soft spring system of Equation (1.63) could oscillate around any of these equilibrium points, depending on the initial conditions. Each of these equilibrium points may also have a different stability property.

The existence of multiple equilibrium points also complicates the notion of stability introduced in Section 1.7. In particular, solutions near each equilibrium point could potentially have different stability behavior. Since the initial conditions may determine the equilibrium around which the solution centers, the behavior of a nonlinear system will depend on the initial conditions. In contrast, for a linear system with fixed parameters the solution form is the same regardless of the initial conditions. This represents another important difference to consider when working with nonlinear components.

1.10 COMPUTING AND SIMULATION IN MATLAB

Modern computer codes such as MATLAB make the visualization and computation of vibration problems available without much programming effort. Such codes can help enhance

understanding through plotting responses, can help find solutions to complex problems lacking closed-form solutions through numerical integration, and can often help with symbolic computations. Plotting certain parametric relations or plotting solutions can often aid in visualizing the nature of relationships or the effect of parameter changes on the response. Most of the plots used in this text are constructed from simple MATLAB commands, as the following examples illustrate. If you are familiar with MATLAB, you may wish to skip this section.

MATLAB is a high-level code, with many built-in commands for numerical integration (simulation), control design, performing matrix computations, symbolic manipulation, etc. MATLAB has two areas to enter information. The first is the command window, which is an active area where the entered command is compiled as it is entered. Using the command window is somewhat like a calculator. The second area is called an m-file, which is a series of commands that are saved and then called from the command window for execution. All of the plots in the figures in this chapter can be reproduced using these simple commands.

Example 1.10.1

Plot the free response of the underdamped system to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$ for values of $m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$, using MATLAB and Equation (1.13).

To enter numbers in the command window, just type a symbol and use an equal sign after the blinking cursor. The following entries in the command window will produce the plot of Figure 1.26. Note that the prompt symbol `'>>'` is provided by MATLAB and the information following it is code typed in by

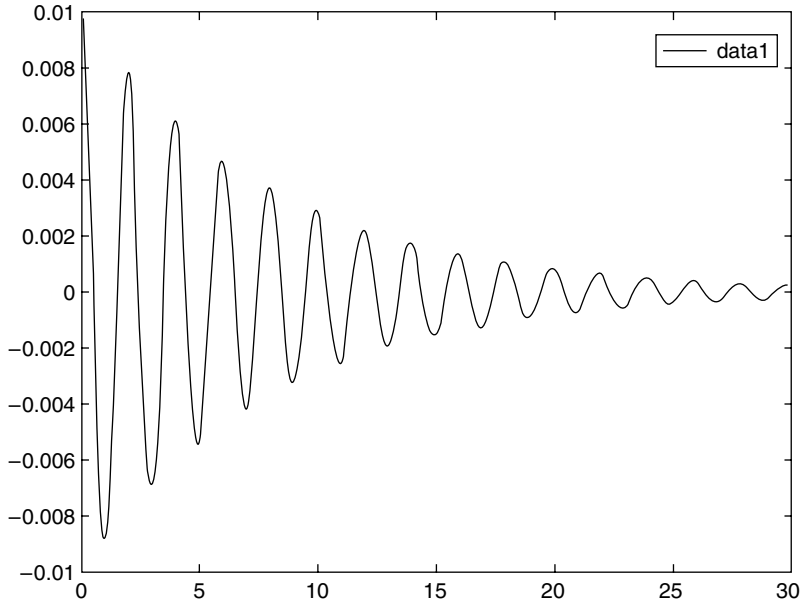

Figure 1.26 Response of an underdamped system ($m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$) to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$, plotted using MATLAB.

the user. The symbol % is used to indicate comments, so that anything following this symbol is ignored by the code and is included to help explain the situation. A semicolon typed after a command suppresses the command from displaying the output. MATLAB uses matrices and vectors so that numbers can be entered and computed in arrays. Thus, there are two types of multiplication. The notation a*b is a vector operation demanding that the number of rows of a be equal to the number of columns of b. The product a.*b, on the other hand, multiplies each element of a by the corresponding element in b.

```
>> clear % used to make sure no previous values are stored
>> %assign the initial conditions, mass, damping and stiffness
>> x0=0.01;v0=0.0;m=100;c=25;k=1000;
>> %compute omega and zeta, display zeta to check if underdamped
>> wn=sqrt(k/m);z=c/(2*sqrt(k*m))
z =
0.0395
>> %compute the damped natural frequency
>> wd=wn*sqrt(1-z^2);
>> t=(0:0.01:15*(2*pi/wn));%set the values of time from 0 in
increments of 0.01 up to 15 periods
>> x=exp(-z*wn*t).*(x0*cos(wd*t)+((v0+z*wn*x0)/wd)*sin(wd*t));
% computes x(t)
>> plot(t,x)%generates a plot of x(t) vs t
```

The MATLAB code used in this example is not the most efficient way to plot the response and does not show the detail of labeling the axis, etc., but is given as a quick introduction.

The next example illustrates the use of m-files in a numerical simulation. Instead of plotting the closed-form solution given in Equation (1.13), the equation of motion can be numerically integrated using the ode command in MATLAB. The ode45 command uses a fifth-order Runge–Kutta, automated time step method for numerically integrating the equation of motion (see, for instance, Pratap, 2002).

In order to use numerical integration, the equations of motion must first be placed in first-order, or state-space, form, as done in Equation (1.69). This state-space form is used in MATLAB to enter the equations of motion.

Vectors are entered in MATLAB by using square brackets, spaces, and semicolons. Spaces are used to separate columns, and semicolons are used to separate rows, so that a row vector is entered by typing

```
>> u = [1  -1  2]
```

which returns the row

```
u =   1  -1  2
```

and a column vector is entered by typing

```
>> u = [1;  -1;  2]
```

which returns the column

```
u =
    1
    -1
    2
```

To create a list of formulae in an m-file, choose 'New' from the file menu and select 'm-file'. This will display a text editor window, in which you can enter commands. The following example illustrates the creation of an m-file and how to call it from the command window for numerical integration of the equation of motion given in example 1.10.1.

Example 1.10.2

Numerically integrate and plot the free response of the underdamped system to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$ for values of $m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$, using MATLAB and equation (1.13).

First create an m-file containing the equation of motion to be integrated and save it. This is done by selecting 'New' and 'm-File' from the File menu in MATLAB, then typing

```
----------------------
Function xdot=f2(t,x)
c=25; k = 1000; m = 100;
% set up a column vector with the state equations
xdot=[x(2); -(c/m)*x(2)-(k/m)*x(1)];
----------------------
```

This file is now saved with the name `f2.m`. Note that the name of the file must agree with the name following the equal sign in the first line of the file. Now open the command window and enter the following:

```
>> ts=[0 30]; % this enters the initial and final time
>> x0 =[0.01 0]; % this enters the initial conditions
>> [t, x]=ode45('f2',ts,x0);
>> plot(t,x(:,1))
```

The third line of code calls the Runge–Kutta program ode45 and the state equations to be integrated contained in the file named `f2.m`. The last line plots the simulation of the first state variable $x_1(t)$ which is the displacement, denoted by `x(:,1)` in MATLAB. The plot is given in Figure 1.27.

Note that the plots of Figures 1.26 and 1.27 look the same. However, Figure 1.26 was obtained by simply plotting the analytical solution, whereas the plot of Figure 1.27 was obtained by numerically integrating the equation of motion. The numerical approach can be used successfully to obtain the solution of a nonlinear state equation, such as Equation (1.63), just as easily.

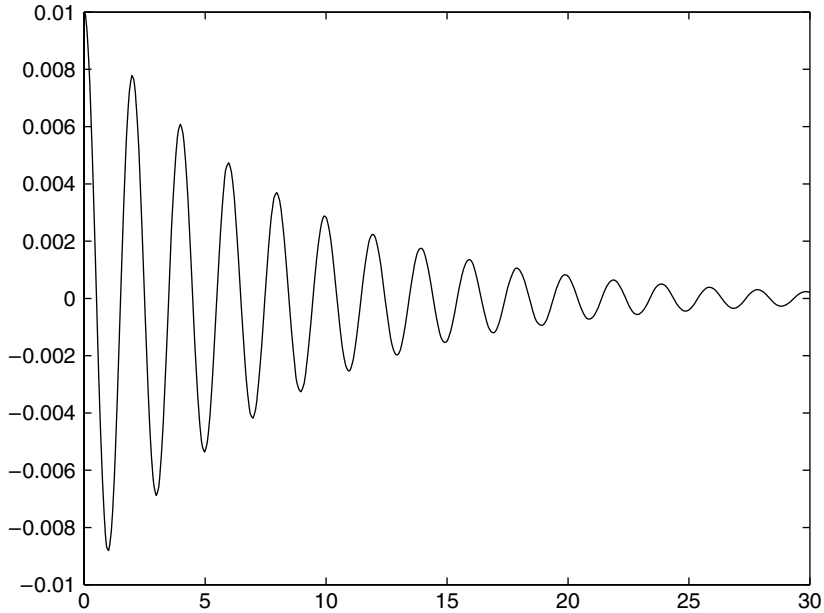

Figure 1.27 Plot of the numerical integration of the underdamped system of example 1.10.1 resulting from the MATLAB code given in example 1.10.2.

The forced response can also be computed using numerical simulation, and this is often more convenient than working through an analytical solution when the forcing functions are discontinuous or not made up of simple functions. Again, the equations of motion (this time with the forcing function) must be placed in state-space form. The equation of motion for a damped system with a general applied force is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

In state-space form this expression becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{1.73}$$

where $f(t) = F(t)/m$ and $F(t)$ is any function that can be integrated. The following example illustrates the procedure in MATLAB.

Example 1.10.3

Use MATLAB to compute and plot the response of the following system:

$$100\ddot{x}(t) + 10\dot{x}(t) + 500x(t) = 150\cos 5t, \qquad x_0 = 0.01, \qquad v_0 = 0.5$$

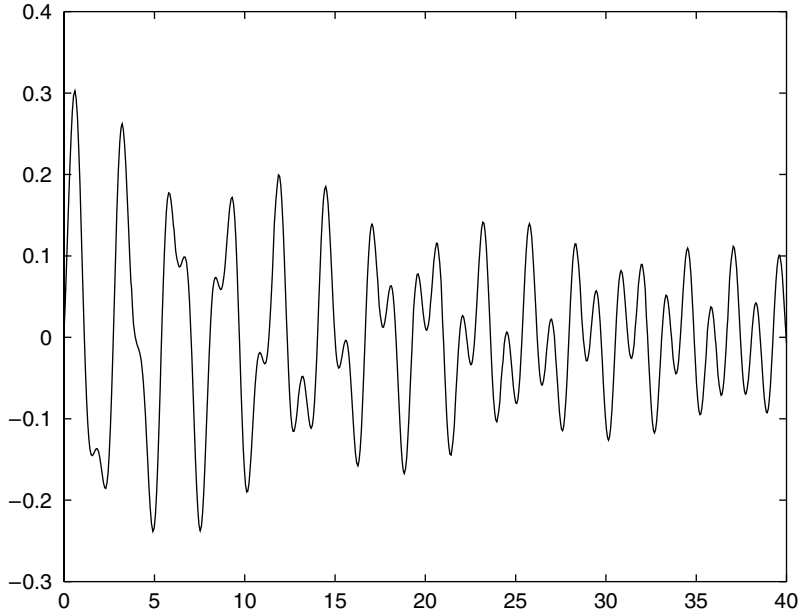

Figure 1.28 A plot of the numerical integration of the damped forced system resulting from the MATLAB code given in example 1.10.3.

The MATLAB code for computing these plots is given. First an m-file is created with the equation of motion given in first-order form:

```
---------------------------------------------
function v=f(t,x)
m=100; k=500; c=10; Fo=150; w=5;
v=[x(2); x(1)*-k/m+x(2)*-c/m + Fo/m*cos(w*t)];
---------------------------------------------
```

Then the following is typed in the command window:

```
>>clear all
>>xo=[0.01; 0.5]; %enters the initial conditions
>>ts=[0 40]; %enters the initial and final times
>>[t,x]=ode45('f',ts,xo); %calls the dynamics and integrates
>>plot(t,x(:,1)) %plots the result
```

This code produces the plot given in Figure 1.28. Note that the influence of the transient dynamics dies off owing to the damping after about 20 s.

Such numerical integration methods can also be used to simulate the nonlinear systems discussed in the previous section. Use of high-level codes in vibration analysis such as MATLAB

is now commonplace and has changed the way vibration quantities are computed. More detailed codes for vibration analysis can be found in Inman (2001). In addition there are many books written on using MATLAB (such as Pratap, 2002) as well as available online help.

CHAPTER NOTES

This chapter attempts to provide an introductory review of vibrations and to expand the discipline of vibration analysis and design by intertwining elementary vibration topics with the disciplines of design, control, stability, and testing. An early attempt to relate vibrations and control at an introductory level was made by Vernon (1967). More recent attempts have been made by Meirovitch (1985, 1990) and by Inman (1989) – the first edition of this text. Leipholz and Abdel-Rohman (1986) take a civil engineering approach to structural control. The latest attempts to combine vibration and control are by Preumont (2002) and Benaroya (2004) who also provides an excellent treatment of uncertainty in vibrations. The information contained in Sections 1.2 and 1.3, and in part of Section 1.4 can be found in every introductory text on vibrations, such as my own (Inman, 2001) and such as the standards by Thomson and Dahleh (1993), Rao (2004), and Meirovitch (1986). A complete summary of most vibration-related topics can be found in Braun, Ewins, and Rao (2002) and in Harris and Piersol (2002).

A good reference for vibration measurement is McConnell (1995). The reader is encouraged to consult a basic text on control such as the older text by Melsa and Schultz (1969), which contains some topics omitted from modern texts, or by Kuo and Golnaraghi (2003), which contains more modern topics integrated with MATLAB. These two texts also provide background to specifications and transfer functions given in Sections 1.4 and 1.5 as well as feedback control discussed in Section 1.8. A complete discussion of plant identification as presented in Section 1.6 can be found in Melsa and Schultz (1969). The excellent text by Fuller, Elliot, and Nelson (1996) examines the control of high-frequency vibration. Control is introduced here not as a discipline by itself but rather as a design technique for vibration engineers. A standard reference on stability is Hahn (1967), which provided the basic ideas for Section 1.7. The topic of flutter and self-excited vibrations is discussed in Den Hartog (1985). Nice introductions to nonlinear vibration can be found in Virgin (2000), in Worden and Tomlinson (2001), and in the standards by Nayfeh and Mook (1978) and Nayfeh and Balachandra (1995).

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PROBLEMS

1.1 Derive the solution of $m\ddot{x} + kx = 0$ and sketch your result (for at least two periods) for the case $x_0 = 1$, $v_0 = \sqrt{5}$, and $k/m = 4$.

1.2 Solve $m\ddot{x} - kx = 0$ for the case $x_0 = 1$, $v_0 = 0$, and $k/m = 4$, for $x(t)$ and sketch the solution.

1.3 Derive the solutions given in the text for $\zeta > 1$, $\zeta = 1$, and $0 < \zeta < 1$ with x_0 and v_0 as the initial conditions (i.e., derive Equations 1.14 through 1.16 and the corresponding constants).

1.4 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the solution.

1.5 Prove that $\zeta = 1$ corresponds to the smallest value of c such that no oscillation occurs. (*Hint*: Let $\lambda = -b$, b a positive real number, and differentiate the characteristic equation.)

1.6 Calculate t_p, OS, T_d, M_p, and BW for a system described by

$$2\ddot{x} + 0.8\dot{x} + 8x = f(t)$$

where $f(t)$ is either a unit step function or a sinusoidal, as required.

1.7 Derive an expression for the forced response of the undamped system

$$m\ddot{x}(t) + kx(t) = F_0 \sin \omega t, \qquad x(0) = x_0, \qquad \dot{x}(0) = v_0$$

to a sinusoidal input and nonzero initial conditions. Compare your result with Equation (1.21) with $\zeta = 0$.

1.8 Compute the total response to the system

$$4\ddot{x}(t) + 16x(t) = 8 \sin 3t, \qquad x_0 = 1\,\text{mm}, \qquad v_0 = 2\,\text{mm/s}$$

1.9 Calculate the maximum value of the peak response (magnification factor) for the system in Figure 1.18 with $\zeta = 1/\sqrt{2}$.

1.10 Derive Equation (1.22).

1.11 Calculate the impulse response function for a critically damped system.

1.12 Solve for the forced response of a single-degree-of-freedom system to a harmonic excitation with $\zeta = 1.1$ and $\omega_n^2 = 4$. Plot the magnitude of the steady state response versus the driving frequency. For what value of ω_n is the response a maximum (resonance)?

1.13 Calculate the compliance transfer function for the system described by the differential equation

$$a\dddddot{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = f(t)$$

where $f(t)$ is the input and $x(t)$ is a displacement. Also, calculate the frequency response function for this system.

1.14 Derive Equation (1.51).

1.15 Plot (using a computer) the unit step response of a single-degree-of-freedom system with $\omega_n^2 = 4, k = 1$ for several values of the damping ratio ($\zeta = 0.01, 0.1, 0.5$, and 1.0).

1.16 Let ω_p denote the frequency at which the peak response occurs [Equation (1.22)]. Plot ω_p/ω_n versus ζ and ω_d/ω_n versus ζ and comment on the difference as a function of ζ.

1.17 For the system of problem 1.6, construct the Bode plots for (a) the inertance transfer function, (b) the mobility transfer function, (c) the compliance transfer function, and (d) the Nyquist diagram for the compliance transfer function.

1.18 Discuss the stability of the following system: $2\ddot{x}(t) - 3\dot{x}(t) + 8x(t) = -3\dot{x}(t) + \sin 2t$.

1.19 Using the system of problem 1.6, refer to Equation (1.62) and choose the gains K, g_1, and g_2 so that the resulting closed-loop system has a 5% overshoot and a settling time of less than 10.

1.20 Calculate an allowable range of values for the gains K, g_1, and g_2 for the system of problem 1.6, such that the closed-loop system is stable and the formulae for overshoot and peak time of an underdamped system are valid.

1.21 Compute a feedback law with full state feedback [of the form given in Equation (1.62)] that stabilizes (makes asymptotically stable) the system $4\ddot{x}(t) + 16x(t) = 0$ and causes the closed-loop settling time to be 1 s.

1.22 Compute the equilibrium positions of the pendulum equation $m\ell^2\ddot{\theta}(t) + mg\ell \sin \theta(t) = 0$.

1.23 Compute the equilibrium points for the system defined by

$$\ddot{x} + \beta\dot{x} + x + x^2 = 0$$

1.24 The linearized version of the pendulum equation is given by

$$\ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$$

Use numerical integration to plot the solution of the nonlinear equation of problem 1.22 and this linearized version for the case where

$$g = 0.01\ell, \qquad \theta(0) = 0.1\,\text{rad}, \qquad \dot{\theta}(0) = 0.1\,\text{rad/s}$$

Compare your two simulations.

2
Lumped-parameter Models

2.1 INTRODUCTION

Many physical systems cannot be modeled successfully by the single-degree-of-freedom model discussed in Chapter 1. That is, to describe the motion of the structure or machine, several coordinates may be required. Such systems are referred to as *lumped-parameter systems* to distinguish them from the distributed-parameter systems considered in Chapters 9 through 12. Such systems are also called *lumped-mass systems* and sometimes *discrete systems* (referring to mass not time). Each lumped mass potentially corresponds to six degrees of freedom. Such systems are referred to as multiple-degree-of-freedom systems (often abbreviated MDOF). In order to keep a record of each coordinate of the system, vectors are used. This is done both for ease of notation and to enable vibration theory to take advantage of the power of linear algebra. This section organizes the notation to be used throughout the rest of the text and introduces several common examples.

Before the motions of such systems are considered, it is important to recall the definition of a matrix and a vector as well as a few simple properties of each. Vectors were used in Sections 1.9 and 1.10, and are formalized here. If you are familiar with vector algebra, skip ahead to Equation (2.7). Let $\mathbf{q}$ denote a vector of dimension n defined by

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \tag{2.1}$$

Here, q_i denotes the ith element of vector $\mathbf{q}$. This is not to be confused with $\mathbf{q}_i$, which denotes the ith vector in a set of vectors. Two vectors $\mathbf{q}$ and $\mathbf{r}$ of the same dimension (n in this case) may be summed under the rule

$$\mathbf{q} + \mathbf{r} = \mathbf{s} = \begin{bmatrix} q_1 + r_1 \\ q_2 + r_2 \\ \vdots \\ q_n + r_n \end{bmatrix} \tag{2.2}$$

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

and multiplied under the rule (dot product or inner product)

$$\mathbf{q} \cdot \mathbf{r} = \sum_{i=1}^{n} q_i r_i = \mathbf{q}^T \mathbf{r} \tag{2.3}$$

where the superscript T denotes the transpose of the vector. Note that the inner product of two vectors, $\mathbf{q}^T\mathbf{q}$, yields a scalar. With $\mathbf{q}$ given in Equation (2.1) as a column vector, $\mathbf{q}^T$ is a row vector given by $\mathbf{q}^T = [q_1 \; q_2 \cdots q_n]$.

The product of a scalar, a, and a vector, $\mathbf{q}$, is given by

$$a\mathbf{q} = [aq_1 \quad aq_2 \quad \cdots \quad aq_n]^T \tag{2.4}$$

If the zero vector is defined as a vector of proper dimension whose entries are all zero, then rules (2.2) and (2.4) define a linear vector space (see Appendix B) of dimension n.

A matrix, A, is defined as a rectangular array of numbers (scalars) of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

consisting of m rows and n columns. Matrix A is said to have the dimensions $m \times n$. For the most part, the equations of motion used in vibration theory result in real-valued square matrices of dimensions $n \times n$. Each of the individual elements of a matrix are labeled as a_{ik}, which denotes the element of the matrix in the position of the intersection of the ith row and kth column.

Two matrices of the same dimensions can be summed by adding the corresponding elements in each position, as illustrated by a 2×2 example:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \tag{2.5}$$

The product of two matrices is slightly more complicated and is given by the formula $C = AB$, where the resulting matrix C has elements given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{2.6}$$

and is only defined if the number of columns of A is the same as the number of rows of B, which is n in Equation (2.6). Note that the product BA is not defined for this case unless A and B are of the same dimensions. In most cases, the product of two square matrices or the product of a square matrix and a vector are used in vibration analysis. Note that a vector is just a rectangular matrix with the smallest dimension being 1.

With this introduction, the equations of motion for lumped-parameter systems can be discussed. These equations can be derived by several techniques, such as Newton's laws and Lagrange's equations (Goldstein, 2002), linear graph methods (Rosenberg and Karnopp, 1983, or Shearer, Murphy, and Richardson, 1967), or finite element methods (Hughes, 2000).

These analytical models can be further refined by comparison with experimental data (Friswell and Mottershead, 1995). Modeling techniques will not be discussed in this text, since the orientation is analysis.

All the methods just mentioned yield equations that can be put in the following form:

$$A_1\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + A_3\mathbf{q} = \mathbf{f}(t) \tag{2.7}$$

which is a vector differential equation with matrix coefficients. Here, $\mathbf{q} = \mathbf{q}(t)$ is an n vector of time-varying elements representing the displacements of the masses in the lumped-mass model. The vectors $\ddot{\mathbf{q}}$ and $\dot{\mathbf{q}}$ represent the accelerations and velocities respectively. The overdot means that each element of $\mathbf{q}$ is differentiated with respect to time. The vector $\mathbf{q}$ could also represent a generalized coordinate that may not be an actual physical coordinate or position but is related, usually in a simple manner, to the physical displacement. The coefficients A_1, A_2, and A_3 are n square matrices of constant real elements representing the various physical parameters of the system. The n vector $\mathbf{f} = \mathbf{f}(t)$ represents applied external forces and is also time varying. The system of Equation (2.7) is also subject to initial conditions on the initial displacement $\mathbf{q}(0) = \mathbf{q}_0$ and initial velocities $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$.

The matrices A_i will have different properties depending on the physics of the problem under consideration. As will become evident in the remaining chapters, the mathematical properties of these matrices will determine the physical nature of the solution $\mathbf{q}(t)$, just as the properties of the scalars m, c, and k determined the nature of the solution $x(t)$ to the single-degree-of-freedom system of Equation (1.6) in Chapter 1.

In order to understand these properties, there is need to classify square matrices further. The *transpose* of a matrix A, denoted by A^T, is the matrix formed by interchanging the rows of A with the columns of A. A square matrix is said to be *symmetric* if it is equal to its transpose, i.e., $A = A^T$. Otherwise it is said to be *asymmetric*. A square matrix is said to be *skew-symmetric* if it satisfies $A = -A^T$. It is useful to note that any real square matrix may be written as the sum of a symmetric matrix and a skew-symmetric matrix. To see this, notice that the symmetric part of A, denoted by A_s, is given by

$$A_s = \frac{A^T + A}{2} \tag{2.8}$$

and the skew-symmetric part of A, denoted by A_{ss}, is given by

$$A_{ss} = \frac{A - A^T}{2} \tag{2.9}$$

so that $A = A_s + A_{ss}$.

With these definitions in mind, Equation (2.7) can be rewritten as

$$M\ddot{\mathbf{q}} + (D + G)\dot{\mathbf{q}} + (K + H)\mathbf{q} = \mathbf{f} \tag{2.10}$$

where $\mathbf{q}$ and $\mathbf{f}$ are as before but

$M = M^T$ = mass, or inertia, matrix
$D = D^T$ = viscous damping matrix (usually denoted by C)

$G = -G^T$ = gyroscopic matrix
$K = K^T$ = stiffness matrix
$H = -H^T$ = circulatory matrix (constraint damping)

Some physical systems may also have asymmetric mass matrices (see, for instance, Soom and Kim, 1983).

In the following sections, each of these matrix cases will be illustrated by a physical example indicating how such forces may arise. The physical basis for the nomenclature is as expected. The mass matrix arises from the inertial forces in the system, the damping matrix arises from dissipative forces proportional to velocity, and the stiffness matrix arises from elastic forces proportional to displacement. The nature of the skew-symmetric matrices G and H is pointed out by the examples that follow.

Symmetric matrices and the physical systems described by Equation (2.10) can be further characterized by defining the concept of the definiteness of a matrix. A matrix differs from a scalar in many ways. One way in particular is the concept of sign, or ordering. In the previous chapter it was pointed out that the sign of the coefficients in a single-degree-of-freedom system determined the stability of the resulting motion. Similar results will hold for Equation (2.10) when the 'sign of a matrix' is interpreted as the definiteness of a matrix (this is discussed in Chapter (4). The definiteness of an $n \times n$ symmetric matrix is defined by examining the sign of the scalar $\mathbf{x}^T A\,\mathbf{x}$, called the *quadratic* form of A, where $\mathbf{x}$ is an arbitrary n-dimensional real vector. Note that, if $A = I$, the identity matrix consisting of ones along the diagonal and all other elements zero, then

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T I\mathbf{x} = \mathbf{x}^T\mathbf{x} = x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2$$

which is clearly quadratic.

In particular, the symmetric matrix A is said to be:

- *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for all nonzero real vectors $\mathbf{x}$ and $\mathbf{x}^T A\mathbf{x} = 0$ if and only if $\mathbf{x}$ is zero;
- *positive semidefinite* (or *nonnegative definite*) if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all nonzero real vectors $\mathbf{x}$ (here, $\mathbf{x}^T A\mathbf{x}$ could be zero for some nonzero vector $\mathbf{x}$);
- *indefinite* (or *sign variable*) if $(\mathbf{x}^T A\mathbf{x})(\mathbf{y}^T A\mathbf{y}) < 0$ for some pair of real vectors $\mathbf{x}$ and $\mathbf{y}$.

Definitions of *negative definite* and *negative semidefinite* should be obvious from the first two of the above.

In many cases, M, D, and K will be positive definite, a condtion that ensures stability, as illustrated in Chapter 4, and follows from physical considerations, as it does in the single-degree-of-freedom case.

2.2 CLASSIFICATIONS OF SYSTEMS

This section lists the various classifications of systems modeled by Equation (2.10) that are commonly found in the literature. Each particular engineering application may have a slightly different nomenclature and jargon. The definitions presented here are meant only to simplify discussion in this text and are intended to conform with most other references.

These classifications are useful in verbal communication of the assumptions made when discussing a vibration problem. In the following, each word in italics is defined to imply the assumptions made in modeling the system under consideration.

The phrase *conservative system* usually refers to systems of the form

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}(t) \tag{2.11}$$

where M and K are both symmetric and positive definite. However, the system

$$M\ddot{\mathbf{q}}(t) + G\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}(t) \tag{2.12}$$

(where G is skew-symmetric) is also conservative, in the sense of conserving energy, but is referred to as a *gyroscopic conservative system*, or an *undamped gyroscopic system*. Such systems arise naturally when spinning motions are present, such as in a gyroscope, rotating machine, or spinning satellite.

Systems of the form

$$M\ddot{\mathbf{q}}(t) + D\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}(t) \tag{2.13}$$

(where M, D, and K are all positive definite) are referred to as *damped nongyroscopic systems* and are also considered to be damped conservative systems in some instances, although they certainly do not conserve energy. Systems with symmetric and positive definite coefficient matrices are sometimes referred to as *passive* systems.

The classification of systems with asymmetric coefficients is not as straightforward, as the classification depends on more matrix theory than has yet been presented. However, systems of the form

$$M\ddot{\mathbf{q}}(t) + (K + H)\mathbf{q}(t) = \mathbf{f}(t) \tag{2.14}$$

are referred to as *circulatory systems* (Ziegler, 1968). In addition, systems of the more general form

$$M\ddot{\mathbf{q}}(t) + D\dot{\mathbf{q}}(t) + (K + H)\mathbf{q}(t) = \mathbf{f}(t) \tag{2.15}$$

result from dissipation referred to as *constraint damping* as well as external damping in rotating shafts. Combining all of these effects provides motivation to study the most general system of the form of Equation (2.10), i.e.,

$$M\ddot{\mathbf{q}}(t) + (D + G)\dot{\mathbf{q}}(t) + (K + H)\mathbf{q}(t) = \mathbf{f}(t) \tag{2.16}$$

This expression is the most difficult model considered in the first half of the text. To be complete, however, it is appropriate to mention that this model of a structure does not account for time-varying coefficients or nonlinearities, which are sometimes present. Physically, the expression represents the most general forces considered in the majority of linear vibration analysis, with the exception of certain external forces. Mathematically, Equation (2.16) will be further classified in terms of the properties of the coefficient matrices.

2.3 FEEDBACK CONTROL SYSTEMS

The vibrations of many structures and devices are controlled by sophisticated control methods. Examples of the use of feedback control to remove vibrations range from machine tools to tall buildings and large spacecraft. As discussed in Section 1.8, one popular way to control the vibrations of a structure is to measure the position and velocity vectors of the structure and to use that information to drive the system in direct proportion to its positions and velocities. When this is done, Equation (2.16) becomes

$$M\ddot{\mathbf{q}}(t) + (G+D)\dot{\mathbf{q}}(t) + (K+H)\mathbf{q}(t) = -K_p\mathbf{q}(t) - K_v\dot{\mathbf{q}}(t) + \mathbf{f}(t) \tag{2.17}$$

which is the vector version of Equation (1.62). Here, K_p and K_v are called feedback gain matrices.

Obviously, the control system (2.17) can be rewritten in the form of Equation (2.10) by moving the terms $K_p\mathbf{q}$ and $K_v\dot{\mathbf{q}}$ to the left side of system (2.17). Thus, analysis performed on Equation (2.10) will also be useful for studying the vibrations of structures controlled by position and velocity feedback (called *state feedback*).

Control and system theory (see, for instance, Rugh, 1996) are very well developed areas. Most of the work carried out in linear systems has been developed for systems in *state-space* form introduced in Section 1.9 to define equilibrium and in Section 1.10 for numerical integration. The state-space form is

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{2.18}$$

where $\mathbf{x}$ is called the *state vector*, A is the *state matrix*, and B is the *input matrix*. Here, $\mathbf{u}$ is the applied force, or control, vector. Much software and many theoretical developments exist for systems in the form of Equation (2.18). Equation (2.10) can be written in this form by several very simple transformations. To this end, let $\mathbf{x}_1 = \mathbf{q}$ and $\mathbf{x}_2 = \dot{\mathbf{q}}$; then, Equation (2.16) can be written as the two coupled equations

$$\begin{aligned} \dot{\mathbf{x}}_1(t) &= \mathbf{x}_2(t) \\ M\dot{\mathbf{x}}_2(t) &= -(D+G)\mathbf{x}_2(t) - (K+H)\mathbf{x}_1(t) + \mathbf{f}(t) \end{aligned} \tag{2.19}$$

This form allows the theory of control and systems analysis to be directly applied to vibration problems.

Now suppose there exists a matrix, M^{-1}, called the inverse of M, such that $M^{-1}M = I$, the $n \times n$ identity matrix. Then, Equation (2.19) can be written as

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}(K+H) & -M^{-1}(D+G) \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \mathbf{f}(t) \tag{2.20}$$

where the state matrix A is

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}(K+H) & -M^{-1}(D+G) \end{bmatrix}$$

and the input matrix B is

$$B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

and where $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}^T = \begin{bmatrix} \mathbf{q} & \dot{\mathbf{q}} \end{bmatrix}^T$. This expression has the same form as Equation (2.18). The state-space approach has made a big impact on the development of control theory and, to a lesser but still significant extent, on vibration theory. This state-space representation also forms the approach used for numerical simulation and calculation for vibration analysis.

The matrix inverse M^{-1} can be calculated by a number of different numerical methods readily available in most mathematical software packages along with other factorizations. This is discussed in detail in Appendix B. A simple calculation will show that for second-order matrices of the form

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse is given by

$$M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $\det(M) = ad - cb$. This indicates that, if $ad = cb$, then M is called *singular* and M^{-1} does not exist. In general, it should be noted that, if a matrix inverse exists, then it is unique. Furthermore, the inverse of a product of square matrices is given by $(AB)^{-1} = B^{-1}A^{-1}$.

The following selection of examples illustrates the preceding ideas and notations. Additional useful matrix definitions and concepts are presented in the next chapter and as the need arises.

2.4 EXAMPLES

This section lists several examples to illustrate how the various symmetries and asymmetries arise from mechanical devices. No attempt is made here to derive the equations of motion. The derivations may be found in the references listed or in most texts on dynamics. In most cases the equations of motion follow from a simple free body force diagram (Newton's laws).

Example 2.4.1

The first example (Meirovitch, 1980, p. 37) consists of a rotating ring of negligible mass containing an object of mass m that is free to move in the plane of rotation, as indicated in Figure 2.1. In the figure, k_1 and k_2 are both positive spring stiffness values, c is a damping rate (also positive), and Ω is the constant angular velocity of the disc. The linearized equations of motion are

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \ddot{\mathbf{q}} + \left\{ \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} + 2\, m\Omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \dot{\mathbf{q}} + \begin{bmatrix} k_1 + k_2 - m\Omega^2 & 0 \\ 0 & 2k_2 - m\Omega^2 \end{bmatrix} \mathbf{q} = \mathbf{0} \quad (2.21)$$

where $\mathbf{q} = [x(t) \quad y(t)]^T$ is the vector of displacements. Here, M, D, and K are symmetric, while G is skew-symmetric, so the system is a damped gyroscopic system.

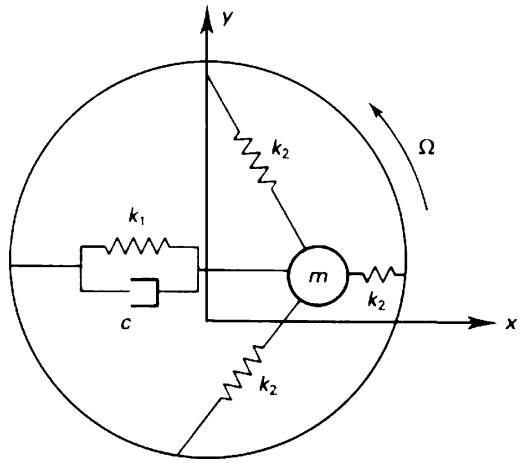

Figure 2.1 Schematic of a simplified model of a spinning satellite.

Note that, for any arbitrary nonzero vector $\mathbf{x}$, the quadratic form associated with M becomes

$$\mathbf{x}^T M\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = m(x_1^2 + x_2^2) > 0$$

Therefore, $\mathbf{x}^T M\mathbf{x}$ is positive for all nonzero choices of $\mathbf{x}$ and the matrix M is (symmetric) positive definite (and nonsingular, meaning that M has an inverse). Likewise, the quadratic form for the damping matrix becomes

$$\mathbf{x}^T \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = cx_1^2 > 0$$

Note here that, while this quadratic form will always be nonnegative, the quantity $\mathbf{x}^T D\mathbf{x} = cx_1^2 = 0$ for the nonzero vector $\mathbf{x} = [0 \quad 1]^T$, so that D is only positive semidefinite (and singular). Easier methods for checking the definiteness of a matrix are given in the next chapter.

The matrix G for the preceding system is obviously skew-symmetric. It is interesting to calculate its quadratic form and note that for any real value of $\mathbf{x}$

$$\mathbf{x}^T G\mathbf{x} = 2\ m\Omega(x_1 x_2 - x_2 x_1) = 0 \tag{2.22}$$

This is true in general. The quadratic form of any-order real skew-symmetric matrix is zero.

Example 2.4.2

A gyroscope is an instrument based on using gyroscopic forces to sense motion and is commonly used for navigation as well as in other applications. One model of a gyroscope is shown in Figure 2.2. This is a three-dimensional device consisting of a rotating disc (with electric motor), two gimbals (hoops), and a platform, all connected by pivots or joints. The disc and the two gimbals each have three moments of inertia – one around each of the principal axes of reference. There is also a stiffness associated with each pivot. Let A, B, and C be the moments of inertia of the disc (rotor);

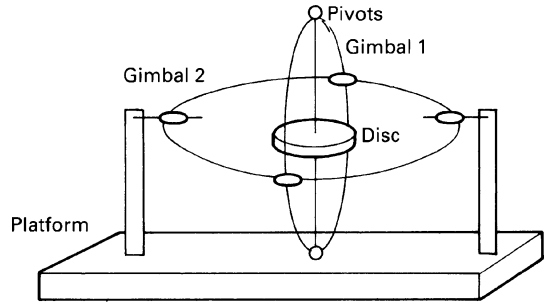

Figure 2.2 Schematic of a simplified model of a gyroscope.

a_i, b_i, and c_i, for $i = 1, 2$, be the principal moments of inertia of the two gimbals; k_{11} and k_{12} be the torsional stiffness elements connecting the driveshaft to the first and second gimbal respectively; k_{21} and k_{22} be the torsional stiffness elements connecting the rotor to the first and second gimbal respectively; and let Ω denote the constant rotor speed. The equations of motion are then given by Burdess and Fox (1978) to be

$$\begin{bmatrix} A+a_1 & 0 \\ 0 & B+b_1 \end{bmatrix} \ddot{\mathbf{q}} + \Omega(A+B-c) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_{11}+k_{22}+2\Omega^2(C-B+c_1-b_1) & 0 \\ 0 & k_{12}+k_{21}+2\Omega^2(C-A+c_2-b_2) \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{2.23}$$

where $\mathbf{q}$ is the displacement vector of the rotor.

Here we note that M is symmetric and positive definite, D and H are zero, G is nonzero skew-symmetric, and the stiffness matrix K will be positive definite if $(C-B+c_1-b_1)$ and $(C-A+c_2-b_2)$ are both positive. This is a conservative gyroscopic system.

Example 2.4.3

A lumped-parameter version of the rod illustrated in Figure 2.3 yields another example of a system with asymmetric matrix coefficients. The rod is called Pflüger's rod, and its equation of motion and the lumped-parameter version of it used here can be found in Huseyin (1978) or by using the methods of Chapter 13. The equations of motion are given by the vector equation

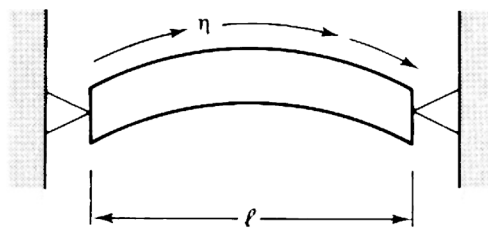

Figure 2.3 Pflüger's rod: a simply supported bar subjected to uniformly distributed tangential forces.

$$\frac{m}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ddot{\mathbf{q}}+\left\{\frac{EI\pi^4}{\ell^3}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 8\end{bmatrix}-\eta\begin{bmatrix}\frac{\pi^2}{4} & \frac{32}{9}\\ \frac{8}{9} & \pi^2\end{bmatrix}\right\}\mathbf{q}=\mathbf{0} \tag{2.24}$$

where η is the magnitude of the applied force, EI is the flexural rigidity, m is the mass density, ℓ is the length of the rod, and $\mathbf{q}(t)=[x_1(t)\ x_2(t)]^T$ represents the displacements of two points on the rod.

Again, note that the mass matrix is symmetric and positive definite. However, owing to the presence of the so-called follower force η, the coefficient of $\mathbf{q}(t)$ is not symmetric. Using Equations (2.8) and (2.9), the stiffness matrix K becomes

$$K=\begin{bmatrix}\frac{EI\pi^4}{2\ell^3}-\frac{\pi}{4}\eta & -\frac{20}{9}\eta\\ -\frac{20}{9}\eta & \frac{8EI\pi^4}{\ell^3}-\eta\pi^2\end{bmatrix}$$

and the skew-symmetric matrix H becomes

$$H=\frac{12\eta}{9}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Example 2.4.4

As an example of the types of matrix that can result from feedback control systems, consider the two-degree-of-freedom system in Figure 2.4. The equations of motion for this system are

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\ddot{\mathbf{q}}+\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}\dot{\mathbf{q}}+\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}\mathbf{q}=\begin{bmatrix}0\\ f_2\end{bmatrix} \tag{2.25}$$

where $\mathbf{q}=[x_1(t)\ x_2(t)]^T$. If a control force of the form $f_2=-g_1x_1-g_2x_2$, where g_1 and g_2 are constant gains, is applied to the mass m_2, then Equation (2.25) becomes

$$\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\ddot{\mathbf{q}}+\begin{bmatrix}c_1+c_2 & -c_2\\ -c_2 & c_2\end{bmatrix}\dot{\mathbf{q}}+\begin{bmatrix}k_1+k_2 & -k_2\\ -k_2+g_1 & k_2+g_2\end{bmatrix}\mathbf{q}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{2.26}$$

Equation (2.26) is analogous to Equation (1.62) for a single-degree-of-freedom system.

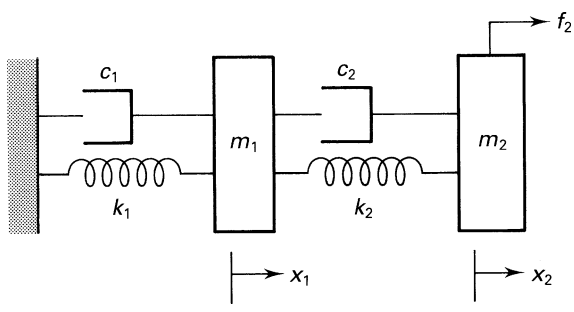

Figure 2.4 Schematic of a two-degree-of-freedom system.

Now the displacement coefficient matrix is no longer symmetric owing to the feedback gain constant g_1. Since just x_1 and x_2 are used in the control, this is called *position feedback*. Velocity feedback could result in the damping matrix becoming asymmetric as well. Without the control, this is a damped symmetric system or nonconservative system. However, with position and/or velocity feedback, the coefficient matrices become asymmetric, greatly changing the nature of the response and, as discussed in Chapter 4, the stability of the system.

These examples are referred to in the remaining chapters of the text, which develops theories to test, analyze, and control such systems.

2.5 EXPERIMENTAL MODELS

Many structures are not configured in nice lumped arrangements, as in examples 2.4.1, 2.4.2, and 2.4.4. Instead, they appear as distributed parameter arrangements (see Chapter 9), such as the rod of Figure 2.3. However, lumped-parameter multiple-degree-of-freedom models can be assigned to such structures on an experimental basis. As an example, a simple beam may be experimentally analyzed for the purpose of obtaining an analytical model of the structure by measuring the displacement at one end that is due to a harmonic excitation ($\sin \omega t$) at the other end and sweeping through a wide range of driving frequencies, ω. Using the ideas of Section 1.6, a magnitude versus frequency relationship similar to Figure 2.5 may result. Because of the three very distinct peaks in Figure 2.5, one is tempted to model the structure as a three-degree-of-freedom system (corresponding to the three resonances). In fact, if each peak is thought of as a single-degree-of-freedom system, using the formulations of Section 1.6 yields a value for m_i, k_i, and c_i (or ω_i and ζ_i) for each of the three peaks ($i = 1, 2$, and 3). A reasonable model for the system *might* then be

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{2.27}$$

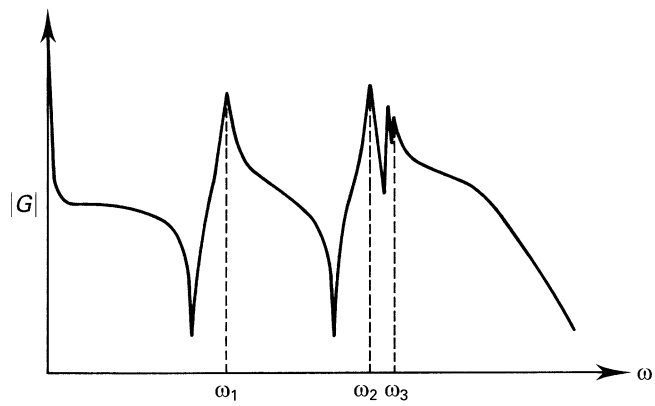

Figure 2.5 Experimentally obtained magnitude versus frequency plot for a simple beam-like structure.

which is referred to as a *physical model*. Alternatively, values of ω_i and ζ_i could be used to model the structure by the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\mathbf{r}} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} \mathbf{r} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \mathbf{r} = \mathbf{0} \tag{2.28}$$

which is referred to as a *modal model*. The problem with each of these models is that it is not clear what physical motion to assign to the coordinates $q_i(t)$ or $r_i(t)$. In addition, as discussed in Chapter 8, it is not always clear that each peak corresponds to a single resonance (however, phase plots of the experimental transfer function can help).

Such models, however, are useful for discussing the vibrational responses of the structure and will be considered in more detail in Chapter 8. The point in introducing this model here is to note that experimental methods can result in viable analytical models of structures directly, and that these models are fundamentally based on the phenomenon of resonance.

2.6 INFLUENCE METHODS

As mentioned in Section 2.1, there are many methods that can be used to determine a model of a structure. One approach is the concept of influence coefficients, which is discussed here because it yields a physical interpretation of the elements of the matrices in Equation (2.7). The influence coefficient idea extends the experiment suggested in Figure 1.2 to multiple-degree-of-freedom systems.

Basically, a coordinate system denoted by the vector $\mathbf{q}(t)$ is chosen arbitrarily, but it is based on as much knowledge of the dynamic behavior of the structure as possible. Note that this procedure will not produce a unique model because of this arbitrary choice of the coordinates. Each coordinate, $x_i(t)$, is used to define a degree of freedom of the structure. At each coordinate a known external force, denoted by $p_i(t)$, is applied. In equilibrium, this force must be balanced by the internal forces acting at that point. These internal forces are modeled to be the internal inertial force, denoted by f_{m_i}, the internal damping force, denoted by f_{d_i}, and the elastic force, denoted by f_{k_i}.

The equilibrium is expressed as

$$p_i(t) = f_{m_i}(t) + f_{d_i}(t) + f_{k_i}(t) \tag{2.29}$$

Since this must be true for each coordinate point, the n equations can be written as the single vector equation

$$\mathbf{p}(t) = \mathbf{f}_m(t) = \mathbf{f}_d(t) + \mathbf{f}_k(t) \tag{2.30}$$

Each of the terms in Equation (2.30) can be further expressed as the sum of forces due to the influence of each of the other coordinates. In particular, k_{ij} is defined to be the constant of proportionality (or slope of Figure 1.3) between the force at point i that is due to a unit displacement at point j ($x_j = 1$). This constant is called the *stiffness influence coefficient*.

Because the structure is assumed to be linear, the elastic force is due to the displacements developed at each of the other coordinates. The resulting force is then the sum

$$f_{k_i} = \sum_{j=1}^{n} k_{ij} x_j(t) \tag{2.31}$$

Writing down all i equations results in the relation

$$\mathbf{f}_k = K\,\mathbf{x} \tag{2.32}$$

where K is the $n \times n$ stiffness matrix or stiffness influence matrix with elements k_{ij} and $\mathbf{x}^T = [x_1 x_2 \ldots x_n]$. In full detail, Equation (2.32) is

$$\begin{bmatrix} f_{k_1} \\ f_{k_2} \\ \vdots \\ f_{k_n} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \cdots k_{1n} \\ k_{21} & k_{22} \cdots k_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ k_{n_1} & k_{n_2} \cdots k_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2.33}$$

The element k_{im} is essentially the force at point i required to produce displacements $x_i = 1$ and $x_m = 0$ for all values m between 1 and n excluding the value of i. Equation (2.33) can be inverted as long as the coefficient matrix K has an inverse. Solving Equation (2.33) for $\mathbf{x}$ then yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ a_{n_1} & a_{n_2} \quad \cdots a_{nn} \end{bmatrix} \begin{bmatrix} f_{k_1} \\ f_{k_2} \\ \vdots \\ f_{k_n} \end{bmatrix} \tag{2.34}$$

or

$$\mathbf{x} = A\,\mathbf{f}_k \tag{2.35}$$

If each f_{k_i} is set equal to zero except for f_{k_m}, which is set equal to unity, then the ith equation of system (2.34) yields

$$x_i = a_{i1} f_{k_1} + a_{i2} f_{k_2} + \cdots + a_{in} f_{k_n} = a_{im} \tag{2.36}$$

Thus a_{im} is the displacement at point i, i.e., x_i, that is due to a unit force applied at point m. The quantity a_{im} is called a *flexibility influence coefficient.*

This procedure can be repeated for the inertial forces and the viscous damping forces. In the case of inertial forces, the *inertial influence coefficient* is defined as the constant of proportionality between the force at point i and the acceleration at point j ($\ddot{\mathbf{x}}_j = 1$) of unit magnitude. These coefficients are denoted by m_{ij} and define the force $\mathbf{f}_m$ by

$$\mathbf{f}_m(t) = M\ddot{\mathbf{x}}(t) \tag{2.37}$$

where the mass matrix M has elements m_{ij}. Likewise, the *damping influence coefficient* is defined to be the constant of proportionality between the force at point i and the velocity at point j ($\dot{x}_j = 1$) of unit magnitude. These coefficients are denoted by d_{ij} and define the force $\mathbf{f}_d$ as

$$\mathbf{f}_d(t) = D\dot{x}(t) \tag{2.38}$$

where the damping matrix D has elements d_{ij}.

Combining the preceding four equations then yields the equations of motion of the structure in the standard form

$$M\ddot{\mathbf{q}}(t) + D\dot{\mathbf{q}}(t) + K\,\mathbf{q}(t) = \mathbf{0} \tag{2.39}$$

where the coefficients have the physical interpretation just given, and $\mathbf{q} = \mathbf{x}$ is used to conform with the generalized notation of earlier sections.

2.7 NONLINEAR MODELS AND EQUILIBRIUM

If one of the springs in Equation (2.21) is stretched beyond its linear region, then Newton's law would result in a multiple-degree-of-freedom system with nonlinear terms. For such systems the equations of motion become coupled nonlinear equations instead of coupled linear equations. The description of the nonlinear equations of motion can still be written in vector form, but this does not result in matrix coefficients, and therefore linear algebra does not help. Rather the equations of motion are written in the state-space form of Equation (1.67), repeated here:

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{2.40}$$

As in Section 1.9, Equation (2.40) is used to define the equilibrium position of the system. Unlike the linear counterpart, there will be multiple equilibrium positions defined by solutions to the nonlinear algebraic equation [Equation (1.68)]

$$F(\mathbf{x}_e) = \mathbf{0}$$

The existence of these multiple equilibrium solutions forms the first basic difference between linear and nonlinear systems. In addition to being useful for defining equilibria, Equation (2.40) is also useful for numerically simulating the response of a nonlinear system with multiple degrees of freedom.

If one of the springs is nonlinear or if a damping element is nonlinear, the stiffness and/or damping terms can no longer be factored into a matrix times a vector but must be left in state-space form. Instead of Equation (2.13), the form of the equations of motion can only be written as

$$M\ddot{\mathbf{q}} + \mathbf{G}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0} \tag{2.41}$$

where $\mathbf{G}$ is some nonlinear vector function of the displacement and velocity vectors. As in the single-degree-of-freedom case discussed in Section 1.9, it is useful to place the system

in Equation (2.41) into state-space form by defining new coordinates corresponding to the position and velocity. To this end, let $\mathbf{x}_1 = \mathbf{q}$ and $\mathbf{x}_2 = \dot{\mathbf{q}}$ and multiply the above by M^{-1}. Then, the equation of motion for the nonlinear system of Equation (2.41) becomes

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{2.42}$$

Here

$$F(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_2 \\ -M^{-1}G(\mathbf{x}_1, \mathbf{x}_2) \end{bmatrix} \tag{2.43}$$

and the $2n \times 1$ state vector $\mathbf{x}(t)$ is

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{2.44}$$

Applying the definition given in Equation (1.68), equilibrium is defined as the vector $\mathbf{x}_e$ satisfying $F(\mathbf{x}_e) = \mathbf{0}$. The solution yields the various equilibrium points for the system.

Example 2.7.1

Compute the equilibrium positions for the linear system of Equation (2.20).
Equation (2.20) is of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \mathbf{f}$$

Equilibrium is concerned with the free response. Thus, set $\mathbf{f} = \mathbf{0}$ in this last expression, and the equilibrium condition becomes $A\mathbf{x} = \mathbf{0}$. As long as matrix A has an inverse, $A\mathbf{x} = \mathbf{0}$ implies that the equilibrium position is defined by $\mathbf{x}_e = \mathbf{0}$. This is the origin with zero velocity: $\mathbf{x}_1 = \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{0}$, or $\mathbf{x} = \mathbf{0}$ and $\dot{\mathbf{x}} = \mathbf{0}$. Physically, this condition is the rest position for each mass.

Much analysis and theory of nonlinear systems focuses on single-degree-of-freedom systems. Numerical simulation is used extensively in trying to understand the behavior of MDOF nonlinear systems. Here, we present a simple example of a two-degree-of-freedom nonlinear system and compute its equilibria. This is just a quick introduction to nonlinear MDOF, and the references should be consulted for a more detailed understanding.

Example 2.7.2

Consider the two-degree-of-freedom system in Figure 2.4, where spring $k_1(q_1)$ is driven into its nonlinear region so that $k_1(q_1) = k_1 q_1 - \beta q_1^3$ and the force and dampers are set to zero. For convenience, let $m_1 = m_2 = 1$. Note that the coordinates are relabeled q_i to be consistent with the state-space coordinates. Determine the equilibrium points.

The equations of motion become

$$m_1\ddot{q}_1 = k_2(q_2 - q_1) - k_1 q_1 + \beta q_1^3$$
$$m_2\ddot{q}_2 = -k_2(q_2 - q_1)$$

Next, define the state variables as $x_1 = q_1$, $x_2 = \dot{x}_1 = q_1$, $x_3 = q_2$, and $x_4 = \dot{x}_3 = q_2$. Then, the equations of motion in first order can be written (for $m_1 = m_2 = 1$) as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = k_2(x_3 - x_1) - k_1 x_1 + \beta x_1^3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -k_2(x_3 - x_1)$$

In vector form this becomes

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ k_2(x_3 - x_1) - k_1 x_1 + \beta x_1^3 \\ x_4 \\ -k_2(x_3 - x_1) \end{bmatrix}$$

Setting $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ yields the equilibrium equations

$$\mathbf{F}(\mathbf{x}_e) = \begin{bmatrix} x_2 \\ k_2(x_3 - x_1) - k_1 x_1 + \beta x_1^3 \\ x_4 \\ -k_2(x_3 - x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This comprises four algebraic equations in four unknowns. Solving yields the three equilibrium points

$$\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{k_1/\beta} \\ 0 \\ \sqrt{k_1/\beta} \\ 0 \\ \sqrt{k_1/\beta} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{k_1/\beta} \\ 0 \\ -\sqrt{k_1/\beta} \\ 0 \end{bmatrix}$$

The first equilibrium vector corresponds to the linear system.

CHAPTER NOTES

The material in Section 2.1 can be found in any text on matrices or linear algebra. The classification of vibrating systems is discussed in Huseyin (1978), which also contains an excellent introduction to matrices and vectors. The use of velocity and position feedback as discussed in Section 2.3 is quite common in the literature but is usually not discussed for multiple-degree-of-freedom mechanical systems in control texts. The experimental models

of Section 2.5 are discussed in Ewins (2000). Influence methods are discussed in more detail in texts on structural dynamics, such as Clough and Penzien (1975).

As mentioned, there are several approaches to deriving the equation of vibration of a mechanical structure, as indicated by the references in Section 2.1. Many texts on dynamics and modeling are devoted to the topic of deriving equations of motion (Meirovitch, 1986). The interest here is in analyzing these models.

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PROBLEMS

For problems 2.1 through 2.5, consider the system described by

$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 6 & 2 \\ 0 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

2.1 Identify the matrices M, C, G, K, and H.

2.2 Which of these matrices are positive definite and why?

2.3 Write the preceding equations in the form $\mathbf{x} = A\,\mathbf{x}$ where $\mathbf{x}$ is a vector of four elements given by $\mathbf{x} = [\mathbf{q}\ \dot{\mathbf{q}}]^T$.

2.4 Calculate the definiteness of M, D, and K from problem 2.1 as well as the values of $\mathbf{x}^T G\,\mathbf{x}$ and $\mathbf{x}^T H\mathbf{x}$ for an arbitrary value of $\mathbf{x}$.

2.5 Calculate M^{-1}, D^{-1}, and K^{-1} as well as the inverse of $D+G$ and $K+H$ from problem 2.1 and illustrate that they are, in fact, inverses.

2.6 Discuss the definiteness of matrix K in example 2.4.3.

2.7 A and B are two real square matrices. Show by example that there exist matrices A and B such that $AB \neq BA$. State some conditions on A and B for which $AB = BA$.

2.8 Show that the ijth element of matrix C, where $C = AB$, the product of matrix A with matrix B, is the inner product of the vector consisting of the ith row of matrix A and the vector consisting of the jth column of matrix B.

2.9 Calculate the solution of Equation (2.27) to the initial conditions given by $\mathbf{q}^T(0) = \mathbf{0}$ and $\dot{\mathbf{q}}(0) = [0\ 1\ 0]^T$.

2.10 **(a)** Calculate the equation of motion in matrix form for the system in Figure 2.4 if the force applied at $f_1 = -g_1 x_2 - g_2 \dot{x}_2$ and $f_2 = -g_3 x_1 - g_4 \dot{x}_1$.

(b) Calculate f_1 and f_2 so that the resulting closed-loop system is diagonal (decoupled).

2.11 Show that, if A and B are any two real square matrices, then $(A + B)^T = A^T + B^T$.

2.12 Show, by using the definition in Equation (2.4), that, if $\mathbf{x}$ is a real vector and a is any real scalar, then $(a\mathbf{x})^T = a\mathbf{x}^T$.

2.13 Using the definition of the matrix product, show that $(AB)^T = B^T A^T$.

2.14 Show that Equation (2.13) can be written in symmetric first-order form $A\dot{\mathbf{x}} + B\mathbf{x} = \mathbf{F}$, where $\mathbf{x} = [\mathbf{q}^T\ \dot{\mathbf{q}}^T]$, $\mathbf{F} = [\mathbf{f}^T\ \mathbf{0}]^T$, and A and B are symmetric.

2.15 Compute the equilibrium positions for the system of example 2.7.2 for the case where the masses m_1 and m_2 are arbitrary and not equal.

3

Matrices and the Free Response

3.1 INTRODUCTION

As illustrated in Chapter 1, the nature of the free response of a single-degree-of-freedom system is determined by the roots of the characteristic equation [Equation (1.8)]. In addition, the exact solution is calculated using these roots. A similar situation exists for the multiple-degree-of-freedom systems described in the previous chapter. Motivated by the single-degree-of-freedom system, this chapter examines the problem of characteristic roots for systems in matrix notation and extends many of the ideas discussed in Chapter 1 to the multiple-degree-of-freedom systems described in Chapter 2. The mathematical tools needed to extend the ideas of Chapter 1 are those of linear algebra, which are introduced here in an informal way, as needed.

Chapter 2 illustrated that many types of mechanical system can be characterized by vector differential equations with matrix coefficients. Just as the nature of the scalar coefficients in the single-degree-of-freedom case determines the form of the response, the nature of the matrix coefficients determines the form of the response of multiple-degree-of-freedom systems.

In fact, if we attempt to follow the method of solving single-degree-of-freedom vibration problems in solving multiple-degree-of-freedom systems, we are led immediately to a standard matrix problem called the algebraic eigenvalue problem. This chapter introduces the matrix eigenvalue problem and applies it to the multiple-degree-of-freedom vibration problems introduced in Chapter 2. The eigenvalues and eigenvectors can be used to determine the time response to initial conditions by the process called modal analysis which is introduced here. The use of high-level codes such as MATLAB is introduced to compute mode shapes and natural frequencies. The chapter concludes with simulation of the time response to initial condition disturbances, using numerical integration as an alternative to modal analysis.

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

3.2 EIGENVALUES AND EIGENVECTORS

This section introduces topics from linear algebra and the matrix eigenvalue problem needed to study the vibrations of multiple-degree-of-freedom systems. Consider first the simple conservative vibration problem of Equation (2.11), repeated here:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$$

for the free response case where $\mathbf{F} = \mathbf{0}$. Since M is assumed to be positive definite, it has an inverse. Premultiplying the equation of motion by the matrix M^{-1} yields the following equation for the free response:

$$\ddot{\mathbf{q}} + M^{-1}K\mathbf{q} = \mathbf{0}$$

Following the mathematical approach of Section 1.2 and the physical notion that the solution should oscillate suggests that a solution may exist of the form of nonzero constant $\mathbf{u}$, in this case a vector, times the exponential $e^{\mu jt}$, i.e., $\mathbf{q}(t) = \mathbf{u}\, e^{\mu jt}$. Substitution of this expression into the preceding equation yields

$$-\mu^2\mathbf{u} + A\mathbf{u} = \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}$$

where $A = M^{-1}K$. Rearrangement of this expression yields the equation

$$A\mathbf{u} = \lambda\mathbf{u}, \qquad \mathbf{u} \neq \mathbf{0}$$

where $\lambda = \mu^2$ and $\mathbf{u}$ cannot be zero. This expression is exactly a statement of the matrix eigenvalue problem. As in the case of the single-degree-of-freedom system, the constants $\lambda = \mu^2$ characterize the natural frequencies of the system. With this as a motivation, the matrix eigenvalue problem is described in detail in this section and applied to the linear vibration problem in Section 3.3. Computational considerations are discussed in Section 3.8.

Square matrices can be characterized by their eigenvalues and eigenvectors, defined in this section. Let A denote an $n \times n$ square matrix. The scalar λ is defined as an *eigenvalue* of matrix A with corresponding *eigenvector* $\mathbf{x}$, which must be nonzero, if λ and $\mathbf{x}$ satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.1}$$

Geometrically, this means that the action of matrix A on vector $\mathbf{x}$ just changes the length of vector $\mathbf{x}$ and does not change its direction or orientation in space. Physically, the eigenvalue λ will yield information about the natural frequencies of the system described by matrix A. It should be noted that, if $\mathbf{x}$ is an eigenvector of A, then so is the vector $\alpha\mathbf{x}$, where α is any scalar. Thus, the magnitude of an eigenvector is arbitrary.

A rearrangement of Equation (3.1) yields

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3.2}$$

where I is the $n \times n$ identity matrix. Since $\mathbf{x}$ cannot be zero, by the definition of an eigenvector, the inverse of the matrix $(A - \lambda I)$ must not exist. That is, there cannot exist a matrix

$(A - \lambda I)^{-1}$ such that, $(A - \lambda I)^{-1}(A - \lambda I) = I$. Otherwise, premultiplying Equation (3.2) by this inverse would mean that the only solution to Equation (3.2) is $\mathbf{x} = \mathbf{0}$, violating the definition of an eigenvector. Matrices that do not have inverses are said to be *singular*, and those that do have an inverse are called *nonsingular*.

Whether or not a matrix is singular can also be determined by examining the *determinant* of the matrix. The determinant of an $n \times n$ matrix A is defined and denoted by

$$\det A = |A| = \sum_{s=1}^{n} a_{rs}|A_{rs}| \tag{3.3}$$

for any fixed r, where a_{rs} is the element of A at the intersection of the rth row and sth column of A and $|A_{rs}|$ is the determinant of the matrix formed from A by striking out the rth row and sth column multiplied by $(-1)^{r+s}$. An illustration of this for $n = 2$ is given in Section 2.3. The value of the determinant of a matrix is a unique scalar. In addition, it is a simple matter to show that

$$|A| = |A^T| \tag{3.4}$$

$$|AB| = |A||B| \tag{3.5}$$

Whether or not the determinant of a matrix is zero is very significant and useful. The following five statements are entirely equivalent:

1. A is nonsingular.
2. A^{-1} exists.
3. $\det A \neq 0$.
4. The only solution of the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
5. Zero is not an eigenvalue of A.

Note that, if $\det(A) = 0$, then A^{-1} does not exist, A is singular, and $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution; i.e., zero is an eigenvalue of A.

Example 3.2.1

The determinant of matrix A is calculated from Equation (3.3) (r is chosen as the fixed value 1) as

$$\det A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} = 1[(1)(3) - (1)(5)] - 3[(0)(3) - (1)(2)] - 2[(0)(5) - (1)(2)] = 8$$

Applying the concept of the determinant of a matrix to the eigenvalue problem stated in Equation (3.2) indicates that, if λ is to be an eigenvalue of matrix A, then λ must satisfy the equation

$$\det(A - \lambda I) = 0 \tag{3.6}$$

This expression results in a polynomial in λ, which is called the *characteristic equation* of matrix A.

Since A is an $n \times n$ matrix, Equation (3.6) will have n roots (or A will have n eigenvalues), which are denoted by λ_i. Then, Equation (3.6) can be rewritten as

$$\det(A - \lambda I) = \prod_{i=1}^{n} (\lambda - \lambda_i) = 0 \tag{3.7}$$

If λ_i happens to be a root that is repeated m_i times, then this becomes

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i}, \qquad \text{where } \sum_{i=1}^{k} m_i = n \tag{3.8}$$

Also, note from examination of Equation (3.2) that any given eigenvalue may have many eigenvectors associated with it. For instance, if $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, and α is any scalar, $\alpha\mathbf{x}$ is also an eigenvector of A with corresponding eigenvalue λ. Eigenvectors have several other interesting properties, many of which are useful in calculating the free response of a vibrating system.

The first property has to do with the concept of linear independence. A set of vectors, denoted by $\{\mathbf{e}_i\}_{i=1}^{n} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, is said to be *linearly independent*, or just *independent*, if

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n = \mathbf{0} \tag{3.9}$$

implies that each of the scalars α_i is zero. If this is not the case, i.e., if there exists one or more nonzero scalars α_i satisfying Equation (3.9), then the set of vectors $\{\mathbf{x}_i\}$ is said to be *linearly dependent*. The set of all linear combinations of all n-dimensional real vectors is called the *span* of the set of all n-dimensional real vectors. A set of n linearly independent vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is said to form a *basis* for the span of vectors of dimension n. This means that, if $\mathbf{x}$ is any vector of dimension n, then there exists a unique representation of vector $\mathbf{x}$ in terms of the basis vectors $\mathbf{e}_i$, given by

$$\mathbf{x} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \cdots + a_n \mathbf{e}_n \tag{3.10}$$

The coefficients a_i are sometimes called the *coordinates* of vector $\mathbf{x}$ in the basis $\{\mathbf{e}_i\}_{i=1}^{n}$. One familiar basis is the basis consisting of unit vectors $\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right)$ of a rectangular coordinate system, which forms a basis for the set of all three-dimensional real vectors.

Another important use of the idea of linear independence is contained in the concept of the *rank* of a matrix. The rank of a matrix is defined as the number of independent rows (or columns) of the matrix when the rows (columns) are treated like vectors. This property is used in stability analysis in Chapter 4, and in control in Chapter 7. Note that a square $n \times n$ matrix is nonsingular if and only if its rank is n (i.e., if and only if it has *full rank*).

If the scalar product, or dot product, of two vectors is zero, i.e., if $\mathbf{x}_i^T \mathbf{x}_j = 0$, then the two vectors are said to be *orthogonal*. If $\mathbf{x}_i^T \mathbf{x}_i = 1$, the vector $\mathbf{x}_i$ is called a *unit vector*. If a set of unit vectors is also orthogonal, i.e., if

$$\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

they are said to be an *orthonormal* set. Here, δ_{ij} is the Kronecker delta. Again, the familiar unit vectors of rectangular coordinate systems are an orthonormal set of vectors. Also, as discussed later, the eigenvectors of a symmetric matrix can be used to form an orthonormal set. This property is used in this chapter and again in Chapter 5 to solve various vibration problems.

Another important property of eigenvectors is as follows. If A is a square matrix and if the eigenvalues of A are distinct, then the eigenvectors associated with those eigenvalues are independent. If A is also symmetric, then an independent set of eigenvectors exist even if the eigenvalues are repeated. Furthermore, if zero is not an eigenvalue of A and A has eigenvalues λ_i with corresponding eigenvectors $\mathbf{x}_i$, then the eigenvectors of A^{-1} are also $\mathbf{x}_i$ and the eigenvalues are λ_i^{-1}. Thus, A and A^{-1} have related eigenvalue problems. Yet another useful result for the eigenvalues, λ_i, of matrix A is that the eigenvalues of $(A \pm \beta I)$ are just $\lambda_i \pm \beta$, where β is any scalar (called a shift).

Matrix A is *similar* to matrix B if there exists a nonsingular matrix P such that

$$A = PBP^{-1} \tag{3.11}$$

In this case, P is referred to as a *similarity transformation* (matrix) and may be used to change vibration problems from one coordinate system, which may be complicated, to another coordinate system that has a simple or canonical form.

The reason that similarity transformations are of interest is that, if two matrices are similar, they will have the same eigenvalues. Another way to state this is that similarity transformations preserve eigenvalues, or that eigenvalues are *invariant* under similarity transformations. Some square matrices are similar to diagonal matrices. Diagonal matrices consist of all zero elements except for those on the diagonal, making them easy to manipulate. The algebra of diagonal matrices is much like that of scalar algebra. This class of matrices is examined in detail next.

If matrix A is similar to a diagonal matrix, denoted by Λ, then A can be written as

$$A = P\Lambda P^{-1} \tag{3.12}$$

Postmultiplying this expression by P yields

$$AP = P\Lambda \tag{3.13}$$

Now, let the vectors p_i, $i = 1, 2, \ldots, n$, be the columns of matrix P, i.e.,

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \cdots \mathbf{p}_n] \tag{3.14}$$

Note that no $\mathbf{p}_i$ can be a zero vector since P is nonsingular. If λ_{ii} denotes the ith diagonal element of diagonal matrix Λ, then Equation (3.13) can be rewritten as the n separate equations

$$A\mathbf{p}_i = \lambda_{ii}\mathbf{p}_i, \qquad i = 1, 2, \ldots, n \tag{3.15}$$

Equations (3.15) state that $\mathbf{p}_i$ is the ith eigenvector of matrix A and that λ_{ii} is the associated eigenvalue, λ_i. The preceding observation can be summarized as follows:

1. If A is similar to a diagonal matrix, the diagonal elements of that matrix are the eigenvalues of A (i.e., $\lambda_i = \lambda_{ii}$).
2. A is similar to a diagonal matrix if and only if A has a set of n linearly independent eigenvectors.
3. If A has distinct eigenvalues, then it is similar to a diagonal matrix.

As an important note for vibration analysis: if A is a real symmetric matrix, then there exists a matrix P such that Equation (3.12) holds.

If the eigenvectors of A are linearly independent, they can be used to form an orthonormal set. Let $\mathbf{s}_i$ denote the orthonormal eigenvectors of A so that $\mathbf{s}_i^T\mathbf{s}_j = \delta_{ij}$, the Kronecker delta. Forming a matrix out of this set of normalized eigenvectors then yields

$$S = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3 \quad \ldots \quad \mathbf{s}_n] \tag{3.16}$$

Here, note that expanding the matrix product S^TS yields

$$S^TS = I \tag{3.17}$$

where I is the $n \times n$ identity matrix, because of the orthonormality of the rows and columns of S. Equation (3.17) implies immediately that $S^T = S^{-1}$. Such real-valued matrices are called *orthogonal matrices*, and Equation (3.12) can be written as

$$A = S\Lambda S^T \tag{3.18}$$

In this case, A is said to be *orthogonally similar* to Λ. (If S is complex valued, then $S^*S = I$, where the asterisk indicates the complex conjugate transpose of S, and S is called a Hermitian matrix.) Orthonormal sets are used to compute the time response of vibrating systems from the eigenvalues and eigenvectors.

Often it is convenient in vibration analysis to modify the concept of orthogonally similar matrices by introducing the concept of a weighting matrix. To this end, the eigenvectors of a matrix K can be normalized with respect to a second positive definite matrix, which in this case is chosen to be the matrix M. That is, the magnitude of the eigenvectors of K, $\mathbf{x}_i$, are chosen such that

$$\mathbf{x}_i^T M \mathbf{x}_j = \delta_{ij} \tag{3.19}$$

In this case the weighted transformation, denoted by S_m, has the following properties:

$$S_m^T M S_m = I \tag{3.20}$$

$$S_m^T K S_m = \mathrm{diag}[\omega_i^2] \tag{3.21}$$

where ω_i^2 denote the eigenvalues of matrix K. This is not to be confused with the diagonal matrix S^TKS, where S is made up of the (not weighted) eigenvectors of matrix K.

3.3 NATURAL FREQUENCIES AND MODE SHAPES

As mentioned previously, the concept of the eigenvalue of a matrix is closely related to the concept of natural frequency of vibration in mechanical structures, just as the roots of the characteristic equation and natural frequency of a single-degree-of-freedom system are related. To make the connection formally, consider again the undamped nongyroscopic conservative system described by

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0} \tag{3.22}$$

subject to initial conditions $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$. Here, the matrices M and K are assumed to be symmetric and positive definite.

In an attempt to solve Equation (3.22), a procedure similar to the method used to solve a single-degree-of-freedom system is employed by assuming a solution of the form

$$\mathbf{q}(t) = \mathbf{u}\, e^{\mu j t} \tag{3.23}$$

Here, $\mathbf{u}$ is a nonzero, unkown vector of constants, μ is a scalar value to be determined, $j = \sqrt{-1}$, and t is, of course, the time. Substitution of Equation (3.23) into Equation (3.22) yields

$$(-M\mu^2 + K)\mathbf{u}\, e^{\mu j t} = \mathbf{0} \tag{3.24}$$

This is identical to the procedure used in Section 1.2 for single-degree-of-freedom systems. Since $e^{\mu j t}$ is never zero for any value of μ or t, Equation (3.24) holds if and only if

$$(-M\mu^2 + K)\mathbf{u} = \mathbf{0} \tag{3.25}$$

This is starting to look very much like the eigenvalue problem posed in Equation (3.2). To make the analogy more complete, let $\mu^2 = \lambda$, so that Equation (3.25) becomes

$$(K - \lambda M)\mathbf{u} = \mathbf{0} \tag{3.26}$$

Since it is desired to calculate nonzero solutions of Equation (3.22), the vector $\mathbf{u}$ should be nonzero. This corresponds very well to the definition of an eigenvector, i.e., that it be nonzero. Eigenvalue problems stated in terms of two matrices of the form $A\mathbf{x} = \lambda B\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, are called *generalized eigenvalue problems*. Now recall, that a nonzero solution $\mathbf{u}$ of Equation (3.26) exists if and only if the matrix $(K - \lambda M)$ is singular or if and only if

$$\det(K - \lambda M) = \mathbf{0} \tag{3.27}$$

Next, note that, since M is positive definite, it must have an inverse. To see this, note that, if M^{-1}does not exist, then there is a nonzero vector $\mathbf{x}$ such that

$$M\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.28}$$

Premultiplying by $\mathbf{x}^T$ results in

$$\mathbf{x}^T M\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.29}$$

which clearly contradicts the fact that M is positive definite (recall the end of Section 2.1).

Since M^{-1} exists, $\det(M^{-1}) \neq 0$ and we can multiply Equation (3.27) by $\det(M^{-1})$ [invoking Equation (3.5)] to obtain

$$\det(M^{-1}K - \lambda I) = 0 \tag{3.30}$$

which is of the same form as Equation (3.6) used to define eigenvalues and yields a polynomial in λ of order n. As will be illustrated, each root of Equation (3.30), or eigenvalue of the matrix $M^{-1}K$, is the square of one of the natural frequencies of Equation (3.22).

There are several alternative ways to relate the eigenvalue problem of Equation (3.1) to the natural frequency problem of Equation (3.22). For instance, since M is positive definite, it has a positive definite square root. That is, there exists a unique positive definite matrix $M^{1/2}$ such that $M^{1/2}M^{1/2} = M$. The eigenvalues of $M^{1/2}$ are $\beta_i^{1/2}$, where β_i are the eigenvalues of M. Both M and its matrix square root have the same eigenvectors. Furthermore, if P is the matrix of eigenvectors of M, then

$$M^{1/2} = P\Lambda_M^{1/2}P^{-1} \tag{3.31}$$

where $\Lambda_M^{1/2}$ is a diagonal matrix with diagonal elements $\beta_i^{1/2}$. Many times in modeling systems, M is already diagonal, in which case the matrix square root is calculated by taking the square root of each of the diagonal elements. Systems with a non-diagonal mass matrix are called *dynamically coupled* systems. The existence of this matrix square root provides an important alternative relationship between matrix eigenvalues and vibrational natural frequencies and allows a direct analogy with the single-degree-of-freedom case. Matrix factorizations, such as the square root, lead to more computationally efficient algorithms (see Section 3.8).

Since $M^{1/2}$ is positive definite, it has an inverse $M^{-1/2}$, and pre- and postmultiplying Equation (3.27) by $\det(M^{-1/2})$ and factoring out -1 yields

$$\det(\lambda I - M^{-1/2}KM^{-1/2}) \quad = \quad 0 \tag{3.32}$$

Equation (3.32) is an alternative way of expressing the eigenvalue problem. The difference between Equations (3.32) and (3.30) is that the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is symmetric and positive definite, whereas $M^{-1}K$ is not necessarily symmetric. Matrix symmetry provides both a theoretical and computational advantage. Specifically, a symmetric matrix is similar to a diagonal matrix consisting of its eigenvalues along the diagonal, and the eigenvectors of a symmetric matrix are linearly independent and orthogonal. The corresponding differential equation then becomes

$$I\ddot{\mathbf{r}}(t) + M^{-1/2}KM^{-1/2}\mathbf{r}(t) = \mathbf{0} \tag{3.33}$$

where $\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t)$ has been substituted into Equation (3.22) and the result premultiplied by $M^{-1/2}$.

As expected and shown later, the numbers λ_i are directly related to the natural frequencies of vibration of the system described by Equation (3.22): $\omega_i^2 = \mu_i^2 = \lambda_i$. It is expected, as in the case of single-degree-of-freedom systems with no damping, that the natural frequencies will be such that the motion oscillates without decay. Mathematically, this result follows from realizing that the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is symmetric and positive definite, ensuring the nature of the natural frequencies and eigenvectors.

To see that a real, symmetric, positive definite matrix such as $\tilde{K} = M^{-1/2}KM^{-1/2}$ has positive real eigenvalues (and, hence, real eigenvectors) requires some simple manipulation of the definitions of these properties. First, note that, if $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, then

$$A\mathbf{x} = \lambda\mathbf{x} \tag{3.34}$$

Assuming that λ and $\mathbf{x}$ are complex and taking the conjugate transpose of this expression yields (because A is symmetric)

$$\mathbf{x}^* A = \mathbf{x}^* \lambda^* \tag{3.35}$$

Premultiplying Equation (3.34) by $\mathbf{x}^*$, postmultiplying Equation (3.35) by $\mathbf{x}$, and subtracting the two yields

$$0 = \mathbf{x}^* A\mathbf{x} - \mathbf{x}^* A\mathbf{x} = (\lambda - \lambda^*)\mathbf{x}^*\mathbf{x}$$

or, since $\mathbf{x} \neq \mathbf{0}$, that $\lambda = \lambda^*$. Hence, λ must be real valued.

Next, consider that A can be written as $A = S\Lambda S^T$. Therefore, for any and all arbitrary vectors $\mathbf{x}$,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T S\Lambda S^T\mathbf{x} = \mathbf{y}^T \Lambda\mathbf{y}$$

where $\mathbf{y} = S^T\mathbf{x}$ is also free to take on any real value. This can be expressed as

$$\mathbf{y}^T \Lambda\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2 > 0$$

since A is positive definite. If the vectors $\mathbf{y}_1 = [1\ 0\ 0\ \cdots\ 0]^T$, $\mathbf{y}_2 = [0\ 1\ 0\ \cdots\ 0]^T$, $\ldots$, $\mathbf{y}_n = [0\ 0\ 0\ \cdots\ 1]^T$ are, in turn, substituted into this last inequality, the result is $\lambda_i > 0$, for each of the n values of index i. Hence, a positive definite symmetric matrix has positive real eigenvalues (the converse is also true).

Applying this fact to Equation (3.32) indicates that each eigenvalue of the mass normalized stiffness matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is a positive real number. From Equation (3.25) we see that the natural frequencies of Equation (3.22) are $\mu = \omega$, where $\omega^2 = \lambda$, a positive real number. Hence, the coefficient of t in Equation (3.23) has the form $\omega = \pm\sqrt{\lambda}j$, just as in the single-degree-of-freedom case. The square roots of λ_i are the natural frequencies of the system, i.e., $\omega_i = \sqrt{\lambda_i}$, where i ranges from 1 to n, n being the number of degrees of freedom. That is, there is one natural frequency for each degree of freedom.

The concept of a positive definite matrix can also be related to conditions on the elements of the matrix in a useful manner. Namely, it can be shown that a symmetric matrix A is positive definite if and only if the leading principal minors of A are positive. That is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}$$

then A is positive definite if and only if

$$a_{11} > 0$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} > 0$$

$$\vdots$$

$$\det A > 0$$

This condition provides a connection between the condition that a matrix be positive definite and the physical parameters of the system. For example, the stiffness matrix of example 2.4.1 will be positive definite if and only if $k_1 + k_2 > m\Omega^2$ and $2k_2 > m\Omega^2$, by the preceding principle minor condition. That is, for $A = K$ in example 2.4.1, the first two conditions yield the two inequalities in k_i, m, and Ω. This provides physical insight as it indicates that stability may be lost if the system spins faster (Ω) than the stiffness can handle. These inequalities are very useful in vibration design and in stability analysis.

Another interesting fact about symmetric matrices is that their eigenvectors form a complete set, or a basis. Recall that a set of real vectors $\{\mathbf{u}_i\}$ of dimension n is a *basis* for the set of all real n-dimensional vectors if and only if they are linearly independent and every other real vector of dimension n can be written as a linear combination of $\mathbf{u}_i$. Thus, the solution $\mathbf{q}(t)$ can be expanded in terms of these eigenvectors. The set of eigenvectors of the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ forms a linearly independent set such that any vector of dimension n can be written as a linear combination of these vectors. In particular, the solution of the vibration problem can be expanded in terms of this basis.

Combining the preceding matrix results leads to the following solution for the response $\mathbf{r}(t)$. There are n solutions of Equation (3.33) of the form

$$\mathbf{r}_k(t) = \mathbf{u}_k\, e^{\mu_k jt} \tag{3.36}$$

As just shown, under the assumption that $M^{-1/2}KM^{-1/2}$ is positive definite, the numbers μ_k must all be of the form

$$\mu_k = \pm\sqrt{\lambda_k} \tag{3.37}$$

where λ_k are the positive eigenvalues of the matrix $M^{-1/2}KM^{-1/2}$. Combining Equation (3.36) and (3.37) it can be seen that each $\mathbf{r}_k(t)$ must have the form

$$\mathbf{r}_k(t) = \left(a_k\, e^{-\sqrt{\lambda_k}jt} + b_k\, e^{+\sqrt{\lambda_k}jt}\right)\mathbf{u}_k \tag{3.38}$$

where a_k and b_k are arbitrary constants. Since $\mathbf{u}_k$ are the eigenvectors of a symmetric matrix, they form a basis, so the n-dimensional vector $\mathbf{r}(t)$ can be expressed as a linear combination of these. That is,

$$\mathbf{r}(t) = \sum_{k=1}^{n} \left(a_k \, e^{-\sqrt{\lambda_k} jt} + b_k \, e^{\sqrt{\lambda_k} jt} \right) \mathbf{u}_k \tag{3.39}$$

where the arbitrary constants a_k and b_k can be determined from the initial conditions $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$. This amounts to solving the $2n$ algebraic equations given by

$$\begin{aligned} \mathbf{r}(0) &= \sum_{k=1}^{n} (a_k + b_k)\mathbf{u}_k \\ \dot{\mathbf{r}}(0) &= j \sum_{k=1}^{n} \sqrt{\lambda_k}(b_k - a_k)\mathbf{u}_k \end{aligned} \tag{3.40}$$

for the $2n$ constants a_k and b_k, $k = 1, \ldots, n$.

Since the symmetric properties of the matrix $M^{-1/2}KM^{-1/2}$ were used to develop the solution given by Equation (3.39), note that the solution expressed in Equation (3.39) is the solution of a slightly different problem to the solution $\mathbf{q}(t)$ of Equation (3.22). The two are related by the transformation

$$\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t) \tag{3.41}$$

which also specifies how the initial conditions in the original coordinates are to be transformed.

Equation (3.39) can be manipulated, using Euler's formulae for trigonometric functions, to become

$$\mathbf{r}(t) = \sum_{k=1}^{n} c_k \sin(\omega_k t + \phi_k) \, \mathbf{u}_k \tag{3.42}$$

where c_k and ϕ_k are constants determined by the initial conditions. This form clearly indicates the oscillatory nature of the system and defines the concept of natural frequency. Here, $\omega_k = +\sqrt{\lambda_k}$ denotes the undamped natural frequencies. Note that the frequencies are always positive because the Euler formula transformation from $e^{\pm\sqrt{\lambda_k} t}$ to $\sin \omega_k t$ effectively uses the $\pm$ sign in defining oscillation at the (positive) frequency ω_k. This expression extends the undamped single-degree-of-freedom result to undamped multiple-degree-of-freedom systems.

To evaluate the constants c_k and ϕ_k, the orthonormality of vectors $\mathbf{u}_k$ is again used. Applying the initial conditions to Equation (3.42) yields

$$\mathbf{r}(0) = \sum_{k=1}^{n} c_k \sin(\phi_k)\mathbf{u}_k \tag{3.43}$$

and

$$\dot{\mathbf{r}}(0) = \sum_{k=1}^{n} c_k \omega_k \cos(\phi_k)\mathbf{u}_k \tag{3.44}$$

Equation (3.41) is used to yield $\mathbf{r}(0) = M^{1/2}\mathbf{q}(0)$ and $\dot{\mathbf{r}}(0) = M^{1/2}\dot{\mathbf{q}}(0)$ from the given initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$. Premultiplying Equation (3.43) by $\mathbf{u}_i^T$ yields

$$\mathbf{u}_i^T\mathbf{r}(0) = \sum_{k=1}^{n} c_k \sin(\phi_k)\mathbf{u}_i^T\mathbf{u}_k$$

Invoking the orthonormality for vectors $\mathbf{u}_i$ yields

$$c_i \sin\phi_i = \mathbf{u}_i^T\mathbf{r}(0) \tag{3.45}$$

Likewise, Equation (3.44) yields

$$c_i \cos\phi_i = \frac{\mathbf{u}_i^T\dot{\mathbf{r}}(0)}{\omega_i} \tag{3.46}$$

Combining Equations (3.45) and (3.46) and renaming the index yields

$$\phi_i = \tan^{-1}\left\{\frac{\omega_i\mathbf{u}_i^T\mathbf{r}(0)}{\mathbf{u}_i^T\dot{\mathbf{r}}(0)}\right\}$$

and

$$c_i = \frac{\mathbf{u}_k^T\mathbf{r}(0)}{\sin\phi_i}$$

Note that, if the initial position $\mathbf{r}(0)$ is zero, then Equation (3.45) would imply that $\phi_i = 0$ for each i, then Equation (3.46) is used to compute the coefficients c_i. Once the constants c_i and ϕ_i are determined, then the index is changed to k to fit into the sum of Equation (3.42) which is written in terms of c_k and ϕ_k.

Next, consider the eigenvectors $\mathbf{u}_k$ to see how they represent the physical motion of the system. Suppose the initial conditions $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ are chosen in such a way that $c_k = 0$ for $k = 2, 3, \cdots, n$, $c_1 = 1$, and $\phi_k = 0$ for all k. Then, the expansion (3.42) reduces to one simple term, namely

$$\mathbf{r}(t) = \sin(\omega_1 t)\mathbf{u}_1 \tag{3.47}$$

This implies that every mass is vibrating with frequency ω_1 or is stationary and that the relative amplitude of vibration of each of the masses is the value of the corresponding element of $\mathbf{u}_1$. Thus, the size and sign of each element of the eigenvector indicates the positions of each mass from its equilibrium position, i.e., the 'shape' of the vibration at any instant of time. Transforming this vector back into the physical coordinate system via $\mathbf{v}_1 = M^{-1/2}\mathbf{u}_1$ allows the interpretation that vector $\mathbf{v}_1$ is the first *mode shape* of the system, or the mode shape corresponding to the first natural frequency. This can clearly be repeated for each of the subscripts k, so that $\mathbf{v}_k$ is the kth mode shape. Hence, the transformed eigenvectors are referred to as the *modes* of vibration of the system. Since eigenvectors are arbitrary to within a multiplicative constant, so are the mode shapes. If the arbitrary constant is chosen so that $\mathbf{v}_k$ is normalized, i.e., so that $\mathbf{v}_k^T\mathbf{v}_k = 1$, and the vector $\mathbf{v}_\mathrm{k}$ is real, the $\mathbf{v}_\mathrm{k}$ is called a *normal mode* of the system. The constants c_k in Equation (3.42) are called *modal participation*

factors because their relative magnitudes indicate how much the indexed mode influences the response of the system.

The procedure just described constitutes a theoretical *modal analysis* of the system of Equation (3.22). Some researchers refer to Equations (3.39) and (3.42) as the *expansion theorem*. They depend on the completeness of the eigenvectors associated with the system, i.e., of the matrix $M^{-1/2}KM^{-1/2}$.

Example 3.3.1

It should be obvious from Equations (3.27) and (3.32) how to calculate the eigenvalues and hence the natural frequencies of the system as they are the roots of the characteristic polynomial following from $\det(\tilde{K} - \lambda I) = 0$. How to calculate the eigenvectors, however, may not be as obvious; thus, calculation is illustrated in this example. Let λ_1 be an eigenvalue of A; then λ_1 and $\mathbf{u}_1 = [x_1 \;\; x_2]^T$ satisfy the vector equation

$$\begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.48}$$

This represents two dependent equations in x_1 and x_2, the two components of the eigenvector $\mathbf{u}_1$. Hence, only their ratio can be determined. Proceeding with the first equation in system (3.48) yields

$$(a_{11} - \lambda_1)x_1 + a_{12}x_2 = 0$$

which is solved for the ratio x_1/x_2. Then, the vector $\mathbf{u}_1$ is 'normalized' so that $\mathbf{u}_1^T\mathbf{u}_1 = x_1^2 + x_2^2 = 1$. The normalization yields specific values for x_1 and x_2. As a consequence of the singularity of $(A - \lambda I)$, the second equation in system (3.48), $a_{21} + x_1 + (a_{22} - \lambda)x_2 = 0$, is dependent on the first and does not yield new information.

Example 3.3.2

This example illustrates the procedure for calculating the free vibrational response of a multiple-degree-of-freedom system by using a *modal expansion*. The procedure is illustrated by a two-degree-of-freedom system, since the procedure for a larger number of degrees of freedom is the same. The purpose of the example is to develop an understanding of the eigenvector problem, and it is not intended to imply that this is the most efficient way to calculate the time response of a system (it is not).

Consider the system described in Figure 2.4 with $c_1 = c_2 = 0$, $m_1 = 9$, $m_2 = 1$, $k_1 = 24$, and $k_2 = 3$. Then, the equation of motion becomes

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

subject to the initial condition $\mathbf{q}(0) = [1 \quad 0]^T$ and $\dot{\mathbf{q}}(0) = [0 \quad 0]^T$ in some set of consistent units. The matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ becomes

$$\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The characteristic equation [equation (3.32)] becomes

$$\lambda^2 - 6\lambda + 8 = 0$$

which has roots $\lambda_1 = 2, \lambda_2 = 4$. The corresponding normalized eigenvectors are computed to be $\mathbf{u}_1 = [1/\sqrt{2} \quad 1/\sqrt{2}]^T$ and $\mathbf{u}_2 = [-1/\sqrt{2} \quad 1/\sqrt{2}]^T$, so that the orthogonal matrix of eigenvectors is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{3.49}$$

Also note that $S^T(M^{-1/2}KM^{-1/2})S = \text{diag}[2 \quad 4]$, as it should. The transformed initial conditions become

$$\mathbf{r}(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

and of course $\dot{\mathbf{r}}(0) = [0 \quad 0]^T$. The values of the constants in Equation (3.42) are found from

$$\phi_1 = \tan^{-1}\left[\frac{\omega_1 \mathbf{u}_1^T \mathbf{r}(0)}{\mathbf{u}_1^T \dot{\boldsymbol{r}}(0)}\right] = \tan^{-1}\infty = \frac{\pi}{2}$$

$$c_1 = \frac{\mathbf{u}_1^T \mathbf{r}(0)}{\sin\phi_1} = \frac{3}{\sqrt{2}}$$

$$\phi_2 = \tan^{-1}\left[\frac{\omega_2 \mathbf{u}_2^T \mathbf{r}(0)}{\mathbf{u}_2^T \dot{\mathbf{r}}(0)}\right] = \tan^{-1}\infty = \frac{\pi}{2}$$

$$c_2 = \frac{\mathbf{u}_2^T \mathbf{r}(0)}{\sin\phi_2} = \frac{-3}{\sqrt{2}}$$

Hence, the solution $\mathbf{r}(t)$ is given by

$$\mathbf{r}(t) = 1.5\cos\sqrt{2}t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1.5\cos 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

In the original coordinates this becomes $\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t)$

$$\mathbf{q}(t) = 1.5\cos\sqrt{2}t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} - 1.5\cos 2t \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Multiplying this out yields the motion of the individual masses:

$$q_1(t) = 0.5\cos\sqrt{2}t + 0.5\cos 2t$$

$$q_2(t) = 1.5\cos\sqrt{2}t - 1.5\cos 2t$$

The two mode shapes are ($\mathbf{v}_i = M^{1/2}\mathbf{u}_i$)

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

If K is not symmetric, i.e., if we have a system of the form

$$M\ddot{\mathbf{q}}(t) + (K + H)\mathbf{q}(t) = \mathbf{0}$$

then proceed by solving an eigenvalue problem of the form

$$M^{-1}(K+H)\mathbf{x}=\lambda\mathbf{x}$$

or

$$A\mathbf{x}=\lambda\mathbf{x}$$

where A is not a symmetric matrix. In this case the eigenvalues λ_i and the eigenvectors $\mathbf{x}_i$ are in general complex numbers. Also, because of the asymmetry, matrix A has a *left eigenvector*, $\mathbf{y}_k$, which satisfies

$$\mathbf{y}_k^T A=\lambda_k\mathbf{y}_k^T$$

and, in general, may not equal the *right eigenvector*, $\mathbf{x}_k$. Now, let $\mathbf{y}_k$ be a left eigenvector and $\mathbf{x}_i$ be a right eigenvector. Then

$$A\mathbf{x}_i=\lambda_i\mathbf{x}_i \qquad \text{or} \qquad \mathbf{y}_k^T A\mathbf{x}_i=\lambda_i\mathbf{y}_k^T\mathbf{x}_i \tag{3.50}$$

and

$$\mathbf{y}_k^T A=\lambda_k\mathbf{y}_k^T \qquad \text{or} \qquad \mathbf{y}_k^T A\mathbf{x}_i=\lambda_k\mathbf{y}_k^T\mathbf{x}_i \tag{3.51}$$

where the ith eigenvalues of $\mathbf{x}_i$ and $\mathbf{y}_i$ are the same. Subtracting Equation (3.51) from Equation (3.50) yields $(\lambda_i-\lambda_k)\mathbf{y}_k^T\mathbf{x}_i=0$, so that, if $\lambda_i\neq\lambda_k$, then $\mathbf{y}_k^T\mathbf{x}_i=0$. This is called *biorthogonality.*

For distinct eigenvalues, the right and left eigenvectors of A each form a linearly independent set and can then be used to express any $n\times 1$ vector, i.e., an expansion theorem still exists. These relations are useful for treating gyroscopic systems, systems with constraint damping, systems with follower forces, and feedback control systems.

3.4 CANONICAL FORMS

The diagonal matrix of eigenvalues of Section 3.3 is considered a canonical, or simple, form of a symmetric matrix. This is so because of the ease of manipulation of a diagonal matrix. For instance, the square root of a diagonal matrix is just the diagonal matrix with nonzero elements equal to the square root of the diagonal elements of the original matrix.

From the point of view of vibration analysis, the diagonal form provides an immediate record of natural frequencies of vibration of systems. In addition, the similarity transformation equation [Equation (3.12)] can be used to solve the undamped vibration problem of Equation (3.33). To see this, let S be the orthogonal similarity transformation associated with the symmetric matrix $\tilde{K}=M^{-1/2}KM^{-1/2}$. Substitution of $\mathbf{r}(t)=S\mathbf{y}(t)$ into Equation (3.33) and premultiplying by S^T yields

$$\ddot{\mathbf{y}}(t)+\Lambda\mathbf{y}(t)=0 \tag{3.52}$$

where Λ is diagonal. Thus, Equation (3.52) represents n scalar equations, each of the form

$$\ddot{y}_i(t) + \omega_i^2 y_i(t) = 0, \qquad i = 1, 2, \ldots, n \tag{3.53}$$

These expressions can be integrated separately using the initial conditions $\mathbf{y}(0) = S^T\mathbf{r}(0)$ and $\dot{\mathbf{y}}(0) = S^T\dot{\mathbf{r}}(0)$ to yield a solution equivalent to Equation (3.42). This argument forms the crux of what is called *modal analysis* and is repeated many times in the following chapters.

Unfortunately, not every square matrix is similar to a diagonal matrix. However, every square matrix is similar to an upper triangular matrix. That is, let matrix A have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; there then exists a nonsingular matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & t_{12} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & t_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & t_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{3.54}$$

The matrix $P^{-1}AP$ is said to be *upper triangular*. If the matrix is symmetric, then the t_{ij} in Equation (3.54) are all zero, and the upper triangular matrix becomes a diagonal matrix.

A classic result in the theory of matrices is known as *Jordan's theorem* and states the following. Let A be $n \times n$ with eigenvalues λ_i of multiplicities m_i, so that

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{m_i}, \qquad \text{where} \quad \sum_{i=1}^{k} m_i = n$$

Then, every matrix A is similar to a block-diagonal matrix of the form

$$J = \begin{bmatrix} \Lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_3 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \Lambda_i \end{bmatrix} \tag{3.55}$$

where each block Λ_i is of the form

$$\Lambda_i = \begin{bmatrix} \lambda_i & a & 0 & \cdots & 0 \\ 0 & \lambda_i & a & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \lambda_i & a \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

Here $a = 0$ or 1, depending on whether or not the associated eigenvectors are dependent. The value of a is determined as follows. If λ_i are distinct, then $a = 0$, always. If λ_i is repeated m_i times but has m_i linearly independent eigenvectors, then $a = 0$. If the eigenvector $\mathbf{x}_i$ is

dependent (degenerate), then $a = 1$. If the preceding matrix describes a vibration problem, the value of a determines whether or not a given system can be diagonalized. Note, then, that in general it is eigenvector 'degeneracy' that causes problems in vibration analysis – not just repeated eigenvalues.

Next, recall again that the determinant of a matrix is invariant under a similarity transformation. Expanding the determinant yields

$$\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

which is the characteristic polynomial and hence is equal to

$$\det(A - \lambda I) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n) \tag{3.56}$$

Thus, the coefficients c_i of the characteristic polynomial must also be invariant under similarity transformations. This fact is used to some advantage.

The *trace* of a matrix A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} \tag{3.57}$$

That is, the trace is the sum of the diagonal entries of the matrix. Some manipulation yields

$$c_1 = -\mathrm{tr}(A) \tag{3.58}$$

and

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i \tag{3.59}$$

Thus, the trace of a matrix is invariant under similarity transformations. Some additional properties of the trace are

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{3.60}$$

For nonsingular matrix P

$$\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP) \tag{3.61}$$

For α and β scalars

$$\mathrm{tr}(\alpha A + \beta B) = \alpha\mathrm{tr}(A) + \beta\mathrm{tr}(B) \tag{3.62}$$

and

$$\mathrm{tr}(A) = \mathrm{tr}(A^T) \tag{3.63}$$

It is interesting to note that the $\mathrm{tr}(A)$ and $\det(A)$ can be used for a check of computational accuracy because they are invariant under similarity transformations.

3.5 LAMBDA MATRICES

Since many structures exhibit velocity-dependent forces, the ideas of Section 3.4 need to be extended to equations of the form

$$A_1\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + A_3\mathbf{q} = \mathbf{0} \tag{3.64}$$

Of course, this expression could be placed in the state-space form of Equation (2.20), and the methods of Section 3.4 can be applied. In fact, many numerical algorithms do exactly that. However, the second-order form does retain more of the physical identity of the problem and hence is worth developing.

Again, assume solutions of Equation (3.64) of the form $\mathbf{q}(t) = \mathbf{u}\,e^{\lambda t}$, where $\mathbf{u}$ is a nonzero vector of constants. Then Equation (3.64) becomes

$$(\lambda^2 A_1 + \lambda A_2 + A_3)\mathbf{u}\,e^{\lambda t} = \mathbf{0}$$

or, since $e^{\lambda t}$ is never zero,

$$(\lambda^2 A_1 + \lambda A_2 + A_3)\mathbf{u} = \mathbf{0}$$

This last expression can be written as

$$D_2(\lambda)\mathbf{u} = \mathbf{0} \tag{3.65}$$

where $D_2(\lambda)$ is referred to as a *lambda matrix* and $\mathbf{u}$ is referred to as a *latent vector*. In fact, in this case $\mathbf{u}$ is called the right latent vector (Lancaster, 1966).

Here, it is important to distinguish between the concept of eigenvalues and eigenvectors of a *matrix* [Equation (3.1)] and eigenvalues and eigenvectors of a *system* [Equation (3.65)] Lancaster (1966) has suggested referring to λ and $\mathbf{u}$ of the system as latent roots and latent vectors respectively, in order to make this distinction clear. Unfortunately, this did not catch on in the engineering literature. In order to be compatible with the literature, the distinction between eigenvectors (of a single matrix) and latent vectors (of the system) must be made from context. Equation (3.65) expresses the system eigenvectors and occasionally is referred to as a nonlinear eigenvalue problem, a matrix polynomial problem, or a lambda matrix problem.

For the existence of nonzero solutions of Equation (3.65), the matrix $D_2(\lambda)$ must be singular, so that

$$\det(D_2(\lambda)) = 0 \tag{3.66}$$

The solutions to this $2n$-degree polynomial in λ are called *latent roots*, eigenvalues, or characteristic values and contain information about the natural frequencies of the system. Note that the solution of Equation (3.66) and the solution of $\det(A - \lambda I) = 0$ are the same. Here, A is the state matrix [see Equation (2.20)] given by

$$A = \begin{bmatrix} 0 & I \\ -A_1^{-1}A_3 & -A_1^{-1}A_2 \end{bmatrix} \tag{3.67}$$

Also, the eigenvectors of A are just $[\mathbf{u}_i \quad \lambda_i \mathbf{u}_i]^T$, where $\mathbf{u}_i$ are the latent vectors of Equation (3.65) and λ_i are the solutions of Equation (3.66).

An $n \times n$ lambda matrix, $D_2(\lambda)$, is said to be *simple* if A_1^{-1} exists and if, for each eigenvalue (latent root) λ_i satisfying Equation (3.65), the rank of $D_2(\lambda_i)$ is $n - \alpha_i$, where α_i is the multiplicity of the eigenvalue λ_i. If this is not true, then $D_2(\lambda)$ is said to be *degenerate*. If each of the coefficient matrices are real and symmetric and if $D_2(\lambda)$ is simple, the solution of Equation (3.64) is given by

$$\mathbf{q}(t) = \sum_{i=1}^{2n} c_i \mathbf{u}_i \, e^{\lambda_i t} \tag{3.68}$$

Here the c_i are $2n$ constants to be determined from the initial conditions, and the $\mathbf{u}_i$ are the right eigenvectors (latent vectors) of $D_2(\lambda)$. Note that, if $A_2 = 0$, Equation (3.65) collapses to the eigenvalue problem of a matrix. The definitions of degenerate and simple still hold in this case.

Since, in general, $\mathbf{u}_i$ and λ_i are complex, the solution $\mathbf{q}(t)$ will be complex. The physical interpretation is as follows. The displacement is the real part of $\mathbf{q}(t)$, and the velocity is the real part of $\dot{\mathbf{q}}(t)$. The terms *modes* and *natural frequencies* can again be used if care is taken to interpret their meaning properly. The damped natural frequencies of the system are again related to the λ_i in the sense that, if the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ are chosen such that $c_i = 0$ for all values of i except $i = 1$, each coordinate $q_i(t)$ will oscillate (if underdamped) at a frequency determined by λ_i. Furthermore, if the $\mathbf{u}_i$ are normalized, i.e., if $\mathbf{u}_i^* \mathbf{u}_i = 1$, then the elements of $\mathbf{u}_i$ indicate the relative displacement and phase of each mass when the system vibrates at that frequency. Here, $\mathbf{u}^*$ denotes the complex conjugate of the transpose of vector $\mathbf{u}$.

In many situations, the coefficient matrices are symmetric and the damping matrix D is chosen to be of a form that allows the solution (2.13) to be expressed as a linear combination of the normal modes, or eigenvectors, of the matrix $\tilde{K}$, which, of course, are real. In this case, the matrix of eigenvectors decouples the equations of motion. In fact the main reason for this assumption is the convenience offered by the analysis of systems that decouple. The advantage in the normal mode case is that the eigenvectors are all real valued. To this end, consider the symmetric damped system of Equation (2.13) and note the following:

1. If $D = \alpha M + \beta K$, where α and β are any real scalars, then the eigenvectors (latent vectors) of Equation (3.65) are the same as the eigenvectors of the same eigenvalue problem with $D = 0$.
2. If $D = \sum_{i=1}^{n} \beta_{i-1} K^{i-1}$, where β_i are real scalars, then the eigenvectors of Equation (2.13) are the same as the eigenvectors of the undamped system ($D = 0$).
3. The eigenvectors of Equation (2.13) are the same as those of the undamped system (with $D = 0$) if and only if $DM^{-1}K = KM^{-1}D$ (Caughey and O'Kelly, 1965).

Systems satisfying any of the above rules are said to be *proportionally damped*, to have *Rayleigh damping*, or to be *normal mode systems*. Such systems can be decoupled by the modal matrix associated with matrix $\tilde{K}$.

Of the cases just mentioned, the third is the most general and includes the other two as special cases. It is interesting to note that case 3 follows from a linear algebra theorem that

states that two symmetric matrices have the same eigenvectors if and only if they commute (Bellman, 1970), i.e., if and only if there exists a similarity transformation simultaneously diagonalizing both matrices. It is also worth noting that, in the normal mode case, the eigenvectors are real, but the reverse is not true (see the discussion of overdamping below). That is, some structures with real-valued eigenvectors are not normal mode systems because the matrix of modal vectors does not decouple the equations of motion (i.e., diagonalize the coefficient matrices). The significance of complex eigenvectors is that the elements are not in phase with each other as they are in the normal mode case. Some researchers have incorrectly stated that, if the damping is small in value, normal modes can be assumed. However, even small amounts of damping can cause condition 3 above to be violated, resulting in complex mode shapes (see, for example, Lallament and Inman, 1995).

As a generic illustration of a normal mode system, let S_m be the matrix of eigenvectors of K normalized with respect to the mass matrix M (i.e., $S_m = M^{-1/2}S$) so that

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T K S_m &= \Lambda_K = \mathrm{diag}[\omega_i^2] \end{aligned} \tag{3.69}$$

where ω_i^2 are the eigenvalues of matrix K and correspond to the square of the natural frequencies of the undamped system. If case 3 holds, then the damping is also diagonalized by the transformation S_m, so that

$$S_m^T D S_m = \mathrm{diag}[2\zeta_i\omega_i] \tag{3.70}$$

where ζ_i are called the modal damping ratios. Then, Equation (3.64) can be transformed into a diagonal system via the following. Let $\mathbf{q}(t) = S_m\mathbf{y}(t)$ in Equation (2.13) and premultiply by S_m^T to get

$$\ddot{y}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = 0, \qquad i = 1, 2, \ldots, n \tag{3.71}$$

where $y_i(t)$ denotes the ith component of vector $\mathbf{y}(t)$. Each of the n equations of system (3.71) is a scalar, which can be analyzed by the methods of Chapter 1 for single-degree-of-freedom systems. In this case the ζ_i are called *modal damping ratios* and the ω_i are the undamped *natural frequencies*, or modal frequencies.

Alternatively, the modal decoupling described in the above paragraph can be obtained by using the mass normalized stiffness matrix. To see this, substitute $\mathbf{q} = M^{-1/2}\mathbf{r}$ into Equation (2.12), multiply by $M^{-1/2}$ to form $\tilde{K} = M^{-1/2}KM^{-1/2}$, compute the normalized eigenvectors of $\tilde{K}$, and use these to form the columns of the orthogonal matrix S. Next, use the substitution $\mathbf{r} = S\mathbf{y}$ in the equation of motion, premultiply by S^T, and Equation (3.71) results. This procedure is illustrated in the following example.

Example 3.5.1

Let the coefficient matrices of Equation (2.13) have the values

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

Calculating $DM^{-1}K$ yields

$$DM^{-1}K = \begin{bmatrix} 30 & -6 \\ -6 & \dfrac{10}{3} \end{bmatrix}$$

which is symmetric and hence equal to $KM^{-1}D$, so that condition 3 is satisfied. From example 3.3.2, the eigenvectors and eigenvalues of matrix $\tilde{K}$ are as follows: $\mathbf{u}_1^T = [1 \quad 1]/\sqrt{2}$, $\lambda_1 = 2$, $\mathbf{u}_2^T = [-1 \quad 1]/\sqrt{2}$, and $\lambda_2 = 4$.

Then $S^T M^{-1/2} D M^{-1/2} S = \text{diag}[2/3 \quad 4/3]$, and $S^T M^{-1/2} K M^{-1/2} S = \text{diag}[2 \quad 4]$. Hence, Equation (2.13) with $\mathbf{f} = \boldsymbol{0}$ is equivalent to the two scalar equations given by

$$\dot{y}_1(t) + (2/3)\dot{y}_1(t) + 2y_1(t) = 0$$

and

$$\ddot{y}_2(t) + (4/3)\dot{y}_2(t) + 4y_2(t) = 0$$

each of which can easily be solved by the methods of Chapter 1. From the displacement coefficient, the frequencies are

$$\omega_1 = \sqrt{2} \text{ rad/s} \qquad \text{and} \qquad \omega_2 = \sqrt{4} = 2 \text{ rad/s}$$

and from the velocity coefficients the damping ratios are

$$\zeta_1 = \frac{2}{3}\frac{1}{2\omega_1} = \frac{1}{3\sqrt{2}} \qquad \text{and} \qquad \zeta_2 = \frac{4}{3}\frac{1}{2\omega_2} = \frac{1}{3}$$

3.6 OSCILLATION RESULTS

The definition of critical damping, overdamping, and underdamping, stated for single-degree-of-freedom systems in Chapter 1, can be extended to some of the lumped-parameter systems of this chapter. In particular, consider the symmetric positive definite system given by

$$\ddot{\mathbf{r}}(t) + \tilde{D}\dot{\mathbf{r}}(t) + \tilde{K}\mathbf{r}(t) = \mathbf{0} \tag{3.72}$$

Here, $\tilde{D} = M^{-1/2} D M^{-1/2}$, $\tilde{K} = M^{-1/2} K M^{-1/2}$, and $\mathbf{r}(t) = M^{1/2}\mathbf{q}(t)$ in Equation (2.13). In a form imitating the single-degree-of-freedom case, a critical damping matrix is defined as $\tilde{D}_{\text{cr}} = 2\tilde{K}^{1/2}$. Then, the following classifications can be derived (Inman and Andry, 1980, and Barkwell and Lancaster, 1992):

1. If $\tilde{D} = \tilde{D}_{\text{cr}}$, then Equation (3.72) is said to be a *critically damped system*, each mode of vibration is critically damped, and each eigenvalue of Equation (3.72) is a repeated negative real number. The response of such systems will not oscillate, and all the eigenvectors are real.

2. If the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ is positive definite and $\tilde{D}\tilde{K} = \tilde{K}\tilde{D}$, then Equation (3.72) is said to be an *overdamped system*, each 'mode' of the structure is overdamped, and each eigenvalue is a negative real number. The response of such systems will not oscillate, and all the eigenvectors are real.
3. If the matrix $\tilde{D}_{\text{cr}} - \tilde{D}$ is positive definite, then Equation (3.72) is said to be an *underdamped system*, each mode of vibration is underdamped, and each eigenvalue is a complex conjugate pair with a negative real part. The response of such systems oscillates with decaying amplitude and the eigenvectors are, in general, complex (unless the matrix $DM^{-1}K$ is symmetric).

A fourth possibility exists for the matrix case. That is, the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ could be indefinite. In this case, Equation (3.72) is said to exhibit *mixed damping*, and at least one mode oscillates and at least one mode does not oscillate. In addition, if A is the state matrix associated with Equation (3.72), then the system is overdamped if and only if A can be factored into the product of two positive definite Hermitian matrices (Nicholson, 1983). In order to relax the condition of normal modes in the overdamped case (case 2 above), Barkwell and Lancaster (1992) showed that Equation (3.72) has all negative real eigenvalues if $\beta_1 > 2\omega_n$, where β_1 is the smallest eigenvalue of the damping matrix $\tilde{D}$ and ω_n is the largest undamped natural frequency.

The determinant condition of Section 3.2 for the positive definiteness of a matrix can be used on the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ to provide a system of nonlinear inequalities in the physical parameters m_i, c_i, and k_i of a given structure. These inequalities can be solved for low-order systems to yield choices of m_i, c_i, and k_i that will cause the system to be overdamped or underdamped as desired. The following example illustrates the process.

Example 3.6.1

Consider the two-degree-of-freedom system of Figure 2.4, which has equations of motion given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \dot{\mathbf{q}}(t) + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \mathbf{q}(t) = \mathbf{0} \tag{3.73}$$

where $\mathbf{q}(t) = [x_1(t) \quad x_2(t)]^T$.

To form the matrix $2\tilde{K}^{1/2}$ requires the computation of the square root of a matrix. This computational burden can be reduced by noting that Bellman (1968) has shown that, if $4\tilde{K} - \tilde{D}^2$ is positive definite, so is the matrix $2\tilde{K}^{1/2} - \tilde{D}$. Hence, it is sufficient to calculate only the square of a matrix instead of the square root of a matrix. To proceed, calculation of the square of the damping matrix in terms of the generic values of the system parameters yields the following:

$$\begin{aligned}
(4\tilde{K} - \tilde{D}^2)_{11} &= 4\frac{k_1 + k_2}{m_1} - \frac{(c_1 + c_2)^2}{m_1^2} - \frac{c_2^2}{m_1 m_2} \\
(4\tilde{K} - \tilde{D}^2)_{12} &= -\frac{4k_2}{\sqrt{m_1 m_2}} + \frac{c_1 c_2 + c_2^2}{m_1\sqrt{m_1 m_2}} + \frac{c_2^2}{m_2\sqrt{m_1 m_2}} \\
(4\tilde{K} - D^2)_{22} &= \frac{4k_2}{m_2} - \frac{c_2^2}{m_2^2} - \frac{c_2^2}{m_1 m_2}
\end{aligned} \tag{3.74}$$

Applying the determinant condition to the matrix defined by Equation (3.74) yields the inequalities

$$4\frac{k_1+k_2}{m_1} > \frac{(c_1+c_2^2)}{m_1^2} + \frac{c_2^2}{m_1 m_2} \tag{3.75}$$

$$\left[\frac{4k_2}{(m_1 m_2)^{1/2}} - \frac{c_1 c_2 + c_2^2}{(m_1^3 m_2)^{1/2}} - \frac{c_2^2}{(m_1 m_2^3)^{1/2}}\right]^2 > \frac{c_1+c_2}{m_1^2} + \frac{c_2^2}{m_1 m_2} - 4\frac{k_1+k_2}{m_1}\frac{c_2^2}{m_2^2} + \frac{c_2^2}{m_1 m_2} - 4\frac{k_2}{m_2}$$

These inequalities have many solutions. One possibility is to choose $m_1 = m_2 = 1$, $c_1 = 2$, $c_2 = 1$, $k_1 = 5$, and $k_2 = 4$. With this choice, the motion should oscillate.

To check to see that this is, in fact, the case, these values of m_i, c_i, and k_i can be substituted into Equation (3.73). The characteristic equation then becomes

$$\lambda^4 + 4\lambda^3 + 15\lambda^2 + 13\lambda + 20 = 0 \tag{3.76}$$

This has roots

$$\left.\begin{aligned}\lambda_1 &= -0.312 - 1.306j\\ \lambda_2 &= -0.312 + 1.306j\end{aligned}\right\} \Rightarrow \omega_1 = 1.343 \text{ rad/s} \quad \text{and} \quad \zeta_1 = 0.232 < 1$$

$$\left.\begin{aligned}\lambda_3 &= -1.688 - 2.870j\\ \lambda_4 &= -1.688 + 2.870j\end{aligned}\right\} \Rightarrow \omega_2 = 3.33 \text{ rad/s} \quad \text{and} \quad \zeta_2 = 0.507 < 1$$

This clearly indicates that the system oscillates as indicated by the theory. Here, the natural frequencies and damping ratios are determined from the complex eigenvalues by solving the two equations $\lambda_{1,2} = -\zeta_1\omega_1 \pm \omega_1\sqrt{1-\zeta_1^2}j$ for the two unknowns ζ_1 and ω_1. Note that the matrix elements in Equation (3.74) and the determinant in Equation (3.75) can be derived using symbolic computations in Mathcad, MATLAB, or Mathematica.

The condition of critical damping is a very special situation and is not easily obtainable. In fact, unlike single-degree-of-freedom structures, not all multiple-degree-of-freedom systems can be made critically damped by adjusting the spring, mass, and/or damping parameters. For instance, consider the example in Figure 2.4. In order for this system to be critically damped, each of the elements of matrix (3.74) must be zero. Since the matrix is symmetric, this yields the three equalities

$$\frac{(c_1+c_2)^2}{m_1^2} + \frac{c_2^2}{m_1 m_2} = 4\frac{k_1+k_2}{m_1} \tag{3.77}$$

$$\frac{c_2^2 + c_1 c_2}{m_1} + \frac{c_2^2}{m_2} = 4k_2 \tag{3.78}$$

$$\frac{c_2^2}{m_2^2} + \frac{c_2^2}{m_1 m_2} = 4\frac{k_2}{m_2} \tag{3.79}$$

Manipulation of these equations shows that all three equalities can be satisfied if and only if one of the pairs (k_1, c_1) or (k_2, c_2) is zero. This means critical damping can result only

if the system is reduced to a single degree of freedom, or perhaps by adding additional components.

If structural changes are allowed, the two-degree-of-freedom system in Figure 2.4 can be made critically damped. For example, consider adding one more dashpot, c_3, and one more spring, k_3, to the system in Figure 2.4 by attaching them from m_2 to ground. The equation of motion then becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{3.80}$$

Choosing the mass matrix to be the identity matrix, the three equalities resulting from setting $\tilde{D}^2 = 4\tilde{K}$ become

$$\begin{aligned} (c_1 + c_2)^2 + c_2^2 &= 4(k_1 + k_2) \\ c_2(c_1 + c_3 + 2c_2) &= 4k_2 \\ (c_2 + c_3)^2 + c_2^2 &= 4(k_2 + k_3) \end{aligned} \tag{3.81}$$

One solution for this system is

$$\begin{aligned} c_1 &= 4, \qquad & k_1 &= 4 \\ c_2 &= 2, \qquad & k_2 &= 6 \\ c_3 &= 4, \qquad & k_3 &= 4 \end{aligned}$$

The characteristic equation then becomes

$$\lambda^4 + 12\lambda^3 + 52\lambda^2 + 96\lambda + 64 = 0 \tag{3.82}$$

which has roots

$$\begin{aligned} \lambda_{1,2} &= -2 \Rightarrow \omega_1 = 2 \qquad \text{and} \qquad \zeta_1 = 1 \\ \lambda_{3,4} &= -4 \Rightarrow \omega_2 = 4 \qquad \text{and} \qquad \zeta_2 = 1 \end{aligned}$$

Hence, each mode is critically damped, as predicted by the theory.

The preceding methods of defining critical damping, overdamping, and underdamping are based on a 'permode' concept of critical damping. That is, a critically damped system is one in which each mode is critically damped. However, as pointed out in problem 1.5, critical damping can be viewed as the smallest value of the damping rate such that the system does not oscillate. This latter approach, taken by Beskos and Boley (1980), can be used for multiple-degree-of-freedom systems to generate critical damping surfaces in spaces defined by the damping parameters of the system. These surfaces can be calculated for two-degree-of-freedom systems of the same structure as in Figure 2.4. Such curves are computed by finding solutions for values of c_1 and c_2 that satisfy

$$\frac{d}{db}[\det(Mb^2 - Db + K)] = 0 \tag{3.83}$$

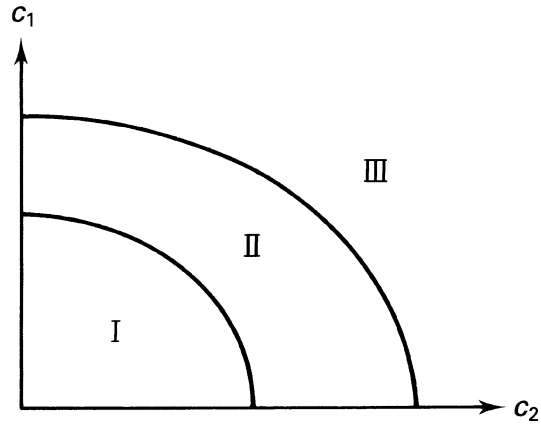

Figure 3.1 Critical damping curves for a two-degree-of-freedom system.

where b is restricted to be a positive real number and d/db indicates the derivative with respect to b. The curves are given in Figure 3.1. Systems with values of c_1 and c_2 lying in region I exhibit oscillation in both modes. In region II, one mode oscillates and one does not. In region III, neither mode oscillates. The two curves, called *critical damping curves*, are the solutions to Equation (3.83) for fixed values of m_i and k_i.

Several extensions of the preceding ideas have been developed in the literature. Papargyri-Beskou, Thessaloniki, and Beskos (2002) present the latest discussion of critical damping and examine a system with an indefinite damping matrix, followed by a comparison of the published definitions. The interest in calculating the critical damping matrix is for comparison and design, as is often the case for single-degree-of-freedom systems.

3.7 EIGENVALUE ESTIMATES

In many instances it is enough to know an approximate value, or estimate, of a particular eigenvalue or how changes in certain parameters affect the natural frequencies. Methods that require less computation than solving the characteristic equation of a given system but yield some information about the eigenvalues of the system may be useful. As an example, consider the single-degree-of-freedom spring–mass system driven by $F_0 \sin \omega t$. If, in a given design situation, one wanted to avoid resonance, it would be enough to know that the natural frequency is less than the driving frequency ω. Also, since the free response of the system is a function of the eigenvalues, estimates of eigenvalues yield some estimates of the nature of the free response of the structure and may lead to design inequalities.

One of the most basic estimates of the eigenvalues of a symmetric matrix is given by Rayleigh's principle. This principle states that, if $\lambda_{\min}$ is the smallest eigenvalue of the symmetric matrix A and $\lambda_{\max}$ is its largest, then for any nonzero vector $\mathbf{x}$

$$\lambda_{\min} < \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} < \lambda_{\max} \tag{3.84}$$

This quotient defines what is called the *Rayleigh quotient* for matrix A; i.e., the Rayleigh quotient is defined as the scalar ratio $R(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$ (see, for instance, Huseyin (1978) for a proof).

The variational characterization of Rayleigh's quotient can also be used to characterize the other eigenvalues of A. If the minimization of the Rayleigh quotient is carried out over all vectors orthogonal to the first eigenvector, the second eigenvalue results. The ith eigenvalue is calculated by

$$\lambda_i = \min_{\substack{\mathbf{x}^T\mathbf{x}=1 \\ \mathbf{x}^T\mathbf{x}_k=0}} (\mathbf{x}^T A\mathbf{x}), \qquad k = 1, 2, \ldots, i-1 \tag{3.85}$$

which states that the ith eigenvalue is obtained by taking the minimum value of $\mathbf{x}^T A\mathbf{x}$ over all vectors $\mathbf{x}$ that satisfy $\mathbf{x}^T\mathbf{x} = 1$ and that are orthogonal to the first $(i-1)$ eigenvectors.

To apply Rayleigh's quotient to the vibration problem of a conservative system

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{3.86}$$

requires little manipulation. Recall that the eigenvalue problem for Equation (3.86) can be written as

$$\lambda M\mathbf{u} = K\mathbf{u}$$

or

$$R(\lambda, \mathbf{u}) = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}} \tag{3.87}$$

where the notation $R(\lambda, \mathbf{u})$ denotes the Rayleigh quotient. Equation (3.87) can be examined for all vectors such that $\mathbf{u}^T M\mathbf{u} = 1$. Alternatively, $R(\lambda, \mathbf{u})$ can be formed for system of equations (3.32) to yield

$$R(\lambda, \mathbf{q}) = \mathbf{q}^T M^{-1/2} K M^{-1/2}\mathbf{q} \tag{3.88}$$

which can be examined for all vectors $\mathbf{q}$ with $\|\mathbf{q}\| = \sqrt{\mathbf{q}^T\mathbf{q}} = 1$, called the *norm* of $\mathbf{q}$.

Example 3.7.1

Consider the system in Figure 2.4 with $c_1 = c_2 = 0$, $m_1 = 1$, $m_2 = 4$, $k_1 = 2$, and $k_2 = 1$. The nondimensional equation of motion is then given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q} = 0$$

where $\mathbf{q} = [x_1 \quad x_2]^T$. Since M is diagonal,

$$M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and $R(\lambda, \mathbf{q})$ from Equation (3.88) becomes

$$R(\lambda, \mathbf{q}) = \mathbf{q}^T \begin{bmatrix} 3 & -1/2 \\ -1/2 & 1/4 \end{bmatrix} \mathbf{q}$$

If a trial vector is chosen (out of thin air and then normalized) of $\mathbf{q} = [0.243 \quad 0.970]^T$, then $R(\lambda, \mathbf{q}) = 0.176$. Since the actual value is $\lambda_1 = 0.1619$, the Rayleigh quotient appears to be a reasonable estimate.

Again, note that the Rayleigh method provides an estimate of λ_1 without having to solve for the roots of the characteristic equation. It should also be noted that the method is not as accurate as it may sometimes appear from the usual textbook examples. If the trial vector $\mathbf{q}$ is 'near' the first eigenvector, the estimate will be fairly close. If not, the estimate will not be as good. For instance, if $\mathbf{q} = [1 \quad 0]^T$ is chosen in the preceding example, then $R(\lambda, \mathbf{q}) = 3$, which is not a very good estimate of λ_1. However, while of little computational value for computing frequencies, the Rayleigh quotient is of use in analysis and design.

Several other results of interest involving eigenvalue inequalities are useful in vibration analysis. One is a method for determining the effect of truncating the degrees of freedom of a system on the eigenvalues of the system. Let the symmetric matrix A be $n \times n$ with eigenvalues, λ_i, ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and let matrix B be formed from matrix A by deleting a row and column. Hence, B is $(n-1)$ by $(n-1)$, so it will have $n-1$ eigenvalues, which are denoted by

$$\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1}$$

It can be shown that these two sets of eigenvalues are *interlaced*, i.e., that

$$\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \lambda_3 < \cdots < \gamma_{n-1} < \lambda_n \tag{3.89}$$

This last statement shows that the natural frequencies of a system decrease as the number of degrees of freedom increase. In fact, if A_r denotes a symmetric $r \times r$ matrix and $\lambda_i(A_r)$ denotes the ith eigenvalue of matrix A_r, then

$$\lambda_i(A_{r+1}) < \lambda_i(A_r) < \lambda_{i+1}(A_{r+1}) \tag{3.90}$$

This is referred to as a *Sturmian separation theorem* (see, for instance, Bellman, 1970) and is useful in illustrating how the order of a vibration model affects the natural frequencies, such as when model reduction is used (defined in Section 6.8).

Another useful result reported by Bellman (1970) is that, if A and B are $n \times n$ symmetric matrices, then

$$\lambda_k(A+B) > \lambda_k(A), \qquad k = 1, 2, \ldots, n \tag{3.91}$$

if B is positive semidefinite, and

$$\lambda_k(A+B) > \lambda_k(A), \qquad k = 1, 2, \ldots, n \tag{3.92}$$

if B is positive definite. Here, $\lambda_k(A+B)$ refers to the kth eigenvalue of the matrix $A+B$, and so on.

The physical parameters of a system are often known only to a certain precision. For instance, mass and stiffness coefficients may be measured quite accurately for most systems, but viscous damping coefficients are very hard to measure and are not always known to a high degree of accuracy.

A symmetric matrix with error in its elements can be written as the sum

$$B = A + E_e \tag{3.93}$$

where B is a known symmetric matrix with known eigenvalues

$$\beta_1 < \beta_2 < \cdots < \beta_n$$

and A is a symmetric matrix with unknown eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and E_e is a symmetric matrix representing the errors in matrix B. The objective is to estimate λ_i given the numbers β_i, without knowing too much about matrix E_e. It can be shown that

$$|\beta_i - \lambda_i| < \|E_e\| \tag{3.94}$$

where $\|E_e\|$ denotes the Euclidian norm of matrix E_e, defined as the square root of the sum of the squares of each element of E_e. It is easy to see that $\|E_e\| < n\varepsilon$, where n is the dimension of E_e and ε is the absolute value of the largest element in matrix E_e. Combining these two inequalities yields

$$|\beta_i - \lambda_i| < n\varepsilon \tag{3.95}$$

Inequality (3.95) can be used to measure the effects of errors in the parameters of a physical system on the eigenvalues of the system. For instance, let $\tilde{K}$ be the mass normalized stiffness matrix of the actual system associated with Equation (3.33), which is measured by some experiment. Let B denote the matrix consisting of all measured values, and let E_e be the matrix consisting of all the measured errors. Then, from expression (3.95), with $A = \tilde{K}$ and with eigenvalues ω_i^2, the inequality becomes $|\beta_i - \omega_i^2| < n\varepsilon$, or $-n\varepsilon < \omega_i^2 < \beta_i + n\varepsilon$, which in turn can be written as

$$\beta_i - n\varepsilon < \omega_i^2 < \beta_i + n\varepsilon \tag{3.96}$$

This last expression indicates how the *actual* natural frequencies, ω_i, are related to the calculated natural frequencies, $\beta_i^{1/2}$, and the measurement error, ε. Note that the assumption of symmetry will be satisfied for the matrix E_i since each element is the sum of the errors of the stiffness elements in that position so that the ijth element of E_i will contain the same measurement error as the jith element of E_i.

A fundamental theorem from linear algebra that yields simple estimates of the eigenvalues of a matrix from knowledge only of its elements is attributed to Gerschgorin (Todd, 1962). Simply stated, let a_{ij} denote the ijth element of a matrix A. Then every eigenvalue of A lies inside at least one of the circles in the complex plane centered at a_{ii} of radius

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{3.97}$$

If a disc has no point in common with any other disc, it contains only one eigenvalue. The following example serves to illustrate the statement of Gerschgorin's theory for a symmetric matrix.

Example 3.7.2

Let matrix A be

$$A = \begin{bmatrix} 2.5 & -1 & 0 \\ -1 & 5 & -\sqrt{2} \\ 0 & -\sqrt{2} & 10 \end{bmatrix}$$

Then, using formula (3.97), define three circles in the plane. The first one has its center at 2.5 and a radius $r_1 = |a_{12}| + |a_{13}| = 1$, the second has its center at 5 with a radius $r_2 = |a_{21}| + |a_{23}| = (1 + \sqrt{2})$, and the third is centered at 10 with a radius of $\sqrt{2}$. The circles are illustrated in Figure 3.2. The actual eigenvalues of the system are

$$\lambda_1 = 2.1193322$$
$$\lambda_2 = 5.00$$
$$\lambda_3 = 10.380678$$

which lie inside the Gerschgorin circles, as illustrated in Figure 3.2.

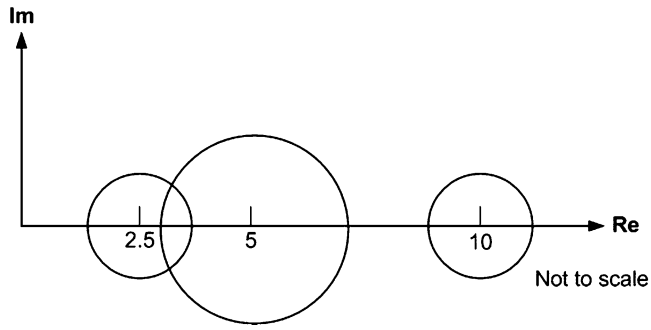

Figure 3.2 Gerschgorin circles and eigenvalues.

In the course of the development of a prototype, a system is built, analyzed, and finally tested. At that point, small adjustments are made in the design to fine-tune the system so that the prototype satisfies all the response specifications. Once these design changes are made, it may not be desirable or efficient to recalculate the eigensolution. Instead, a perturbation technique may be used to show how small changes in the elements of a matrix affect its eigensolution.

Perturbation methods are based on approximations of a function obtained by writing down a Taylor series expansion (see any introductory calculus text) for a function about some point. The equivalent statement for matrix and vector functions is more difficult to derive. However, with proper assumptions, a similar expansion can be written down for the eigenvalue problem.

In the following, let A denote an $n \times n$ symmetric matrix with distinct eigenvalues, denoted by μ_i, and refer to A as the *unperturbed matrix*. Define the matrix $A(\varepsilon)$ by $A(\varepsilon) = A + \varepsilon B$. Matrix $A(\varepsilon)$ is called the *perturbed matrix*. Note that $A(0) = A$. Furthermore, denote the eigenvalues of $A(\varepsilon)$ by $\lambda_i(\varepsilon)$ and the corresponding eigenvectors by $\mathbf{x}_i(\varepsilon)$. It is clear that, as ε approaches zero, $\lambda_i(\varepsilon)$ approaches μ_i and $\mathbf{x}_i(\varepsilon)$ approaches $\mathbf{x}_i$ for each value of index i. Here, μ_i and $\mathbf{x}_i$ are the eigenvalues and eigenvectors of A respectively (see, for instance, Lancaster, 1969). For sufficiently small ε and symmetric A and B, the expansions for $\lambda_i(\varepsilon)$ and $\mathbf{x}_i(\varepsilon)$ are

$$\lambda_i(\varepsilon) = \lambda_i + \varepsilon\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \cdots \tag{3.98}$$

and

$$\mathbf{x}_i(\varepsilon) = \mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \varepsilon^2\mathbf{x}_i^{(2)} + \cdots \tag{3.99}$$

where $\mathbf{x}_i^T\mathbf{x}_i = 1$. Here, the parenthetical superscript (k) denotes the kth derivative, with respect to the parameter ε, evaluated at $\varepsilon = 0$ and multiplied by $(1/k!)$. That is,

$$\lambda_i^{(k)} = \left(\frac{1}{k!}\right)\left[\frac{d^k\lambda_i}{d\varepsilon^k}\right]_{\varepsilon=0}$$

Here, differentiation of vector $\mathbf{x}$ is defined by differentiating each element of $\mathbf{x}$.

Next, consider the ith eigenvalue problem for the perturbed matrix

$$A(\varepsilon)\mathbf{x}_i(\varepsilon) = \lambda_i(\varepsilon)\mathbf{x}_i(\varepsilon) \tag{3.100}$$

Substitution of Equations (3.98) and (3.99) into Equation (3.100) yields

$$(A + \varepsilon B)(\mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \varepsilon^2\mathbf{x}_i^{(2)} + \cdots) = (\lambda_i + \varepsilon\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \cdots)(\mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \cdots) \tag{3.101}$$

Multiplying out this last expression and comparing coefficients of the powers of ε yields several useful relationships. The result of comparing the coefficients of ε^0 is just the eigenvalue problem for the unperturbed system. The coefficient of ε^1, however, yields the expression

$$(\lambda_i I - A)\mathbf{x}_i^{(1)} = (B - \lambda_i^{(1)} I)\mathbf{x}_i \tag{3.102}$$

Premultiplying this by $\mathbf{x}_i^T$ (suppressing the index) results in

$$\mathbf{x}^T(B-\lambda^{(1)}I)\mathbf{x}=\mathbf{x}^T(\lambda I-A)\mathbf{x}^{(1)}=0 \tag{3.103}$$

The last term in Equation (3.103) is zero, since $\mathbf{x}^T$ is the left eigenvector of A, i.e., $\mu\mathbf{x}_i^T=\mathbf{x}_i^T A$. Hence, the first term in the perturbation of the eigenvalue (recall that $\mathbf{x}^T\mathbf{x}=1$) becomes

$$\lambda_i^{(1)}=\mathbf{x}_i^T B\mathbf{x}_i \tag{3.104}$$

Equation (3.104) indicates how the eigenvalues of a matrix, and hence the natural frequencies of an undamped system, change as the result of a small change, εB, in the matrix values. This is illustrated in example 3.7.3. The preceding formulae can be used to calculate the eigenvalues of the perturbation matrix in terms of the perturbation matrix itself and the known eigensolution of the unperturbed system defined by A. Equation (3.98) can be used to yield the eigenvalues of the 'new,' or perturbed, system by making the approximations $\lambda_i(\varepsilon)=\mu_i+\varepsilon\lambda_i^{(1)}$ and using Equation (3.104). This method is good for small values of ε.

Perturbation schemes can also be used to calculate the effect of the perturbation on the eigenvectors as well. In addition, the method can be easily used for nongyroscopic conservative systems of the forms given in Equation (3.32). It has also been used for damped systems and for systems with gyroscopic forces. Example 3.7.3 illustrates its use for systems in the form of Equation (3.33).

Example 3.7.3

This example illustrates the use of perturbation calculations to find the result of making a small perturbation to a given system [here A is perturbed to $A(\varepsilon)$]

$$M^{-1/2}KM^{-1/2}=A=\begin{bmatrix}3 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 5\end{bmatrix} \quad \text{and} \quad A(\varepsilon)=\begin{bmatrix}3.1 & -1.1 & 0\\ -1.1 & 1.1 & -1\\ 0 & -1 & 5\end{bmatrix}$$

Suppose the eigensolution of A is known, i.e.,

$$\begin{aligned}
\lambda_1&=0.3983\\
\lambda_2&=3.3399\\
\lambda_3&=5.2618\\
x_i&=[0.3516 \quad 0.9148 \quad 0.1988]^T\\
x_2&=[-0.9295 \quad 0.3159 \quad 0.1903]^T\\
x_3&=[0.1113 \quad -0.2517 \quad 0.9614]^T
\end{aligned}$$

Given this information, the eigensolution of the new system $A(\varepsilon)$ is desired, where

$$\varepsilon B=A(\varepsilon)-A=(0.1)\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Here, $\varepsilon = 0.1$ is small, so that the series in Equation (3.98) converges and can be truncated. Equation (3.104) yields

$$\varepsilon\lambda_1^{(1)} = \mathbf{x}_1^T \varepsilon B \mathbf{x}_1 = 0.03172$$
$$\varepsilon\lambda_2^{(1)} = \mathbf{x}_2^T \varepsilon B \mathbf{x}_2 = 0.15511$$
$$\varepsilon\lambda_3^{(1)} = \mathbf{x}_3^T \varepsilon B \mathbf{x}_3 = 0.01317$$

Then, the new (perturbed) eigenvalues are $\lambda_i(\varepsilon) = \lambda_i + \varepsilon\lambda_i^{(1)}$

$$\lambda_1(\varepsilon) = 0.43002 \quad (0.4284)$$
$$\lambda_2(\varepsilon) = 3.55410 \quad (3.4954)$$
$$\lambda_3(\varepsilon) = 5.27497 \quad (5.2762)$$

Here, the actual values are given in parentheses for comparison.

The methods presented in this section are not really needed to compute eigenvalues. Rather, the methods of the following section should be used for computing accurate eigenvalues and modal data. The eigenvalue approximations and bounds presented in this section are significant analytical tools that can be used in design and redesign to understand how changes in the system or system model affect modal data.

3.8 COMPUTATION EIGENVALUE PROBLEMS IN MATLAB

The availability of cheap, high-speed computing and the subsequent development of high-level mathematically oriented computer codes (MATLAB, Mathcad, and Mathematic in particular) almost negate the need for eigenvalue approximation methods and schemes presented in the previous section. The very nature of many computational schemes demands that the analytical formulation change. The following presents some alternative formulations to matrix-related computations based on the available codes. The details of the various algorithms used in these codes are left to the references (Meirovitch, 1980; Golub and Van Loan, 1996; Datta, 1995). Table 3.1 lists various MATLAB commands useful in computing natural frequencies, damping ratios, and mode shapes.

The best way to compute a matrix inverse is not to. Rather, Gaussian elimination can be used effectively to solve for the inverse of a matrix. The matrix inverse can be thought of as the solution to a system of n linear equations in n variables written in the matrix form $A\mathbf{x} = \mathbf{b}$. Solving this by Gaussian elimination yields the effective inverse $\mathbf{x} = A^{-1}\mathbf{b}$.

The best way to compute the eigenvalues and eigenvectors of a matrix is to use one of the many eigenvalue routines developed by the numerical linear algebra community and packaged nicely in a variety of commercial codes. These are both numerically superior to computing the roots of the polynomial derived from $\det(\lambda I - A)$ and applicable to systems of much larger order.

Table 3.1 Sample MATLAB matrix commands for solving the eigenvalue problem.

`M = [1 0; 0 4]`	creates the mass matrix of example 3.7.1
`Chol(M)`	computes the Cholesky factor of matrix M
`Sqrtm(M)`	computes the matrix square root of M
`inv(M)`	computes the inverse of matrix M
`M\I`	computes the inverse of matrix M, using Gaussian elimination
`d = eig(A)`	returns a vector $\mathbf{d}$ containing the eigenvalues of A
`[V,D] = eig(A)`	returns a matrix V of eigenvectors and a matrix D of eigenvalues
`[V,D] = eig(A,'nobalance')`	returns a matrix V of eigenvectors and a matrix D of eigenvalues without balancing
`d = eig(A,B)`	returns a vector $\mathbf{d}$ of eigenvalues, using the generalized problem $A\mathbf{x} = \lambda B\mathbf{x}$ (works for a singular B matrix)
`[V,D] = eig(A,B)`	returns a matrix D of eigenvalues and a matrix V of mode shapes, solving the generalized problem $A\mathbf{x} = \lambda B\mathbf{x}$

The matrix square root can be computed by using the function of a matrix approach, which is trivial for diagonal matrices (as is often, but not always, the case for the mass matrix). However, for nondiagonal matrices, the square root involves solving the eigenvalue problem for the matrix. This is given in Equation (3.31) and repeated here. If M is a positive definite matrix, then its eigenvalues μ_i are all positive numbers, and its eigenvectors $\mathbf{u}_i$ form an orthonormal set and can be used to form an orthogonal matrix $S = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n]$ such that $S^T MS = \text{diag}(\mu_i)$. Then, any scalar function f of matrix M can be computed by

$$f(M) = S\text{diag}[f(\mu_1) \quad f(\mu_2) \ \cdots \ f(\mu_n)]S^T \tag{3.105}$$

In particular, the inverse and matrix square root of any positive definite matrix can be computed with Equation (3.105).

An alternative to the eigenvalue decomposition of Equation (3.105) is to use the Cholesky decomposition, or Cholesky factors, of a positive definite matrix. Cholesky noted that every positive definite matrix can be factored into the product of an upper triangular matrix R and its transpose: $M = R^T R$. In this case it follows that

$$(R^T)^{-1} M R^{-1} = I$$

Hence, the Cholesky factor R behaves like a square root. In fact, if M is diagonal, $R = R^T$ is the square root of M.

The most efficient way to compute the undamped eigenvalues is to use the Cholesky factors. In this case the transformations of Equations (3.33) and (3.72) become

$$\tilde{K} = (R^T)^{-1} K R^{-1} \qquad \text{and} \qquad \tilde{C} = (R^T)^{-1} C R^{-1}$$

So far, several different approaches to computing the natural frequencies and mode shapes of a conservative system have been presented. These are summarized in Table 3.2, along with a computational 'time' measured by listing the floating-point operations per second (flops) for a given example in MATLAB.

Note from Table 3.2 that using the Cholesky factor R requires the least flops to produce the eigenvalues and eigenvectors. The next 'fastest calculation' is using Gaussian elimination

Table 3.2 Comparison of the computing 'time' required to calculate eigenvalues and eigenvectors for the various methods for a conservative system.

Method	Flops
`inv (R')*K*inv (R)`	118
`M\K`	146
`inv (M)*K`	191
`inv (sqrtm(M))*K* inv (sqrtm(M))`	228
`[V,D]=eig (K,M)`	417

to compute $M^{-1}K$, but this becomes an asymmetric matrix so that the eigenvectors are not orthogonal, and hence an additional computational step is required.

The eigenvalue problem can also be placed into a number of state matrix forms, and these are now presented. The first and most common case is given by Equation (2.20). The associated eigenvalue problem for the state matrix is asymmetric and in general gives complex eigenvalues and eigenvectors. In addition, the eigenvectors of the state matrix are twice as long and related to the eigenvectors $\mathbf{u}_i$ in second-order form by

$$A\mathbf{z} = \lambda\mathbf{z}, \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \Rightarrow \mathbf{z}_i = \begin{bmatrix} \mathbf{u}_i \\ \lambda_i \mathbf{u}_i \end{bmatrix} \tag{3.106}$$

The eigenvalues, however, are exactly the same.

Other state-space approaches can be formulated by rearranging the equations of motion in state-space form. For instance, in Equation (3.64) let

$$\mathbf{y}_1 = \mathbf{q} \qquad \text{and} \qquad \mathbf{y}_2 = \dot{\mathbf{q}}$$

This then implies that

$$\dot{\mathbf{y}}_1 = \mathbf{y}_2$$

and hence

$$-K\dot{\mathbf{y}}_1 = -K\mathbf{y}_2$$

Then the equation of motion can be written as

$$M\dot{\mathbf{y}}_2 = -C\mathbf{y}_2 - K\mathbf{y}_1$$

Combining the last two expressions yields the state-space system and symmetric generalized eigenvalue problem:

$$\underbrace{\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}}_{A} \begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix}}_{B} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \Rightarrow \lambda A\mathbf{y} = B\mathbf{y}$$

which does not require a matrix inverse.

Alternative forms of solving the eigenvalue problem can be useful for special cases, such as a nearly singular mass matrix. Such formulae can also be useful for analysis. Once the state-space eigenvalue problem is solved, the data need to be related to natural frequencies, damping ratios, and mode shapes of the physical system. This can be done in the case of an underdamped system by representing all of the eigenvalues as the complex pairs

$$\lambda_i = -\zeta_i\omega_i - \omega_i\sqrt{1-\zeta_i^2}j \qquad \text{and} \qquad \lambda_{i+1} = -\zeta_i\omega_i + \omega_i\sqrt{1-\zeta_i^2}j$$

Comparing this form with the complex form $\lambda_i = \alpha_i + \beta_i j = \mathrm{Re}(\lambda_i) + \mathrm{Im}(\lambda_i)j$ shows that the modal frequencies and damping ratios can be determined by

$$\begin{aligned} \omega_i &= \sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2} \\ \zeta_i &= \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}} = \frac{-\mathrm{Re}(\lambda_i)}{\sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}} \end{aligned} \tag{3.107}$$

The mode shapes are taken as the first n values of the $2n$ state vector by the relationship given in Equation (3.106). The mode shapes in this case are likely to be complex valued even if the condition for normal modes to exist is satisfied ($DM^{-1}K = KM^{-1}D$). In this case there will be a normalizing condition on $\mathbf{u}$ in Equation (3.106) that will normalize the modes to be real valued. If, however, $DM^{-1}K \neq KM^{-1}D$, then vector $\mathbf{u}$ will be complex, meaning that the masses pass through their equilibrium out of phase with each other.

3.9 NUMERICAL SIMULATION OF THE TIME RESPONSE IN MATLAB

The time response can be computed by calculating the eigenvalues and eigenvectors of the system and then forming the summation of modes as outlined in example 3.3.2. This same procedure also works for the damped case as long as the damping is proportional. However, for systems that do not have proportional damping (the nonsymmetric $KM^{-1}C$ matrix), the modal summations are overcomplex values, which can occasionally lead to confusion. In these cases, numerical simulation can be performed to compute the time response directly without computing the eigenvalues and eigenvectors. The method follows directly from the material in Section 1.10 with the state-space model of Equations (2.20) and (3.106). For any class of second-order systems, the equations of motion can be written in state-space form as given in Equation (2.20) and repeated here (for the free response case, $\mathbf{f}(t) = \mathbf{0}$):

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}(K+H) & -M^{-1}(D+G) \end{bmatrix}$$

To solve this using numerical integration, the Runge–Kutta ode command in MATLAB is used. The ode command uses a fifth-order Runge–Kutta automated time step method for numerically integrating the equation of motion (see, for instance, Inman, 2001). The following example illustrates the procedure.

Example 3.9.1

Compute the response of the system

$$M=\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix},\qquad D=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix},\qquad G=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\qquad K=\begin{bmatrix}10 & -4\\ -4 & 4\end{bmatrix}$$

to the initial conditions

$$\mathbf{x}(0)=\begin{bmatrix}0.1\\ 0\end{bmatrix}\text{ m},\qquad \dot{\mathbf{x}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix}\text{ m/s}$$

using MATLAB numerical integration.

In order numerically to integrate the equations of motion in MATLAB using Runge–Kutta, an m-file containing the system dynamics must first be created and stored (see example 1.10.2). The following file sets up the equations of motion in state-space form:

```
function v=f391(t,x)
M=[4 0; 0 3];D=[2 -1;-1 1];G=[0 1; -1 0];K=[10 -4;-4 4];
A=[zeros(2) eye(2);-inv(M)*K -inv(M)*(D+G)];
v=A*x;
```

This function must be saved under the name f391.m. Note that the command `zeros(n)` produces an $n \times n$ matrix of zeros and that the matrix `eye(n)` creates an $n \times n$ identity matrix. Once this is saved, the following is typed in the command window:

```
EDU>clear all
EDU>xo=[0.1;0;0;0];
EDU>ts=[0 40];
EDU>[t,x]=ode45('f391',ts,xo);
EDU>plot(t,x(:,1),t,x(:,2),'--')
```

This returns the plot shown in Figure 3.3. Note that the command `x(:,1)` pulls off the record for $x_1(t)$ and the command ode45 calls a fifth-order Runge–Kutta program. The command `ts=[0 40];` tells the code to integrate from 0 to 40 time units (seconds in this case).

The plot illustrated in Figure 3.3 can also be labeled and titled using additional plotting commands in MATLAB. For instance, typing `,title('displacement versus time')` after the plot command in the code in example 3.9.1 would add a title to the plot.

This numerical solution technique also still applies if the system is nonlinear. In this case the state-space formulation becomes a nonlinear vector rather than a matrix. This form was illustrated in Equations (1.66) and (1.67), and again in Section 2.7. An example of the state-space form of a nonlinear system is given in example 2.7.2.

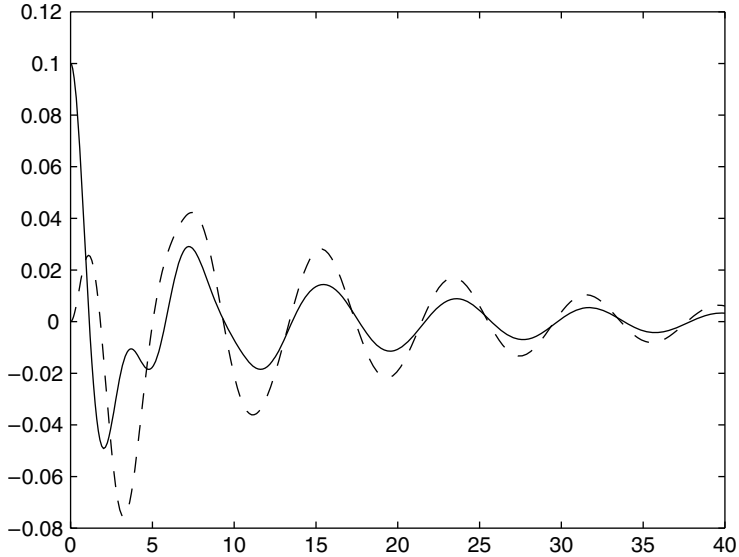

Figure 3.3 Response $q_1(t)$ versus time (solid line) and response $q_2(t)$ versus time (dashed line) as computed in MATLAB using numerical integration.

CHAPTER NOTES

The material of Section 3.2 can be found in any text concerning linear algebra or matrices, such as Lancaster (1969). An excellent quick summary of relevant matrix results is available in the first chapter of Huseyin (1978). A very good historical account and development can be found in Bellman (1960, 1970). An explanation of mode shapes and undamped natural frequencies in Section 3.3 can be found in any modern vibration text. Most linear algebra and matrix texts devote several chapters to canonical forms (Section 3.4); for instance, both Lancaster (1966) and Bellman (1970) do. The development of lambda matrices of Section 3.5 stems mostly from the book and work of Lancaster (1966), who has published extensively in that area. The idea of decoupling the equations of motion is based on the result of commuting matrices discussed in Bellman (1960) and was set straight in the engineering literature by Caughey and O'Kelly (1965). The extension of critical damping and the like to multiple-degree-of-freedom systems of Section 3.6 comes directly from Inman and Andry (1980), which contains all the references up to that date. Since then, several results have appeared that examine more efficient means of computing a critical damping matrix. Nicholson and Inman (1983) provide a review of oscillation results. Barkwell and Lancaster (1992) corrected the overdamping condition by pointing out that the result initially reported (Inman and Andry, 1980) was only a local condition. Papargyri-Beskou, Thessaloniki, and Beskos (2002) provide interesting examples and results regarding critical damping. The material of Section 3.7 follows the pattern presented in Meirovitch (1980); however, Rayleigh quotients are discussed in every vibration text and most texts on matrices – in particular, Bellman (1970) and Lancaster (1969). Bellman (1970) also treats the lacing of eigenvalues in a rigorous fashion. Gerschgorin's result is also to be found in many texts on matrices. An excellent treatment of perturbation methods can be found in Kato (1966).

The results presented in Section 3.7 on perturbation of eigenvalues are due to Lancaster (1969). Other applications of perturbation results to vibration problems are presented in Hagedorn (1983) and Meirovitch and Ryland (1979). Key papers in the development of linear systems and control using linear algebra can be found in Patel, Laub, and Van Dooren (1994). Information and sample codes for solving dynamics problems in MATLAB can be found in Soutas-Little and Inman (1999) or by simply typing 'MATLAB' into Google.

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PROBLEMS

3.1 Check if the four vectors given by $\mathbf{x}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^T$, and $\mathbf{x}_4 = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^T$ are independent.

3.2 Select a basis for R^3, which denotes the set of all 3×1 vectors with real elements, from the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$, $\mathbf{x}_3 = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}^T$, and $\mathbf{x}_4 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$.

3.3 Determine whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

is singular or not by calculating the value of its determinant.

3.4 Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

3.5 Consider the following system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

with initial conditions $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

(a) Calculate the eigenvalues of the system.
(b) Calculate the eigenvectors and normalize them.
(c) Use (a) and (b) to write the solution $\mathbf{x}(t)$ for the preceding initial conditions.
(d) Sketch $x_1(t)$ versus t and $x_2(t)$ versus $\mathbf{t}$.
(e) What is the solution if $\mathbf{x}(0) = [0 \quad 0]^T$ and $\dot{\mathbf{x}}(0) = [0 \quad 0]^T$?

3.6 Calculate the natural frequencies of the following system:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

3.7 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and calculate its eigenvalues and eigenvectors. Are the left and right eigenvectors the same? Are they orthogonal? Are they biorthogonal?

3.8 Does the following system have normal modes (i.e., does it decouple)?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 15 & -3 \\ -3 & 3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

3.9 Does the system in problem 3.8 oscillate? Why or why not?

3.10 Consider the following system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \mathbf{x} = 0$$

(a) Calculate the eigenvalues of the system.
(b) Calculate the system eigenvectors and normalize them.
(c) Show that the eigenvectors can be used to diagonalize the system.
(d) Calculate the modal damping ratios and damped and undamped natural frequencies.
(e) Calculate the free response for $\mathbf{x}^T(0) = [1\ 0], \dot{\mathbf{x}}^T(0) = [0\ 0]$.
(f) Plot the responses $x_1(t)$ and $x_2(t)$ as well as $||\mathbf{x}(t)||$.

3.11 Calculate the eigenvalues for the matrix

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

what are the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}?$$

Think before you calculate anything.

3.12 For the matrix in problem 3.11, calculate $\mathbf{x}^T A \mathbf{x}_1 / \mathbf{x}_1^T \mathbf{x}_1$ and $\mathbf{x}_2^T A \mathbf{x}_2 / \mathbf{x}_2^T \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the eigenvectors of A. Next, choose *five* different values of vector $\mathbf{x}$ and calculate the five scalars $\mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x}$ for your five choices. Compare all of these numbers with the values of the eigenvalues computed in problem 3.11. Can you draw any conclusions?

3.13 Consider the following model of a machine part that has equations of motion given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \mathbf{x} = 0$$

Let $k_1 = 2$ and $k_2 = 1$. The elements of M are known precisely, whereas the elements of K are known only to within 0.01 at worst. (Everything here is dimensionless.) Note that the machine will fail if it is disturbed by a driving frequency equal to one of the natural frequencies of the system. If there is a disturbance to this system of frequency $\sqrt{0.15}\, j$ ($\lambda = 0.15$) will this system fail? Why or why not? Try to work this out with a minimum of calculation.

3.14 Referring to problem 3.13, suppose that, in order to satisfy a given manufacturing change, the spring coefficient k_1 is required to change from 2 to 2.1 units. How will this affect the natural frequencies of the system? Give a quantitative answer without recalculating the eigenvalues, that is, use perturbation results.

3.15 If m_1 is neglected in problem 3.13, i.e., if the order of the model is reduced by one, by what would you expect the natural frequency of the new system to be bounded? Check your result by calculation.

3.16 Show that Gerschgorin's theory works for the matrices

$$R = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

3.17 Show that the solution of Equations (3.77) through (3.79) requires either $c_1 = k_1 = 0$ or $c_2 = k_2 = 0$.

3.18 Prove that, if A is similar to a diagonal matrix, then the eigenvectors of A form a linearly independent set.

3.19 Derive the relationship between S_m of Equation (3.20) and matrix S of Equation (3.17).

3.20 Show that the matrices $M, M^{1/2}, M^{-1}$ and M^2 all have the same eigenvectors. How are the eigenvalues related?

3.21 Prove that, if a real symmetric matrix has positive eigenvalues, then it must be positive definite.

3.22 Derive Equation (3.40). Let $n = 3$, and solve symbolically for the constants of integration.

3.23 Derive Equation (3.42) from Equation (3.39).

3.24 Let S be the matrix of eigenvectors of the symmetric matrix A. Show that $S^T AS$ is diagonal and compare it with SAS^T.

3.25 Derive the relationship between the modal matrix S of example 3.3.2 and the matrix S_m of Equation (3.21).

3.26 Use perturbation to calculate the effect on the eigenvalues of matrix A given in example 3.7.2 by making the following changes in A: change a_{11} by 0.1, a_{12} and a_{21} by 0.1, and a_{22} by 0.2.

3.27 A geometric interpretation of the eigenvector problem for a 2×2 matrix is that the eigenvectors determine the principal axis of an ellipse. Calculate matrix A for the quadratic form $2x_1^2 + 2x_1x_2 + 2x_2^2 = 3 = \mathbf{x}^T A\mathbf{x}$. Then use the eigenvector of A to determine the principal axis for the ellipse.

3.28 Show that the eigenvalues for the first-order form [Equation (2.20)] are equivalent to the latent roots of Equation (3.65) by noting that

$$\det \begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det A \ \det[B - CA^{-1}D]$$

as long as A^{-1} exists, for the case where $G = H = 0$.

3.29 Show that the generic system of Equation (3.73) has normal modes if and only if

$$\frac{c_1}{c_2} = \frac{k_1}{k_2}.$$

3.30 Consider the system defined by the following coefficient matrices:

$$M = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{bmatrix}, \qquad K = \begin{bmatrix} 2000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$$

Compute the eigenvalues, eigenvectors, and natural frequencies.

3.31 Consider again the system of problem 3.30 and determine the effects of damping. Suppose a damping matrix of the form

$$C = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 30 & -20 \\ 0 & -20 & 20 \end{bmatrix}$$

is added to the system of problem 3.30. Is the system overdamped, underdamped, critically damped, or does it exhibit mixed damping? Does the system have normal modes or not?

3.32 Compute the eigenvalues and eigenvectors for the system of problem 3.31. Also compute the natural frequencies and mode shapes. If you worked out problem 3.31, do your computations agree with the results obtained there?

3.33 Compute the response of the system defined in problem 3.31 to the initial displacement $\mathbf{x}(0) = \begin{bmatrix} 0.01 & 0 & 0 & -0.01 \end{bmatrix}^T$ and zero initial velocity.

3.34 Consider the system of problem 3.30 with a gyroscopic term added of the form

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the eigenvalues and eigenvectors. What are the natural frequencies?

3.35 Compute the time response of the system of problem 3.34 to the initial displacement $\mathbf{x}(0) = \begin{bmatrix} 0.01 & 0 & 0 & -0.01 \end{bmatrix}^T$ and zero initial velocity.

3.36 Show that the coefficient c_i in Equation (3.42) can be written as

$$c_i = \pm \frac{1}{\omega_i} \sqrt{\mathbf{r}^T(0)\mathbf{u}_i\mathbf{u}_i^T\mathbf{r}(0) + \dot{\mathbf{r}}^T(0)\mathbf{u}_i\mathbf{u}_i^T\dot{\mathbf{r}}(0)}$$

4
Stability

4.1 INTRODUCTION

A rough idea concerning the concept of stability was introduced for single-degree-of-freedom systems in the first chapter. It was pointed out that the sign of the coefficients of the acceleration, velocity, and displacement terms determined the stability behavior of a given single-degree-of-freedom system. That is, if the coefficients have the proper sign, the motion will always remain within a given bound. This idea is extended in this chapter to the multiple-degree-of-freedom systems described in the previous two chapters. As in the case of the oscillatory behavior discussed in Chapter 3, the criterion based on the sign of the coefficients is translated into a criterion based on the definiteness of certain coefficient matrices.

It should be noted that no universal definition of stability exists, but rather variations are adopted depending on the nature of the particular problem under consideration. However, all definitions of stability are concerned with the response of a system to certain disturbances and whether or not the response stays within certain bounds.

4.2 LYAPUNOV STABILITY

The majority of the work done on the stability behavior of dynamical systems is based on a formal definition of stability given by Lyapunov (see, for instance, Hahn, 1963). This definition is stated with reference to the equilibrium point, $\mathbf{x}_0$, of a given system. In the case of the linear systems considered in this chapter, the equilibrium point can always be taken to be the zero vector. In addition, the definition of Lyapunov is usually stated in terms of the state vector of a given system rather than in physical coordinates directly, so that the equilibrium point refers to both the position and velocity.

Let $\mathbf{x}(0)$ represent the vector of initial conditions for a given system (both position and velocity). The system is said to have a *stable equilibrium* if, for any arbitrary positive number ε, there exists some positive number $\delta(\varepsilon)$ such that, whenever $||\mathbf{x}(0)|| < \delta$, then $||\mathbf{x}(t)|| < \varepsilon$ for all values of $t > 0$. A physical interpretation of this mathematical definition is that, if the initial state is within a certain value, i.e., $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then the motion stays within another bound for all time, i.e., $||\mathbf{x}(t)|| < \varepsilon$. Here, $||\mathbf{x}(t)||$, called the norm of $\mathbf{x}$, is defined by $||\mathbf{x}(t)|| = (\mathbf{x}^T\mathbf{x})^{1/2}$.

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

To apply this definition to the single-degree-of-freedom system of Equation (1.1), note that $\mathbf{x}(t) = [x(t)\ \dot{x}(t)]^T$. Hence

$$||\mathbf{x}(t)|| = (\mathbf{x}^T\mathbf{x})^{1/2} = \sqrt{x^2(t) + \dot{x}^2(t)}$$

For the sake of illustration, let the initial conditions be given by $\mathbf{x}(0) = 0$ and $\dot{\mathbf{x}}(0) = \omega_n = \sqrt{k/m}$. Then the solution is given by $x(t) = \sin \omega_n t$. Intuitively, this system has a stable response as the displacement response is bounded by 1, and the velocity response is bounded by ω_n. The following simple calculation illustrates how this solution satisfies the Lyapunov definition of stability.

First, note that

$$\|\mathbf{x}(0)\| = (\mathbf{x}^2(0) + \dot{\mathbf{x}}^2(0))^{1/2} = (0 + \omega_n^2)^{1/2} = \omega_n \tag{4.1}$$

and that

$$\|\mathbf{x}(t)\| = [\sin^2 \omega_n t + \omega_n^2 \cos^2 \omega_n t]^{1/2} < (1 + \omega_n^2)^{1/2} \tag{4.2}$$

These expressions show exactly how to choose δ as a function of ε for this system. From Equation (4.2) note that, if $(1 + \omega_n^2)^{1/2} < \varepsilon$, then $\|x(t)\| < \varepsilon$. From Equation (4.1) note that, if $\delta(\varepsilon)$ is chosen to be $\delta(\varepsilon) = \varepsilon\omega_n(1 + \omega_n^2)^{-1/2}$, then the definition can be followed directly to show that, if

$$\|\mathbf{x}(0)\| = \omega_n < \delta(\varepsilon) = \frac{\varepsilon\omega_n}{\sqrt{1 + \omega_n^2}}$$

is true, then $\omega_n < \varepsilon\omega_n/\sqrt{1 + \omega_n^2}$. This last expression yields

$$\sqrt{1 + \omega_n^2} < \varepsilon$$

That is, if $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then $\sqrt{1 + \omega_n^2} < \varepsilon$ must be true, and Equation (4.2) yields that

$$\|\mathbf{x}(t)\| \leq \sqrt{1 + \omega_n^2} < \varepsilon$$

Hence, by a judicious choice of the function $\delta(\varepsilon)$, it has been shown that, if $||\mathbf{x}(0)|| < \delta(\varepsilon)$, then $||\mathbf{x}(t)|| < \varepsilon$ for all $t > 0$. This is true for any arbitrary choice of the positive number ε.

The preceding argument demonstrates that the undamped harmonic oscillator has solutions that satisfy the formal definition of Lyapunov stability. If dissipation, such as viscous damping, is included in the formulation, then not only is this definition of stability satisfied, but also

$$\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0 \tag{4.3}$$

Such systems are said to be *asymptotically stable*. As in the single-degree-of-freedom case, if a system is asymptotically stable it is also stable. In fact, by definition, a system is asymptotically stable if it is stable and the norm of its response goes to zero as t becomes large. This can be seen by examining the definition of a limit (see Hahn, 1963).

The procedure for calculating $\delta(\varepsilon)$ is similar to that of calculating ε and δ for limits and continuity in beginning calculus. As in the case of limits in calculus, this definition of

stability does not provide the most efficient means of checking the stability of a given system. Hence, the remainder of this chapter develops methods to check the stability properties of a given system that require less effort than applying the definition directly.

There are many theories that apply to the stability of multiple-degree-of-freedom systems, some of which are discussed here. The most common method of analyzing the stability of such systems is to show the existence of a Lyapunov function for the system. A *Lyapunov function*, denoted by $V(\mathbf{x})$, is a real scalar function of the vector $\mathbf{x}(t)$, which has continuous first partial derivatives and satisfies the following two conditions:

1. $V(\mathbf{x}) > 0$ for all values of $\mathbf{x}(t) \neq 0$.
2. $\dot{V}(\mathbf{x}) < 0$ for all values of $\mathbf{x}(t) \neq 0$.

Here, $\dot{V}(\mathbf{x})$ denotes the time derivative of the function $V(\mathbf{x})$. Based on this definition of a Lyapunov function, several extremely useful stability results can be stated. The first result states that, if there exists a Lyapunov function for a given system, then that system is *stable*. If, in addition, the function $\dot{V}(\mathbf{x})$ is strictly less than zero, then the system is *asymptotically stable*. This is called the *direct*, or *second, method* of Lyapunov. It should be noted that, if a Lyapunov function cannot be found, nothing can be concluded about the stability of the system, as the Lyapunov theorems are only sufficient conditions for stability.

The stability of a system can also be characterized by the eigenvalues of the system. In fact, it can easily be shown that a given linear system is stable if and only if it has no eigenvalue with a positive real part. Furthermore, the system will be asymptotically stable if and only if all of its eigenvalues have negative real parts (no zero real parts allowed). These statements are certainly consistent with the discussion in Section 4.1. The correctness of the statements can be seen by examining the solution using the expansion theorem (modal analysis) of the previous chapter [Equation (3.68)]. The eigenvalue approach to stability has the attraction of being both necessary and sufficient. However, calculating all the eigenvalues of the state matrix of a system is not always desirable.

The preceding statements about stability are not always the easiest criteria to check. In fact, use of the eigenvalue criteria requires almost as much calculation as computing the solution of the system. The interest in developing various different stability criteria is to find conditions that (1) are easier to check than calculating the solution, (2) are stated in terms of the physical parameters of the system, and (3) can be used to help design and/or control systems to be stable. Again, these goals can be exemplified by recalling the single-degree-of-freedom case, where it was shown that the sign of the coefficients m, c, and k determine the stability behavior of the system. To this end, more convenient stability criteria are examined on the basis of the classifications of a given physical system stated in Chapter 2.

4.3 CONSERVATIVE SYSTEMS

For conservative systems of the form

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.4}$$

where M and K are symmetric, a simple stability condition results – namely if M and K are positive definite, the eigenvalues of K are all positive, and hence the eigenvalues of the

system are all purely imaginary. The solutions are then all linear combinations of terms of the form $e^{\pm\omega_n jt}$, or, by invoking Euler's formula, all terms are of the form $A\sin(\omega_n t + \phi)$. Thus, from the preceding statement, the system of Equation (4.4) is stable, since both the displacement and velocity response of the system are always less than some constant (A and $\omega_n A$ respectively) for all time and for any initial conditions.

Also, note that, if K has a negative eigenvalue, then the system has a positive real exponent. In this case one mode has a temporal coefficient of the form e^{at}, $a > 0$, which grows without bound, causing the system to become unstable (note that, in this case, $\delta(\varepsilon)$ cannot be found).

The condition that K be positive definite can be coupled with the determinant condition, discussed in Section 3.3, to yield inequalities in the system parameters. In turn, these inequalities can be used as design criteria. It should be pointed out that, in most mechanical systems, K will be positive definite or positive semidefinite, unless some applied or external force proportional to displacement is present. In control theory, the applied control force is often proportional to position, as indicated in Equation (2.17) and example 2.4.4.

It is instructive to note that the function $V(\mathbf{q})$ defined by (the energy in the system)

$$V(\mathbf{q}) = \frac{1}{2}[\dot{\mathbf{q}}^T(t)M\dot{\mathbf{q}}(t) + \mathbf{q}^T(t)K\mathbf{q}(t)] \tag{4.5}$$

serves as a Lyapunov function for the system in Equation (4.4). To see this, note first that $V(\mathbf{q}) > 0$, since M and K are positive definite, and that

$$\frac{d}{dt}[V(\mathbf{q})] = \dot{\mathbf{q}}^T M\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T K\mathbf{q} \tag{4.6}$$

Now, if $\mathbf{q}(t)$ is a solution of Equation (4.4), it must certainly satisfy Equation (4.4). Thus, premultiplying Equation (4.4) by $\dot{\mathbf{q}}^T$ yields

$$\dot{\mathbf{q}}^T M\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T K\mathbf{q} = 0 \tag{4.7}$$

This, of course, shows that $\dot{V}(\mathbf{q}) = 0$. Hence, $V(\mathbf{q})$ is a Lyapunov function and, by the second method of Lyapunov, the equilibrium of the system described by Equation (4.4) is stable.

In cases where K may be positive semidefinite, the motion corresponding to the zero eigenvalue of K is called a *rigid body mode* and corresponds to a translational motion. Note that in this case Equation (4.5) is not a Lyapunov function because $V(\mathbf{q}) = 0$ for $\mathbf{q} \neq \mathbf{0}$, corresponding to the singularity of matrix K. Since the other modes are purely imaginary, such systems may still be considered well behaved because they consist of stable oscillations superimposed on the translational motion. This is common with moving mechanical parts. This explains why the concept of stability is defined differently in different situations. For instance, in aircraft stability, some rigid body motion is desirable.

4.4 SYSTEMS WITH DAMPING

As in the single-degree-of-freedom system case, if damping is added to a stable system (4.4), the resulting system can become asymptotically stable. In particular, if M, D, and K are all symmetric and positive definite, then the system

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.8}$$

is asymptotically stable. Each of the eigenvalues of Equation (4.8) can be shown to have a negative real part. Again, since the conditions of stability are stated in terms of the definiteness of the coefficient matrices, the stability condition can be directly stated in terms of inequalities involving the physical constants of the system.

To see that this system has a stable equilibrium by using the Lyapunov direct method, note that $V(\mathbf{q})$ as defined by Equation (4.5) is still a Lyapunov function for the damped system of Equation (4.8). In this case, the solution $\mathbf{q}(t)$ must satisfy

$$\dot{\mathbf{q}}^T M\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T K\mathbf{q} = -(\dot{\mathbf{q}}^T D\dot{\mathbf{q}}) \tag{4.9}$$

which comes directly from Equation (4.8) by premultiplying by $\dot{\mathbf{q}}^T(t)$. This means that the time derivative of the proposed Lyapunov function, $\dot{V}(\mathbf{q})$, is given by Equation (4.9) to be

$$\frac{d}{dt}V(\mathbf{q}(t)) = -\dot{\mathbf{q}}^T D\dot{\mathbf{q}} < 0 \tag{4.10}$$

This is negative for all nonzero values of $\mathbf{q}(t)$ because matrix D is positive definite. Hence, $V(\mathbf{q})$ defined by Equation (4.5) is in fact a Lyapunov function for the system described by Equation (4.8), and the system equilibrium is stable. Furthermore, since the inequality in expression (4.10) is strict, the equilibrium of the system is asymptotically stable.

An illustration of an asymptotically stable system is given in example 2.4.4. The matrices M, D, and K are all positive definite. In addition, the solution of problem 3.10 shows that both elements of the vector $\mathbf{q}(t)$ are combinations of the functions $e^{-at}\sin\omega_n t$, $a > 0$. Hence, each element goes to zero as t increases to infinity, as the definition (4.3) indicates it should.

4.5 SEMIDEFINITE DAMPING

An interesting situation occurs when the damping matrix in Equation (4.8) is only positive semidefinite. Then the above argument for the existence of a Lyapunov function is still valid, so that the system is stable. However, it is not clear whether or not the system is asymptotically stable. There are two equivalent answers to this question of asymptotic stability for systems with a semidefinite damping matrix.

The first approach is based on the *null space* of the matrix D. The null space of matrix D is the set of all nonzero vectors $\mathbf{x}$ such that $D\mathbf{x} = \mathbf{0}$, i.e., the set of those vectors corresponding to the zero eigenvalues of matrix D. Since D is semidefinite in this situation, there exists at least one nonzero vector $\mathbf{x}$ in the null space of D. Moran (1970) showed that, if D is semidefinite in Equation (4.8), then the equilibrium of Equation (4.8) is asymptotically stable if and only if none of the eigenvectors of matrix K lies in the null space of D.

This provides a convenient, necessary, and sufficient condition for asymptotic stability of semidefinite systems, but it requires the computation of the eigenvectors for K or at least the null space of D.

Physically, this result makes sense because, if there is an eigenvector of matrix K in the null space of D, the vector also becomes an eigenvector of the system. Furthermore, this eigenvector results in a zero damping mode for the system, and hence a set of initial conditions exists that excites the system into an undecaying harmonic motion.

The second approach avoids having to solve an eigenvector problem to check for asymptotic stability. Walker and Schmitendorf (1973) showed that the system of Equation (4.8) with semidefinite damping will be asymptotically stable if and only if

$$\text{Rank}\begin{bmatrix} D \\ DK \\ DK^2 \\ \vdots \\ DK^{n-1} \end{bmatrix} = n \tag{4.11}$$

where n is the number of degrees of freedom of the system. The rank of a matrix is the number of linearly independent rows (or columns) the matrix has (see Appendix B). This type of rank condition comes from control theory considerations and is used and explained again in Chapter 7.

These two approaches are equivalent. They essentially comment on whether or not the system can be transformed into a coordinate system in which one or more modes are undamped. It is interesting to note that, if D is semidefinite and $KM^{-1}D$ is symmetric, then the system is not asymptotically stable. This results since, as pointed out in section 3.5, if $KM^{-1}D = DM^{-1}K$, then $\tilde{K}$ and $\tilde{D}$ have the same eigenvectors and the system can be decoupled. In this decoupled form there will be at least one equation with no velocity term corresponding to the zero eigenvalue of D. The solution of this equation will not go to zero with time, and hence the system cannot be asymptotically stable.

4.6 GYROSCOPIC SYSTEMS

The stability properties of gyroscopic systems provide some very interesting and unexpected results. First, consider an undamped gyroscopic system of the form

$$M\ddot{\mathbf{q}} + G\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.12}$$

where M and K are both positive definite and symmetric and where G is skew-symmetric. Since the quadratic form $\dot{\mathbf{q}}^T G\dot{\mathbf{q}}$ is zero for any choice of q, the Lyapunov function for the previous system [Equation (4.5)] still works for Equation (4.12), and the equilibrium of Equation (4.12) is stable.

If matrix K in Equation (4.12) is indefinite, semidefinite, or negative definite, the system may still be stable. This is a reflection of the fact that gyroscopic forces can sometimes be used to stabilize an unstable system. A child's spinning top provides an example of such a situation. The vertical position of the top is unstable until the top is spun, providing a stabilizing gyroscopic force.

Easy-to-use conditions are not available to check if Equation (4.12) is stable when K is not positive definite. Hagedorn (1975) has been able to show that, if K is negative definite and if the matrix $4K - GM^{-1}G$ is negative definite, then the system is definitely unstable. Several authors have examined the stability of Equation (4.12) when the dimension of the system is $n = 2$. Teschner (1977) showed that, if $n = 2, K$ is negative definite, and $4K - GM^{-1}G$ is positive definite, then the system is stable. Inman and Saggio (1985) showed that, if $n = 2$, K is negative definite, det $K > 0$, and the trace of $4K - GM^{-1}G$ is positive, then the system is stable.

Huseyin, Hagedorn, and Teschner (1983) showed that, for any degree-of-freedom system, if $4K - GM^{-1}G$ is positive definite and if the matrix $(GM^{-1}K - KM^{-1}G)$ is positive semidefinite, then the system is stable. In addition, they showed that, if $GM^{-1}K = KM^{-1}G$, then the system is stable if and only if the matrix $4K - GM^{-1}G$ is positive definite. These represent precise conditions for the stability of undamped gyroscopic systems. Most of these ideas result from Lyapunov's direct method. The various results on gyroscopic systems are illustrated in example 4.6.1. Bernstein and Bhat (1995) give additional examples and summarize known stability conditions up to 1994.

Example 4.6.1

Consider a simplified model of a mass mounted on a circular, weightless rotating shaft that is also subjected to an axial compression force. This system is described by Equation (4.12) with

$$M = I, \qquad G = 2\gamma \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} c_1 - \gamma^2 - \eta & 0 \\ 0 & c_2 - \gamma^2 - \eta \end{bmatrix}$$

where γ represents the angular velocity of the shaft and η the axial force. The parameters c_1 and c_2 represent the flexural stiffness in two principal directions, as illustrated in Figure 4.1.

It is instructive to consider this problem first for fixed rotational speed ($\gamma = 2$) and for $\eta = 3$. Then the relevant matrices become ($M = I$)

$$K = \begin{bmatrix} c_1 - 7 & 0 \\ 0 & c_2 - 7 \end{bmatrix}$$

$$4K - GM^{-1}G = 4\begin{bmatrix} c_1 - 3 & 0 \\ 0 & c_2 - 3 \end{bmatrix}$$

Figure 4.2 shows plots of stable and unstable choices of c_1 and c_2 using the previously mentioned theories. To obtain the various regions of stability illustrated in Figure 4.2, consider the following calculations:

1. K positive definite implies that $c_1 - 7 > 0$ and $c_2 - 7 > 0$, a region of stable operation.
2. det$K > 0$ implies that $(c_1 - 7)(c_2 - 7) > 0$, or that $c_1 < 7, c_2 < 7$. The tr$(4K - GM^{-1}G) > 0$ implies that $4[(c_1 - 3) + (c_2 - 3)] > 0$, or that $c_1 + c_2 > 6$, which again yields a region of stable operation.
3. $4K - GM^{-1}G$ negative definite implies that $c_1 < 3$ and $c_2 < 3$, a region of unstable operation.
4. $4K - GM^{-1}G$ positive definite implies that $c_1 > 3$ and $c_2 > 3$, a region of either stable or unstable operation depending on other considerations. If, in addition, the matrix

$$GM^{-1}K - KM^{-1}G = 4\begin{bmatrix} 0 & c_1 - c_2 \\ c_1 - c_2 & 0 \end{bmatrix}$$

is zero, i.e., if $c_1 = c_2$, then the system is stable. Thus, the line $c_1 = c_2$ represents a region of stable operation for $c_1 = c_2 > 3$ and unstable operation for $c_1 = c_2 < 3$.

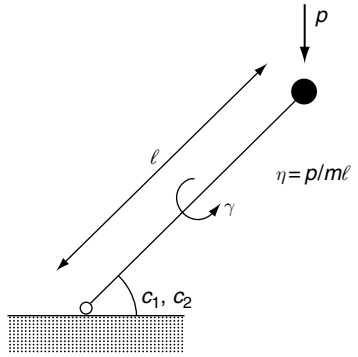

Figure 4.1 Schematic of a rotating shaft subject to an axial compression force.

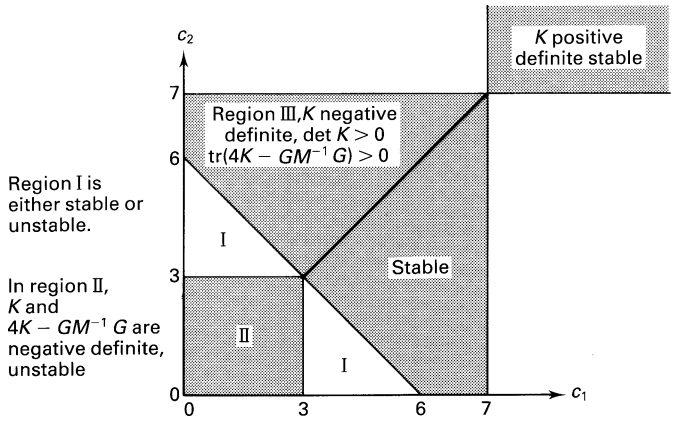

Figure 4.2 Regions of stable and unstable operation of a conservative gyroscopic system as a function of stiffness coefficients.

4.7 DAMPED GYROSCOPIC SYSTEMS

As the previous section illustrated, gyroscopic forces can be used to stabilize an unstable system. The next logical step is to consider adding damping to the system. Since added positive definite damping has caused stable symmetric systems to become asymptotically stable, the same effect is expected here. However, this turns out *not* to be the case in all circumstances.

Consider a damped gyroscopic system of the form

$$M\ddot{\mathbf{q}} + (D+G)\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.13}$$

where $M = M^T > 0$, $D = D^T$, $G = -G^T$, and $K = K^T$. The following results are due to Kelvin, Tait, and Chetaev and are referred to as the KTC theorem by Zajac (1964, 1965):

1. If K and D are both positive definite, the system is asymptotically stable.
2. If K is not positive definite and D is positive definite, the system is unstable.
3. If K is positive definite and D is positive semidefinite, the system may be stable or asymptotically stable. The system is asymptotically stable if and only if none of the eigenvectors of the undamped gyroscopic system is in the null space of D. Also, proportionally damped systems will be stable.

Hughes and Gardner (1975) showed that the Walker and Schmitendorf rank condition [Equation (4.11)] also applies to gyroscopic systems with semidefinite damping and positive definite stiffness. In particular, let the state matrix A and the 'observer' matrix C be defined and denoted by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}G \end{bmatrix}, \qquad C = [0 \quad D]$$

Then the equilibrium of Equation (4.13) is asymptotically stable if the rank of the $2n \times 2n^2$ matrix $R^T = \left[C^T \; A^T C^T \cdots (A^T)^{n-1} C^T \right]$ is full, i.e., rank $R^T = 2n$. Systems that satisfy either this rank condition or Equation (4.11) are said to be *pervasively damped*, meaning that the influence of the damping matrix D pervades each of the system coordinates. Each mode of a pervasively damped system is damped, and such systems are asymptotically stable.

Note that condition 2 points out that, if one attempts to stabilize an unstable system by adding gyroscopic forces to the system and at the same time introduces viscous damping, the system will remain unstable. A physical example of this is again given by the spinning top if the friction in the system is modeled as viscous damping. With dissipation considered, the top is in fact unstable and eventually falls over after precessing because of the effects of friction.

4.8 CIRCULATORY SYSTEMS

Next, consider those systems that have asymmetries in the coefficient of the displacement term. Such systems are called *circulatory*. A physical example is given in example 2.4.3. Other examples can be found in the fields of aeroelasticity, thermoelastic stability, and in control (see example 2.4.4). The equation of motion of such systems takes the form

$$M\ddot{\mathbf{q}} + (K + H)\mathbf{q} = \mathbf{0} \tag{4.14}$$

where $M = M^T$, $K = K^T$, and $H = -H^T$. Here, K is the symmetric part of the position coefficient and H is the skew-symmetric part. Results and stability conditions for circulatory systems are not as well developed as for symmetric conservative systems.

Since damping is not present, the stability of Equation (4.14) will be entirely determined by the matrix $A_3 = K + H$, as long as M is nonsingular. In fact, it can be shown (see Huseyin, 1978, p. 174) that Equation (4.14) is stable if and only if there exists a symmetric and positive definite matrix P such that $PM^{-1}A_3$ is symmetric and positive definite. Furthermore, if the

matrix $PM^{-1}A_3$ is symmetric, there is no flutter instability. On the other hand, if such a matrix P does not exist, Equation (4.14) can be unstable both by flutter and by divergence. In this case, the system will have some complex eigenvalues with positive real parts.

The preceding results are obtained by considering an interesting subclass of circulatory systems that results from a factorization of the matrix $M^{-1}A_3$. Taussky (1972) showed that any real square matrix can be written as the product of two symmetric matrices. That is, there exist two real symmetric matrices S_1 and S_2 such that $M^{-1}A_3 = S_1S_2$. With this factorization in mind, all asymmetric matrices $M^{-1}A_3$ can be classified into two groups: those for which at least one of the matrix factors, such as S_1, is positive definite and those for which neither of the factors is positive definite. Matrices for which S_1 (or S_2) is positive definite are called *symmetrizable matrices* or *pseudosymmetric matrices*. The corresponding systems are referred to as *pseudoconservative systems*, *pseudosymmetric systems*, or *symmetrizable systems* and behave essentially like symmetric systems. One can think of this transformation as a change of coordinate systems to one in which the physical properties are easily recognized.

In fact, for $M^{-1}A_3 = S_1S_2$, S_1 positive definite, the system described by Equation (4.14) is stable if and only if S_2 is positive definite. Furthermore, if S_2 is not positive definite, instability can only occur through divergence, and no flutter instability is possible. Complete proofs of these statements can be found in Huseyin (1978), along with a detailed discussion. The proof follows from the simple idea that, if $M^{-1}A_3$ is symmetrizable, then the system is mathematically similar to a symmetric system. Thus, the stability problem is reduced to considering that of the symmetric matrix S_2.

The similarity transformation is given by the matrix $S_2^{1/2}$, the positive definite square root of matrix S_1. To see this, premultiply Equation (4.14) by $S_1^{-1/2}$, which is nonsingular. This yields

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{-1/2}(M^{-1}A_3)\mathbf{q} = \mathbf{0} \tag{4.15}$$

which becomes

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{-1/2}S_1S_2\mathbf{q} = \mathbf{0}$$

or

$$S_1^{-1/2}\ddot{\mathbf{q}} + S_1^{1/2}S_2\mathbf{q} = \mathbf{0} \tag{4.16}$$

Substitution of $\mathbf{q} = S_1^{1/2}\mathbf{y}$ into this last expression yields the equivalent symmetric system

$$\ddot{\mathbf{y}} + S_1^{1/2}S_2S_1^{1/2}\mathbf{y} = \mathbf{0} \tag{4.17}$$

Thus, there is a nonsingular transformation $S_1^{1/2}$ relating the solution of symmetric problems given by Equation (4.17) to the asymmetric problem of Equation (4.14). Because the transformation is nonsingular, the eigenvalues of Equations (4.14) and (4.17) are the same. Thus, the two representations have the same stability properties. Here, the matrix $S_1^{1/2}S_2S_1^{1/2}$ is seen to be symmetric by taking its transpose, i.e., $(S_1^{1/2}S_2S_1^{1/2})^T = S_1^{1/2}S_2S_1^{1/2}$. Thus, if S_2 is positive definite, then $S_1^{1/2}S_2S_1^{1/2}$ is positive definite (and symmetric) so that Equation (4.17) is stable. Methods for calculating the matrices S_1 and S_2 are discussed in the next section.

Note that, if the system is not symmetrizable, i.e., if S_1 is not positive definite, then $S_1^{1/2}$ does not exist and the preceding development fails. In this case, instability of Equation (4.14) can be caused by either flutter or divergence.

4.9 ASYMMETRIC SYSTEMS

For systems that have both asymmetric velocity and stiffness coefficients not falling into any of the previously mentioned classifications, several different approaches are available. The first approach discussed here follows the idea of a pseudosymmetric system introduced in the previous section, and the second approach follows methods of constructing Lyapunov functions. The systems considered in this section are of the most general form [Equation (2.7) with $\mathbf{f} = \mathbf{0}$]

$$A_1\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + A_3\mathbf{q} = \mathbf{0} \tag{4.18}$$

where A_1 is assumed to be nonsingular, $A_2 = D + G$, and $A_3 = K + H$. Since A_1 is nonsingular and since A_2 and A_3 are symmetric, it is sufficient to consider the equivalent system

$$\ddot{\mathbf{q}} + A_1^{-1}A_2\dot{\mathbf{q}} + A_1^{-1}A_3\mathbf{q} = \mathbf{0} \tag{4.19}$$

The system described by Equation (4.19) can again be split into two classes by examining the factorization of the matrices $A_1^{-1}A_2$ and $A_1^{-1}A_3$ in a fashion similar to the previous section. First note that there exists a factorization of these matrices of the form $A_1^{-1}A_2 = T_1T_2$ and $A_1^{-1}A_3 = S_1S_2$, where the matrices S_1, S_2, T_1, and T_2 are all symmetric. This is always possible because of the result of Taussky just mentioned, i.e., any real square matrix can always be written as the product of two symmetric matrices. Then, the system in Equation (4.19) is similar to a symmetric system if and only if there exists at least one factorization of $A_1^{-1}A_2$ and $A_1^{-1}A_3$ such that $S_1 = T_1$, which is positive definite. Such systems are called *symmetrizable*. Under this assumption, it can be shown that the equilibrium position of Equation (4.18) is asymptotically stable if the eigenvalues of the matrix $A_1^{-1}A_2$ and the matrix $A_1^{-1}A_3$ are all positive real numbers. This corresponds to requiring the matrices S_2 and T_2 to be positive definite.

Deciding if the matrices $A_1^{-1}A_2$ and $A_1^{-1}A_3$ are symmetrizable is, in general, not an easy task. However, if the matrix A_2 is proportional, i.e., if $A_2 = \alpha A_1 + \beta A_3$, where α and β are scalars, and if $A_1^{-1}A_3$ is symmetrizable, then $A_1^{-1}A_2$ is also symmetrizable, and there exists a common factor S_1T_1. It can also be shown that, if two real matrices commute and one of them is symmetrizable, then the other matrix is also symmetrizable, and they can be reduced to a symmetric form simultaneously.

Several of the usual stability conditions stated for symmetric systems can now be stated for symmetrizable systems. If $A_1^{-1}A_2$ has nonnegative eigenvalues (i.e., zero is allowed) and if $A_1^{-1}A_3$ has positive eigenvalues, Equation (4.18) is asymptotically stable if and only if the $n^2 \times n$ matrix

$$R = \begin{bmatrix} A_1^{-1}A_2 \\ A_1^{-1}A_2(A_1^{-1}A_3) \\ A_1^{-1}A_2(A_1^{-1}A_3)^2 \\ \vdots \\ A_1^{-1}A_2(A_1^{-1}A_3)^{n-1} \end{bmatrix} \tag{4.20}$$

has rank n. This, of course, is equivalent to the statement made by Moran (1970) for symmetric systems, mentioned in section 4.5, that the system is asymptotically stable if and only if none of the eigenvectors of $A_1^{-1}A_3$ lies in the null space of $A_1^{-1}A_2$.

The KTC theorem can also be extended to systems with asymmetric but symmetrizable coefficients. However, the extension is somewhat more complicated. Consider the matrix $S = (A_1^{-1}A_2)T_1 + T_1(A_1^{-1}A_2)^T$, and note that S is symmetric. If S is positive definite and if A_3 is nonsingular, then Equation (4.18) is stable if and only if all of the eigenvalues of $A_1^{-1}A_3$ are positive numbers. If $S_1 \neq T_1$, the matrix $A_1^{-1}A_2$ contains a gyroscopic term, and this result states the equivalent problem faced in using gyroscopic forces to stabilize an unstable system, that it cannot be done in the presence of damping. Hence, in the case where $S_1 \neq T_1$, the stability of the system is determined by the eigenvalues of T_2 (which are the eigenvalues of A_3) for systems with a symmetrizable stiffness coefficient matrix.

The following two examples serve to illustrate the above discussion as well as indicate the level of computation required.

Example 4.9.1

The preceding results are best understood by considering some examples. First, consider a system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} 10 & 8 \\ 0 & 1 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

Here note that

$$A_1^{-1}A_2 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1.2461 & -0.2769 \\ -0.2769 & 0.3115 \end{bmatrix} \begin{bmatrix} 2.8889 & 5.7778 \\ 5.7778 & 11.5556 \end{bmatrix}$$

$$A_1^{-1}A_3 = \begin{bmatrix} 10 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.2461 & -0.2769 \\ -0.2769 & 0.3115 \end{bmatrix} \begin{bmatrix} 10 & 8.8889 \\ 8.8889 & 11.1111 \end{bmatrix}$$

so that $T_1 = S_1$, and the coefficient matrices have a common factor. Then the eigenvalue problem associated with this system is similar to a symmetric eigenvalue problem. An illustration on how to calculate the symmetric factors of a matrix is given in example 4.9.3.

According to the previous theorems, the stability of this equation may be indicated by calculating the eigenvalues of $A_1^{-1}A_2$ and of $A_1^{-1}A_3$. The eigenvalues of $A_1^{-1}A_2$ in this example are $\lambda_{1,2} = 0, 4$, and those of $A_1^{-1}A_3$ are $\lambda_{1,2} = 1, 10$. Hence, $A_1^{-1}A_3$ has positive real eigenvalues and $A_1^{-1}A_2$ has nonnegative real eigenvalues. Because of the singularity of the matrix $A_1^{-1}A_2$, knowledge of the rank of the matrix equation [Equation (4.20)] is required in order to determine if the system is asymptotically stable or just stable. The matrix of Equation (4.20) is

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 20 & 20 \\ 10 & 10 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which obviously has rank $= 2$, the value of n. Here, the symbol $\sim$ denotes column (or row) equivalence, as discussed in Appendix B. Thus, the previous result states that the equilibrium of this example is asymptotically stable. This is in agreement with the eigenvalue calculation for the system, which yields

$$\lambda_{1,2} = -1 \pm 2j$$

$$\lambda_{3,4} = -1 \pm j$$

showing clearly that the equilibrium is in fact asymptotically stable, as predicted by the theory.

Example 4.9.2

As a second example, consider the asymmetric problem given by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 9 & 20 \\ 3 & 8 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} -5 & -6 \\ -4 & 0 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

Premultiplying this by A_1^{-1} yields

$$I\ddot{\mathbf{q}} + \begin{bmatrix} 6 & 12 \\ 3 & 8 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} -1 & -6 \\ -4 & 0 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

The eigenvalues of $A_1^{-1}A_2$ are $\lambda_{1,2} = 7 \pm (1/2)\sqrt{148}$ and those of $A_1^{-1}A_3$ are $\lambda_{1,2} = -1/2 \pm (1/2)\sqrt{97}$. Thus, both coefficient matrices have real distinct eigenvalues and are therefore symmetrizable. However, a simple computation shows that there does not exist a factorization of $A_1^{-1}A_2$ and $A_1^{-1}A_3$ such that $T_1 = S_1$.

Thus, the generalized KTC theorem must be applied. Accordingly, if the matrix $(A_1^{-1}A_2)T_1 + T_1(A_1^{-1}A_2)^T$ is positive definite, then the equilibrium of this system is unstable, since $A_1^{-1}A_3$ has a negative eigenvalue. To calculate T_2, note that $(A_1^{-1}A_3) = T_1T_2$, where T_1 is positive definite and hence nonsingular. Thus, multiplying by T_1^{-1} from the right results in the matrix T_2 being given by $T_2 = T_1^{-1}(A_1^{-1}A_3)$.

Let T_1^{-1} be a general generic symmetric matrix denoted by

$$T_1^{-1} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where it is desired to calculate a, b, and c so that T_1 is positive definite. Thus

$$T_2 = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} -1 & -6 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} -a-4b & -6a \\ -b-4c & -6b \end{bmatrix}$$

Requiring T_2 to be symmetric and T_1 to be positive definite yields the following relationships for a, b, and c:

$$ac > b^2$$

$$6a = b + 4c$$

This set of equations has multiple solutions; one convenient solution is $a = 2$, $b = 0$, and $c = 3$. Then T_1 becomes

$$T_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Thus, $(A_1^{-1}A_2)T_1 + T_1(A_1^{-1}A_2)^T$ becomes

$$(A_1^{-1}A_2)T_1 + T_1(A_1^{-1}A_2)^T = \begin{bmatrix} 6 & \frac{11}{2} \\ \frac{11}{2} & \frac{16}{3} \end{bmatrix}$$

which is positive definite. Thus, the equilibrium must be unstable.

This analysis again agrees with calculation of the eigenvalues, which are $\lambda_1 = 0.3742$, $\lambda_2 = -13.5133$, and $\lambda_{3,4} = -0.4305 \pm 0.2136j$, indicating an unstable equilibrium, as predicted.

Example 4.9.3

The question of how to calculate the factors of a symmetrizable matrix is discussed by Huseyin (1978) and Ahmadian and Chou (1987). Here it is shown that the matrices $A_1^{-1}A_3 = S_1S_2$ and $A_1^{-1}A_2 = T_1T_2$ of example 4.9.2 do not have any common factorization such that $T_1 = S_1$. Hence, $A_1^{-1}A_2$ and $A_1^{-1}A_3$ cannot be simultaneously symmetrized by the same transformation.

It is desired to find a symmetric positive definite matrix P such that $PA_1^{-1}A_2$ and $PA_1^{-1}A_3$ are both symmetric. To that end, let

$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

Then

$$PA_1^{-1}A_2 = \begin{bmatrix} 6a+3b & 12a+8b \\ 6b+3c & 12b+8c \end{bmatrix}$$

and

$$PA_1^{-1}A_3 = \begin{bmatrix} -a-4b & -6a \\ -b-4c & -6b \end{bmatrix}$$

Symmetry of both matrices then requires that

$$\begin{aligned} 6b+3c &= 12a+8b \\ b+4c &= 6a \end{aligned} \tag{4.21}$$

Positive definiteness of P requires

$$\begin{aligned} a &> 0 \\ ac &> b^2 \end{aligned} \tag{4.22}$$

It will be shown that the problem posed by Equations (4.21) and (4.22) does not have a solution. Equations (4.21) may be written in matrix form as

$$\begin{bmatrix} -2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = 6a \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which has the unique solution

$$\begin{bmatrix} b \\ c \end{bmatrix} = a \begin{bmatrix} 2.73 \\ 2.18 \end{bmatrix}$$

for all values of a. Thus, $b = 2.73a$ and $c = 2.18a$, so that $b^2 = 7.45a^2$ and $ac = 2.18a^2$. Then

$$ac = 2.18a^2 < 7.45a^2 = b^2$$

and condition (4.22) cannot be satisfied.

Other alternatives exist for analyzing the stability of asymmetric systems. Walker (1970), approaching the problem by looking for Lyapunov functions, was able to state several results for the stability of Equation (4.18) in terms of a fourth matrix R. If there exists a symmetric positive definite matrix R such that $RA_1^{-1}A_3$ is symmetric and positive definite (this is the same as requiring $A_1^{-1}A_3$ to be symmetrizable), then the system is stable if the symmetric part of $RA_1^{-1}A_2$ is positive semidefinite and asymptotically stable if the symmetric part of $RA_1^{-1}A_2$ is strictly positive definite. This result is slightly more general than the symmetrizable results just stated in that it allows the equivalent symmetric systems to have gyroscopic forces.

In addition to these results, Walker (1970) showed that, if there exists a symmetric matrix R such that $RA_1^{-1}A_2$ is skew-symmetric and $RA_1^{-1}A_3$ is symmetric, and such that R and $RA_1^{-1}A_3$ have the same definiteness, then the system is stable but not asymptotically stable.

Another approach to the stability of Equation (4.18), not depending on symmetrizable coefficients, has been given by Mingori (1970). He showed that, if the coefficient matrices M, D, G, H, and K satisfy the commutivity conditions

$$\begin{aligned} HD^{-1}M &= MD^{-1}H \\ HD^{-1}G &= GD^{-1}H \\ HD^{-1}K &= KD^{-1}H \end{aligned}$$

then the stability of the system is determined by the matrix

$$Q = HD^{-1}MD^{-1}H - GD^{-1}H + K$$

This theory states that the system is stable, asymptotically stable, or unstable if the matrix Q possesses nonnegative, positive, or at least one negative eigenvalue respectively. Although the problem addressed is general, the restrictions are severe. For instance, this method cannot be used for systems with semidefinite damping (D^{-1} does not exist).

Other more complicated and more general stability conditions are due to Walker (1974) and an extension of his work by Ahmadian and Inman (1986). The methods are developed by using Lyapunov functions to derive stability and instability conditions on the basis of the direct method. These are stated in terms of the symmetry and definiteness of certain matrices consisting of various combinations of the matrices A_1, A_2, and A_3. These conditions offer a variety of relationships among the physical parameters of the system, which can aid in designing a stable or asymptotically stable system.

4.10 FEEDBACK SYSTEMS

One of the major reasons for using feedback control is to stabilize the system response. However, most structures are inherently stable to begin with, and control is applied to improve performance. Unfortunately, the introduction of active control can effectively destroy the symmetry and definiteness of the system, introducing the possibility of instability. Thus, checking the stability of a system after a control is designed is an important step. A majority of the work in control takes place in state space (first-order form). However, it is interesting to treat the control problem specifically in 'mechanical' or physical coordinates in order to take advantage of the natural symmetries and definiteness in the system. Lin (1981) developed

a theory for closed-loop asymptotic stability for mechanical structures being controlled by velocity and position feedback. The systems considered here have the form (see Section 2.3)

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}_f + \mathbf{f} \tag{4.23}$$

where $M = M^T$ is positive definite, $A_2 = D + G$ is asymmetric, and K is symmetric. Here, the vector $\mathbf{f}$ represents external disturbance forces (taken to be zero in this section) and the vector $\mathbf{f}_f$ represents the control force derived from the action of r force actuators represented by

$$\mathbf{f}_f = B_f\mathbf{u} \tag{4.24}$$

where the $r \times 1$ vector $\mathbf{u}$ denotes the r inputs, one for each control device (actuator), and B_f denotes the $n \times r$ matrix of weighting factors (influence coefficients or actuator gains) with structure determined by the actuator locations. In order to be able to feed back the position and velocity, let $\mathbf{y}$ be an $s \times 1$ vector of sensor outputs denoted and defined by

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}} \tag{4.25}$$

Here, C_p and C_v are $s \times n$ matrices of displacement and velocity influence coefficients respectively, with structure determined by the sensor locations and where s is the number of sensors. Equation (4.25) represents those coordinates that are measured as part of the control system and is a mathematical model of the transducer and signal processing used to measure the system response. The input vector $\boldsymbol{u}$ is chosen to be of the special form

$$\mathbf{u}(t) = -G_f\mathbf{y} = -G_fC_p\mathbf{q} - G_fC_v\dot{\mathbf{q}} \tag{4.26}$$

where the $r \times s$ matrix G_f consists of constant feedback gains. This form of control law is called *output feedback*, because the input is proportional to the measured output or response, $\mathbf{y}$.

In Equation (4.24) the matrix B_f reflects the location on the structure of any actuator or device being used to supply the forces $\mathbf{u}$. For instance, if an electromechanical or piezoelectric actuator is attached to mass m_1 in Figure 2.4, and if it supplies a force of the form $F_0 \sin \omega t$, the vector $\mathbf{u}$ reduces to the scalar $u = F_0 \sin(\omega t)$ and the matrix B_f reduces to a vector $B_f^T = [1 \quad 0]$. Alternatively, the control force can be written as a column vector $\mathbf{u}$, and B_f can be written as a matrix

$$\mathbf{u} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \qquad \text{and} \qquad B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

If, on the other hand, there are two actuators, one attached to m_1 supplying a force $F_1 \sin(\omega_1 t)$ and one at m_2 supplying a force $F_2 \sin(\omega_2 t)$, then the vector $\mathbf{u}$ becomes $\mathbf{u}^T = [F_1 \sin(\omega_1 t) \quad F_2 \sin(\omega_2 t)]$ and the matrix B_f becomes $B_f = I$, the 2×2 identity matrix. Likewise, if the positions x_1 and x_2 are measured, the matrices in Equation (4.25) become $C_\mathrm{p} = I$ and $C_\mathrm{v} = 0$, the 2×2 matrix of zeros. If only the position x_1 is measured and the control force is applied to x_2, then

$$B_f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Making the appropriate substitutions in the preceding equations and assuming no external disturbance (i.e., $\mathbf{f} = \mathbf{0}$) yields an equivalent homogeneous system, which includes the effect of the controls. It has the form

$$M\ddot{\mathbf{q}} + (D + G + B_f G_f C_v)\dot{\mathbf{q}} + (K + B_f G_f C_p)\mathbf{q} = 0 \tag{4.27}$$

For the sake of notation, define the matrices $D^* = B_f G_f C_v$ and $K^* = B_f G_f C_p$. Note that, since the number of actuators r is usually much smaller than the number of modeled degrees of freedom n (the dimension of the system), the matrices K^* and D^* are usually singular. Since, in general, $D + D^*$ and $K + K^*$ may not be symmetric or positive definite, it is desired to establish constraints on any proposed control law to ensure the symmetry and definiteness of the coefficient matrices and hence the stability of the system (see problem 4.11). These constraints stem from requiring $D + D^*$ and $K + K^*$ to be symmetric positive definite. The problem of interest in control theory is how to choose the matrix G_f so that the response $\mathbf{q}$ has some desired property (performance and stability). Interest in this section focuses on finding constraints on the elements of G_f so that the response $\mathbf{q}$ is asymptotically stable or at least stable. The stability methods of this chapter can be applied to Equation (4.27) to develop these constraints. Note that the matrices $B_f G_f C_p$ and $B_f G_f C_v$ are represented as the matrices K_p and K_v respectively in Equation (2.17).

Collocated control refers to the case where the sensors are located at the same physical location as the actuators. If the sensors or the actuators add no additional dynamics, then collocated controllers provide improved stability of the closed-loop system. As seen above, the closed-loop system coefficients D^* and K^* generally lose their symmetry for many choices of B_f, C_f, B_v, and C_v. If, however, the gain matrices G_f and G_v are symmetric, and if $B_f^T = C_f$ and $B_v^T = C_v$, then the matrices D^* and K^* remain symmetric. The symmetry then results in the possibility of choosing the gain matrices so that D^* and K^* remain positive definite, ensuring closed-loop stability for stable open-loop systems (D and K positive definite). Placing sensors and actuators at the same location causes $B_f^T = C_f$ and $B_v^T = C_v$, so that collocated control enhances closed-loop stability. The controller design consists of choosing gains G_f and G_v that are symmetric and positive definite (or at least semidefinite) with collocated sensors and actuators to ensure a stable closed-loop response.

Example 4.10.1

Consider the two-degree-of-freedom system in figure 2.4, with a control force applied to m_1 and a measurement made of x_2 so that the control system is not collocated. Then the input matrix, output matrix, and symmetric control gain matrix are

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_p = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

Note that this is not collocated because $B_f^T \neq C_f$. The closed-loop system of Equation (4.27) becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 + g_1 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has an asymmetric displacement coefficient, implying the potential loss of stability. If, on the other hand, a control force is applied to m_1 and a measurement made of x_1, then the control system is collocated and the input matrix, output matrix, and control gain matrix are

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

so that $B_f^T = C_f$ and the closed-loop system of Equation (4.27) becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 + g_1 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is symmetric and stable for any choice of g_1 such that $k_1 + k_2 + g_1 > 0$.

The topic of control and Equation (4.27) is discussed in more detail in section 6.6 and in Chapter 7. Historically, most of the theory developed in the literature for the control of systems has been done using a state-space model of the structure. The next section considers the stability of systems in the state variable coordinate system.

4.11 STABILITY IN STATE SPACE

In general, if none of the stability results just mentioned is applicable, the problem can be cast in first-order form as given in Section 2.3. The system of Equation (4.18) then has the form

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{4.28}$$

where A is a $2n \times 2n$ state matrix and $\mathbf{x}$ is a $2n$ state vector. In this setting, it can easily be shown that the system is asymptotically stable if all the eigenvalues of A have negative real parts and is unstable if A has one or more eigenvalues with positive real parts.

The search for stability by finding a Lyapunov function in first-order form leads to the *Lyapunov equation*

$$A^T B + BA = -C \tag{4.29}$$

where C is positive semidefinite and B is the symmetric, positive definite, unknown matrix of the desired (scalar) Lyapunov function:

$$V(\mathbf{x}) = \mathbf{x}^T B \mathbf{x} \tag{4.30}$$

Do not confuse the arbitrary matrices B and C used here with the B and C used for input and output matrices. To see that $V(\mathbf{x})$ is, in fact, the desired Lyapunov function, note that differentiation of Equation (4.30) yields

$$\frac{d}{dt}[V(\mathbf{x})] = \dot{\mathbf{x}}^T B \mathbf{x} + \mathbf{x}^T B \dot{\mathbf{x}} \tag{4.31}$$

Substitution of the state equation [Equation (4.28)] into Equation (4.31) yields

$$\begin{aligned}\frac{d}{dt}[V(\mathbf{x})] &= \mathbf{x}^T A^T B\mathbf{x} + \mathbf{x}^T BA\mathbf{x} \\ &= \mathbf{x}^T (A^T B + BA)\mathbf{x} \\ &= -\mathbf{x}^T C\mathbf{x}\end{aligned} \tag{4.32}$$

Here, taking the transpose of Equation (4.28) yields $\dot{\mathbf{x}}^T = \mathbf{x}^T A^T$, which is used to remove the time derivative in the second term. Hence, if $V(\mathbf{x})$ is to be a Lyapunov function, matrix C must be positive semidefinite. The problem of showing stability by this method for a system represented by matrix A then becomes one, given the symmetric positive semidefinite matrix C, of finding a positive definite matrix B such that Equation (4.29) is satisfied. This approach involves solving a system of linear equations for the $n(n+1)/2$ elements b_{ik} of matrix B.

As explained by Walker (1974), Hahn (1963) has shown that, for a given choice of symmetric positive definite matrix C, there exists a unique solution, i.e., there exists a symmetric matrix B satisfying Equation (4.29) if the eigenvalues of A, λ_i, satisfy $\lambda_i + \lambda_k \neq 0$ for all $i, k = 1, 2, \ldots, 2n$. Furthermore, matrix B is positive definite if and only if each eigenvalue of A has a negative real part, in which case the system is asymptotically stable. Matrix B is indefinite if and only if at least one eigenvalue of A has a positive real part, in which case the equilibrium of the system is unstable. Many theoretical and numerical calculations in stability theory are based on the solution of Equation (4.29). Walker (1974) has shown that this system of linear equations has a unique solution.

Solving for the eigenvalues of Equation (4.28) can involve writing out the characteristic equation of the system. In such cases where this can be done analytically and the coefficients of the characteristic equation are available, a simple stability condition exists, namely if the characteristic equation is written in the form

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \tag{4.33}$$

then the system is asymptotically stable if and only if the principal minors of the $n \times n$ *Hurwitz matrix* defined by

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & & & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & & & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_n \end{bmatrix}$$

are all positive. In addition, if any of the coefficients a_i are nonpositive (i.e., negative or zero), then the system may be unstable. This is called the *Hurwitz test*.

Writing out the (determinant) principal minors of the Hurwitz matrix yields nonlinear inequalities in the coefficients that provide relationships in the physical parameters of the system. If these inequalities are satisfied, asymptotic stability is ensured.

Example 4.11.1

As an illustration of the Hurwitz method, consider determining the asymptotic stability of a system with the characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

The Hurwitz matrix is

$$\begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{bmatrix}$$

From the Hurwitz test, $a_1 > 0$, $a_2 > 0$, and $a_3 > 0$ must be satisfied. From the principal minors of the Hurwitz matrix, the inequalities

$$a_1 > 0$$

$$a_1 a_2 - a_3 > 0$$

$$a_1(a_2 a_3) - a_3^2 = a_1 a_2 a_3 - a_3^2 > 0$$

must be satisfied. The above set reduces to the conditions that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1 a_2 - a_3 > 0$ be satisfied for the system to be asymptotically stable.

4.12 STABILITY BOUNDARIES

An alternative way of looking at stability has been summarized by Huseyin (1978) and involves examining the characteristic equation as a surface from which stability properties can be deduced by plotting various stability boundaries. These methods are especially useful when examining stability questions that arise because of an applied load. A typical example is the case of a circulatory force given in example 4.6.1 above.

The point of view taken here is that a system without an applied load is represented by a symmetric system. For example

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{4.34}$$

where M and K are positive definite and symmetric, i.e., the system is stable. This system is now subjected to a load proportional to position and results in the equation

$$M\ddot{\mathbf{q}} + (K - \eta E)\mathbf{q} = \mathbf{0} \tag{4.35}$$

where η is a parameter characterizing the magnitude of the applied load and E is a matrix representing the point, or points, of application of the load. If there are several loads present, they can be indexed, $\eta_k E_k$, summed, and included in the equation of motion as

$$M\ddot{\mathbf{q}} + (K - \sum \eta_k E_k)\mathbf{q} = \mathbf{0} \tag{4.36}$$

In some sense, this equation is similar to the feedback control systems described in the previous sections, the difference being that the extra term $\eta_k E_k$ results in this case from

some physical loading of a structure, whereas the position feedback term $G_f C_p$ in Equation (4.26) results from adding actuators to the structure. In the case of feedback, the matrix $G_f C_p$ is found that causes the system to have a desired (stable) response. In the case of Equation (4.35), it is desired to see how the stability properties are affected by changes in the scalar parameter η_k.

The characteristic equation associated with Equation (4.36) is an equation (if no damping or gyroscopic terms are present) in λ^2 and the variable η_k. This is denoted by $\Delta(\lambda^2, \eta_k)$ and defined by

$$\Delta(\lambda^2, \eta_k) = \det(M\lambda^2 + K - \eta_k E_k) = 0 \tag{4.37}$$

In most circumstances, $\eta_k = 0$ corresponds to a stable state. Then the problem is to find values of η_k at which the system loses stability. The initially stable system may, in general, lose stability by either divergence or flutter. Many of the investigations using this method focus on determining which way stability is lost. The locus of points in the η_k space corresponding to zero roots, or divergence (recall Section 1.7), is called the *divergence boundary*. On the other hand, flutter instability corresponds to repeated roots with degenerate eigenvectors. The locus of points corresponding to repeated roots generates the *flutter boundary*. Together, these two curves comprise the *stability boundary*.

Huseyin (1978) showed that the flutter condition results from those values of η such that

$$\frac{\partial \Delta}{\partial \lambda^2} = 0 \tag{4.38}$$

A majority of Huseyin's text is devoted to various ways of computing stability boundaries for various classifications of systems. These curves allow design work to be done by examining the relationship of η to the stability of the system.

CHAPTER NOTES

The classification of systems in this chapter is motivated by the text of Huseyin (1978). This text provides a complete list of references for each type of system mentioned here, with the exception of the material on control systems. In addition, Huseyin's text provides an in-depth discussion of each topic. The material of Section 4.2 is standard Lyapunov (also spelled Liapunov in older literature) stability theory, and the definitions are available in most texts. The reader who understands limits and continuity from elementary calculus should be able to make the connection to the definition of stability. The material in Sections 4.3 and 4.4 is also standard fare and can be found in most texts considering stability of mechanical systems. The material of Section 4.5 on semidefinite damping results from several papers (as referenced) and is not usually found in text form. The material on gyroscopic systems presented in Section 4.6 is also from several papers. The material on damped gyroscopic systems is interesting because it violates instinct by illustrating that adding damping to a structure may not always make it 'more stable.' The next section deals with asymmetric but symmetrizable systems. The material of Section 4.9 is taken from Inman (1983). The major contributors to the theories (Walker and Huseyin) developed separate methods, which turned out to be quite similar and, in fact, related. The paper by Ahmadian and Chou (1987) should

be consulted for methods of calculating symmetric factors of a matrix. Feedback control systems are presented in this chapter (Section 4.10) before they are formally introduced in Chapter 7 to drive home the fact that the introduction of a control to a system adds energy to it and can make it unstable. The topic of Section 4.11 also presents material from control texts. It is important to note that the controls community generally thinks of a stable system as one that is defined as asymptotically stable in this text, i.e., one with eigenvalues with negative real parts. Systems are said to be *marginally stable* by the controls community if the eigenvalues are all purely imaginary. This is called stable in this and other vibration texts. Recent survey articles on the stability of second-order systems are provided by Bernstein and Bhat (1995) and Nicholson and Lin (1996). Vidyasagar (2002) includes detailed stability analysis for both linear and nonlinear systems, as well as definitions of other types of stability.

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PROBLEMS

4.1 Consider the system in figure 2.4 with $c_1 = c$, $c_2 = 0$, $f = 0$, $k_1 = k_2 = k$, and $m_1 = m_2 = m$. The equation of motion is

$$m\ell\ddot{\mathbf{x}} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Use Moran's theorem to see if this system is asymptotically stable.

4.2 Repeat problem 4.1 by using Walker and Schmitendorf's theorem.

4.3 Again, consider the system in Figure 2.4, this time with $c_1 = 0$, $c_2 = c$, $f = 0$, $k_1 = k_2 = k$, and $m_1 = m_2 = m$. The equation of motion is

$$m\ell\ddot{\mathbf{x}} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Is this system asymptotically stable? Use any method.

4.4 Discuss the stability of the system of Equation (2.24) using any method. Note that your answer should depend on the relative values of η, m, E, I, and ℓ.

4.5 Calculate a Lyapunov function for the system of example 4.9.1.

4.6 Show that, for a system with symmetric coefficients, if D is positive semidefinite and $DM^{-1}K = KM^{-1}D$, then the system is *not* asymptotically stable.

4.7 Calculate the matrices and vectors B_f, $\mathbf{u}$, C_p and C_v, defined in Section 4.10 for the system in Figure 2.4 for the case where the velocities of m_1 and m_2 are measured and the actuator (f_2) at m_2 supplies a force of $-g_1\dot{\mathbf{x}}_1$. Discuss the stability of this closed-loop system as the gain g_1 is changed.

4.8 The characteristic equation of a given system is

$$\lambda^4 + 10\lambda^3 + \lambda^2 + 15\lambda + 3 = 0$$

Is this system asymptotically stable or unstable? Use a root solver to check your answer.

4.9 Consider the system defined by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{q}} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\dot{\mathbf{q}} + \begin{bmatrix} k_1 & -k_3 \\ k_3 & k_2 \end{bmatrix}\mathbf{q} = \mathbf{0}$$

Assume a value of matrix R from the theory of Walker (1970) of the form

$$R = \begin{bmatrix} m_1 & g \\ g & m_2 \end{bmatrix}$$

and calculate relationships between the parameters m_i, k_i, and g that guarantee the stability of the system. Can the system be asymptotically stable?

4.10 Let

$$A_2 = \begin{bmatrix} c_1 & c_4 \\ c_3 & c_2 \end{bmatrix}$$

in problem 4.9 and repeat the analysis.

4.11 Let $\mathbf{y} = C_p B_f^T \mathbf{q} + C_v B_f^T \dot{\mathbf{q}}$, where C_p and C_v are restricted to be symmetric and show that the resulting closed-loop system with $G_f = I$ and $G = 0$ in Equation (4.27) has symmetric coefficients (Junkins, 1986).

4.12 For the system of problem 4.10, choose feedback matrices B_f, G_f, C_p, and C_v that make the system symmetric and stable (see problem 4.11 for a hint).

4.13 Prove that collocated control is stable for the system of problem 4.10.

4.14 The characteristic equation of a two-link structure with stiffness at each joint and loaded at the end by p_2 and at the joint by p_1 is

$$2\lambda^4 + p_2^2 + \lambda^2 p_1 + 4\lambda^2 p_2 + p_1 p_2 - 8\lambda^2 - p_1 - 4p_2 + 2 = 0$$

where the parameters of the structure are all taken to be unity (see Huseyin, 1978, p. 84). Calculate and sketch the divergence boundary in the $p_1 - p_2$ space. Discuss the flutter condition.

4.15 Use the Hurwitz test to discuss the stability of the system in problem 4.14.

4.16 Consider the system of example 2.4.4 [Equation (2.26)] and compute the B_f, G_f and C_p matrices that correspond to the control law suggested in the example.

4.17 Consider the system of example 4.10.1 with $M = I, C = 0.1K$, and $k_1 = k_2 = 2$. Compute the values for the gain g_1 that make the closed loop stable for the collocated case.

4.18 Consider the system of example 4.10.1 with $M = I, C = 0.1K$, and $k_1 = k_2 = 2$. Compute the values for the gain g_1 that make the closed loop stable for the noncollocated case.

4.19 A common way of improving the response of a system is to add damping via velocity feedback control. Consider the standard two-degree-of-freedom system in Figure 2.4 with open-loop values of $m_1 = m_2 = 1$, $c_1 = c_2 = 0.1$, $k_1 = 4$, and $k_2 = 1$. Add damping to the system using a control system that measures $\dot{x}_1$ and applies a force to the mass m_2 that is proportional to $\dot{x}_1$ (i.e., $-g\dot{x}_1$). Determine values of g that make the closed-loop response asymptotically stable.

4.20 Consider the system of problem 4.5 and calculate the eigenvalues of the state matrix in MATLAB to determine the stability.

5

Forced Response of Lumped-parameter Systems

5.1 INTRODUCTION

Up to this point, with the exception of Section 1.4 and some brief comments about feedback control, only the free response of a system has been discussed. In this chapter the forced response of a system is considered in detail. Such systems are called *nonhomogeneous*. Here, an attempt is made to extend the concepts used for the forced response of a single-degree-of-freedom system to the forced response of a general lumped-parameter system. In addition, the concept of stability of the forced response, as well as bounds on the forced response, is discussed. The beginning sections of this chapter are devoted to the solution for the forced response of a system by modal analysis, and the latter sections are devoted to introducing the use of a forced modal response in measurement and testing. The topic of experimental modal testing is considered in detail in Chapter 8. This chapter ends with an introduction to numerical simulation of the response to initial conditions and an applied force.

Since only linear systems are considered, the superposition principle can be employed. This principle states that the total response of the system is the sum of the free response (the homogeneous solution) plus the forced response (the nonhomogeneous solution). Hence, the form of the transient responses calculated in Chapter 3 are used again as part of the solution of a system subject to external forces and nonzero initial conditions. The numerical integration technique presented at the end of the chapter may also be used to simulate nonlinear system response, although that is not presented.

5.2 RESPONSE VIA STATE-SPACE METHODS

This section considers the state-space representation of a structure given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t) \tag{5.1}$$

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

where A is a $2n \times 2n$ matrix containing the generalized mass, damping, and stiffness matrices as defined in Equation (2.20). The $2n \times 1$ state vector $\mathbf{x}(t)$ contains both the velocity and position vectors and will be referred to as the response in this case. Equation (5.1) reflects the fact that any set of n differential equations of second order can be written as a set of $2n$ first-order differential equations. In some sense, Equation (5.1) represents the most convenient form for solving for the forced response, since a great deal of attention has been focused on solving state-space descriptions numerically (such as the Runge–Kutta method), as discussed in Sections 1.10, 3.9, and 5.8, as well as analytically. In fact, several software packages are available for solving Equation (5.1) numerically on virtually every computing platform. The state-space form is also the form of choice for solving control problems (Chapter 7).

Only a few of the many approaches to solving this system are presented here; the reader is referred to texts on numerical integration and systems theory for other methods. More attention is paid in this chapter to developing methods that cater to the special form of mechanical systems, i.e., systems written in terms of position, velocity, and acceleration rather than in state space.

The first method presented here is simply that of solving Equation (5.1) by using the Laplace transform. Let $\mathbf{X}(0)$ denote the Laplace transform of the initial conditions. Taking the Laplace transform of Equation (5.1) yields

$$s\mathbf{X}(s) = A\mathbf{X}(s) + \mathbf{F}(s) + \mathbf{X}(0) \tag{5.2}$$

where $\mathbf{X}(s)$ denotes the Laplace transform of $\mathbf{x}(t)$ and is defined by

$$\mathbf{X}(s) = \int_0^\infty \mathbf{x}(t)\, e^{-st}\, dt \tag{5.3}$$

Here, s is a complex scalar. Algebraically solving Equation (5.2) for $\mathbf{X}(s)$ yields

$$\mathbf{X}(s) = (sI - A)^{-1}\mathbf{X}(0) + (sI - A)^{-1}\mathbf{F}(s) \tag{5.4}$$

The matrix $(sI - A)^{-1}$ is referred to as the *resolvent matrix*. The inverse Laplace transform of Equation (5.4) then yields the solution $\mathbf{x}(t)$. The form of Equation (5.4) clearly indicates the superposition of the transient solution, which is the first term on the right-hand side of Equation (5.4), and the forced response, which is the second term on the right-hand side of Equation (5.4). The inverse Laplace transform is defined by

$$\mathbf{x}(t) = \pounds^{-1}[X(s)] = \lim_{a\to\infty} \frac{1}{2\pi j} \int_{c-ja}^{c+ja} X(s)\, e^{st}\, ds \tag{5.5}$$

where $j = \sqrt{-1}$. In many cases, Equation (5.5) can be evaluated by using a table such as the one found in Thomson (1960) or a symbolic code. If the integral in Equation (5.5) cannot be found in a table or calculated, then numerical integration can be used to solve Equation (5.1) as presented in Section 5.8.

Example 5.2.1

Consider a simple single-degree-of-freedom system $\ddot{x} + 3\dot{x} + 2x = \mu(t)$, written in the state-space form defined by Equation (5.1) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \qquad \mathbf{x}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

subject to a force given by $f_1 = 0$ and

$$f_2(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

the unit step function, and initial condition given by $\mathbf{x}(0) = [0 \quad 1]^T$. To solve this, first calculate

$$(sI - A) = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

and then determine the resolvent matrix

$$(sI - A)^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Equation (5.4) becomes

$$\mathbf{X}(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^3 + 3s^2 + 2s} \\ \dfrac{s+1}{s^2 + 3s + 2} \end{bmatrix}$$

Taking the inverse Laplace transform by using a table yields

$$\mathbf{x}(t) = \begin{bmatrix} \dfrac{1}{2} - \dfrac{1}{2} e^{-2t} \\ e^{-2t} \end{bmatrix}$$

This solution is in agreement with the fact that the system is overdamped. Also note that $\dot{x}_1 = x_2$, as it should in this case, and that setting $t = 0$ satisfies the initial conditions.

A second method for solving Equation (5.1) imitates the solution of a first-order scalar equation, following what is referred to as the method of 'variation of parameters' (Boyce and DiPrima, 2005). For the matrix case, the solution depends on defining the exponential of a matrix. The *matrix exponential* of matrix A is defined by the infinite series

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{5.6}$$

where $k!$ denotes the k factorial with $0! = 1$ and $A^0 = I$, the $n \times n$ identity matrix. This series converges for all square matrices A. By using the definition of Section 2.1 of a scalar multiplied by a matrix, the time-dependent matrix e^{At} is similarly defined as

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \tag{5.7}$$

which also converges. The time derivative of Equation (5.7) yields

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A \tag{5.8}$$

Note that matrix A and the matrix e^{At} commute because a matrix commutes with its powers.

Following the method of variation of parameters, assume that the solution of Equation (5.1) is of the form

$$\mathbf{x}(t) = e^{At}\mathbf{c}(t) \tag{5.9}$$

where $\mathbf{c}(t)$ is an unknown vector function of time. The time derivative of Equation (5.9) yields

$$\dot{\mathbf{x}}(t) = Ae^{At}\mathbf{c}(t) + e^{At}\dot{\mathbf{c}}(t) \tag{5.10}$$

This results from the product rule and Equation (5.8). Substitution of Equation (5.9) into Equation (5.1) yields

$$\dot{\mathbf{x}}(t) = Ae^{At}\mathbf{c}(t) + \mathbf{f}(t) \tag{5.11}$$

Subtracting Equation (5.11) from Equation (5.10) yields

$$e^{At}\dot{\mathbf{c}}(t) = \mathbf{f}(t) \tag{5.12}$$

Premultiplying Equation (5.12) by e^{-At} (which always exists) yields

$$\dot{\mathbf{c}}(t) = e^{-At}\mathbf{f}(t) \tag{5.13}$$

Simple integration of this differential equation yields the solution for $\mathbf{c}(t)$:

$$\mathbf{c}(t) = \int_0^t e^{-A\tau}\mathbf{f}(\tau)d\tau + \mathbf{c}(0) \tag{5.14}$$

Here, the integration of a vector is defined as integration of each element of the vector, just as differentiation is defined on a per element basis. Substitution of Equation (5.14) into the assumed solution (5.9) produces the solution of Equation (5.1) as

$$\mathbf{x}(t) = e^{At}\int_0^t e^{-A\tau}\mathbf{f}(\tau)\mathrm{d}\tau + e^{At}\mathbf{c}(0) \tag{5.15}$$

Here, $\mathbf{c}(0)$ is the initial condition on $\mathbf{x}(t)$. That is, substitution of $t = 0$ into (5.9) yields

$$\mathbf{x}(0) = e^0\mathbf{c}(0) = I\mathbf{c}(0) = \mathbf{c}(0)$$

so that $\mathbf{c}(0) = \mathbf{x}(0)$, the initial conditions on the state vector. The complete solution of Equation (5.1) can then be written as

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\mathbf{f}(\tau)d\tau \tag{5.16}$$

The first term represents the response due to the initial conditions, i.e., the free response of the system. The second term represents the response due to the applied force, i.e., the steady state response. Note that the solution given in Equation (5.16) is independent of the nature of the viscous damping in the system (i.e., proportional or not) and gives both the displacement and velocity time response.

The matrix e^{At} is often called the *state transition matrix* of the system defined by Equation (5.1). Matrix e^{At} 'maps' the initial condition $\mathbf{x}(0)$ into the new or next position $\mathbf{x}(t)$. While Equation (5.16) represents a closed-form solution of Equation (5.1) for any state matrix A, use of this form centers on the calculation of matrix e^{At}. Many papers have been written on different methods of calculating e^{At} (Moler and Van Loan, 1978).

One method of calculating e^{At} is to realize that e^{At} is equal to the inverse Laplace transform of the resolvent matrix for A. In fact, a comparison of Equations (5.16) and (5.4) yields

$$e^{At} = \pounds^{-1}\{(sI - A)^{-1}\} \tag{5.17}$$

Another interesting method of calculating e^{At} is restricted to those matrices A with diagonal Jordan form. Then it can be shown [recall Equation (3.31)] that

$$e^{At} = Ue^{\Lambda t}U^{-1} \tag{5.18}$$

where U is the matrix of eigenvectors of A, and Λ is the diagonal matrix of eigenvalues of A. Here, $e^{\Lambda t} = \text{diag}\left[e^{\lambda_1 t}\ e^{\lambda_2 t} \cdots e^{\lambda_n t}\right]$, where the λ_i denote the eigenvalues of matrix A.

Example 5.2.2

Compute the matrix exponential e^{At} for the state matrix

$$A = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix}$$

Using the Laplace transform approach of Equation (5.18) requires forming of the matrix $(sI - A)$:

$$sI - A = \begin{bmatrix} s+2 & -3 \\ 3 & s+2 \end{bmatrix}$$

Calculating the inverse of this matrix yields

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s+2}{(s+2)^2+9} & \dfrac{3}{(s+2)^2+9} \\ \dfrac{-3}{(s+2)^2+9} & \dfrac{s+2}{(s+2)^2+9} \end{bmatrix}$$

The matrix exponential is now computed by taking the inverse Laplace transform of each element. This results in

$$e^{At} = \pounds^{-1}\{(sI - A)^{-1}\} = \begin{bmatrix} e^{-2t}\cos 3t & e^{-2t}\sin 3t \\ -e^{-2t}\sin 3t & e^{-2t}\cos 3t \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{bmatrix}$$

5.3 DECOUPLING CONDITIONS AND MODAL ANALYSIS

An alternative approach to solving for the response of a system by transform or matrix exponent methods is to use the eigenvalue and eigenvector information from the free response as a tool for solving for the forced response. This provides a useful theoretical tool as well as a computationally different approach. Approaches based on the eigenvectors of the system are referred to as *modal analysis* and also form the basis for understanding modal test methods (see Chapter 8). Modal analysis can be carried out in either the state vector coordinates of first-order form or the physical coordinates defining the second-order form.

First consider the system described by Equation (5.1). If matrix A has a diagonal Jordan form (Section 3.4), which happens when it has distinct eigenvalues, for example, then matrix A can be diagonalized by its modal matrix. In this circumstance, Equation (5.1) can be reduced to $2n$ independent first-order equations. To see this, let $\mathbf{u}_i$ be the eigenvectors of the state matrix A with eigenvalues λ_i. Let $U = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_{2n}]$ be the modal matrix of matrix A. Then substitute $\mathbf{x} = U\mathbf{z}$ into Equation (5.1) to obtain

$$U\dot{\mathbf{z}} = AU\mathbf{z} + \mathbf{f} \tag{5.19}$$

Premultiplying Equation (5.19) by U^{-1} yields the decoupled system

$$\dot{\mathbf{z}} = U^{-1}AU\mathbf{z} + U^{-1}\mathbf{f} \tag{5.20}$$

each element of which is of the form

$$\dot{z}_i = \lambda_i z_i + F_i \tag{5.21}$$

where z_i is the ith element of the vector $\mathbf{z}$ and F_i is the ith element of the vector $\mathbf{F} = U^{-1}\mathbf{f}$. Equation (5.21) can now be solved using scalar integration of each of the equations subject to the initial condition $z_i(0) = [U^{-1}\mathbf{x}(0)]_i$. In this way, the vector $\mathbf{z}(t)$ can be calculated, and the solution $\mathbf{x}(t)$ in the original coordinates becomes

$$\mathbf{x}(t) = U\mathbf{z}(t) \tag{5.22}$$

The amount of effort required to calculate the solution via this method is comparable with that required to calculate the solution via Equation (5.17). The modal form offered by Equations (5.21) and (5.22) provides a tremendous analytical advantage.

The above process can also be used on systems described in the second-order form. The differences are that the eigenvector–eigenvalue problem is solved in n dimensions instead of $2n$ dimensions and the solution is the position vector instead of the state vector of position and velocity. In addition, the modal vectors used to decouple the equations of motion have important physical significance when viewed in second-order form which is not as apparent in the state-space form.

Consider examining the forced response in the physical or spatial coordinates defined by Equation (2.7). First consider the simplest problem, that of calculating the forced response of an undamped nongyroscopic system of the form

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}(t) \tag{5.23}$$

where M and K are $n \times n$ real positive definite matrices, $\mathbf{q}(t)$ is the vector of displacements, and $\mathbf{f}(t)$ is a vector of applied forces. The system is also subject to an initial position given by $\mathbf{q}(0)$ and an initial velocity given by $\dot{\mathbf{q}}(0)$.

To solve Equation (5.23) by eigenvector expansions, one must first solve the eigenvalue–eigenvector problem for the corresponding homogeneous system. That is, one must calculate λ_i and $\mathbf{u}_i$ such that $(\lambda_i = \omega_i^2)$

$$\lambda_i \mathbf{u}_i = \tilde{K}\mathbf{u}_i \tag{5.24}$$

Note that $\mathbf{u}_i$ now denotes an $n \times 1$ eigenvector of the mass normalized stiffness matrix. From this, the modal matrix S_m is calculated and normalized such that

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T K S_m &= \Lambda = \mathrm{diag}(\omega_i^2) \end{aligned} \tag{5.25}$$

This procedure is the same as that of Section 3.3 except that, in the case of the forced response, the form that the temporal part of the solution will take is not known. Hence, rather than assuming that the dependence is of the form $\sin(\omega t)$, the temporal form is computed from a generic temporal function designated as $y_i(t)$.

Since the eigenvectors $\mathbf{u}_i$ form a basis in an n-dimensional space, any vector $\mathbf{q}(t_1)$, where t_1 is some fixed but arbitrary time, can be written as a linear combination of the vectors $\mathbf{u}_i$; thus

$$\mathbf{q}(t_1) = \sum_{i=1}^{n} y_i(t_1)\mathbf{u}_i = S_m \mathbf{y}(t_1) \tag{5.26}$$

where $\mathbf{y}(t_1)$ is an n-vector with components $y_i(t_1)$ to be determined. Since t_1 is arbitrary, it is reasoned that Equation (5.25) must hold for any t. Therefore

$$\mathbf{q}(t) = \sum_{i=1}^{n} y_i(t)\mathbf{u}_i = S_m \mathbf{y}(t), \qquad t \geq 0 \tag{5.27}$$

This must be true for any n-dimensional vector $\mathbf{q}$. In particular this must hold for solutions of Equation (5.23). Substitution of Equation (5.27) into Equation (5.23) shows that the vector $\mathbf{y}(t)$ must satisfy

$$M S_m \ddot{\mathbf{y}}(t) + K S_m \mathbf{y}(t) = \mathbf{f}(t) \tag{5.28}$$

Premultiplying by S_m^T yields

$$\ddot{\mathbf{y}}(t) + \Lambda \mathbf{y}(t) = S_m^T \mathbf{f}(t) \tag{5.29}$$

Equation (5.28) represents n decoupled equations, each of the form

$$\ddot{\mathbf{y}}_i(t) + \omega_i^2 \mathbf{y}_i(t) = f_i(t) \tag{5.30}$$

where $f_i(t)$ denotes the ith element of the vector $S_m^T \mathbf{f}(t)$.

If K is assumed to be positive definite, each ω_i^2 is a positive real number. Denoting the 'modal' initial conditions by $y_i(0)$ and $\dot{y}_i(0)$, the solutions of Equation (5.30) are calculated by the method of variation of parameters to be

$$y_i(t) = \frac{1}{\omega_i}\int_0^t f_i(t-\tau)\sin(\omega_i\tau)\,d\tau + y_i(0)\cos(\omega_i t) + \frac{\dot{y}_i(0)}{\omega_i}\sin(\omega_i t), \qquad i = 1, 2, 3, \ldots, n \tag{5.31}$$

(see Boyce and DiPrima, 2005, for a derivation).

If K is semidefinite, one or more values of ω_i^2 might be zero. Then, Equation (5.30) would become

$$\ddot{y}_i(t) = f_i(t) \tag{5.32}$$

Integrating Equation (5.32) then yields

$$y_i(t) = \int_0^t \left[\int_0^\tau f_i(s)\,ds\right] d\tau + y_i(0) + \dot{y}_i(0)t \tag{5.33}$$

which represents a rigid body motion.

The initial conditions for the new coordinates are determined from the initial conditions for the original coordinates by the transformation

$$\mathbf{y}(0) = S_m^{-1}\mathbf{q}(0) \tag{5.34}$$

and

$$\dot{\mathbf{y}}(0) = S_m^{-1}\dot{\mathbf{q}}(0) \tag{5.35}$$

This method is often referred to as *modal analysis* and differs from the state-space modal approach in that the computations involve matrices and vectors of size n rather than $2n$. They result in a solution for the position vector rather than the $2n$-dimensional state vector. The coordinates defined by the vector $\mathbf{y}$ are called *modal coordinates, normal coordinates, decoupled coordinates*, and (sometimes) *natural coordinates*. Note that, in the case of a free response, i.e., $f_i = 0$, then $y_i(t)$ is just $e^{\pm\omega_i jt}$, where ω_i is the ith natural frequency of the system, as discussed in Section 3.3.

Alternatively, the modal decoupling described in the above paragraphs can be obtained by using the mass normalized stiffness matrix. To see this, substitute $\mathbf{q} = M^{1/2}\mathbf{r}$ into Equation (2.11), multiply by $M^{-1/2}$ to form $\tilde{K} = M^{-1/2}KM^{-1/2}$, compute the normalized eigenvectors of $\tilde{K}$, and use these to form the columns of the orthogonal matrix S. Next, use the substitution $\mathbf{r} = S\mathbf{y}$ in the equation of motion, premultiply by S^T, and equation (5.30) results. This procedure is illustrated in the following example.

Example 5.3.1

Consider the undamped system of example 3.3.2 and Figure 2.4 subject to a harmonic force applied to m_2, given by

$$\mathbf{f}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t$$

Following the alternative approach, compute the modal force by multiplying the physical force by $S^T M^{-1/2}$:

$$S^T M^{-1/2}\mathbf{f}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin 3t = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 3t$$

Combining this with the results of example 3.32 leads to the modal equations [Equation (5.30)]

$$\ddot{y}_1(t) + 2y_1(t) = \frac{1}{\sqrt{2}}\sin 3t$$

$$\ddot{y}_2(t) + 4y_2(t) = \frac{1}{\sqrt{2}}\sin 3t$$

This of course is subject to the transformed initial conditions, and each equation can be solved by the methods of Chapter 1. For instance, if the initial conditions in the physical coordinates are

$$\mathbf{q}(0) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then in the modal coordinates the initial velocity remains zero but the initial displacement is transformed (solving $\mathbf{q} = M^{-1/2}\,\mathbf{r}$ and $\boldsymbol{r} = S\mathbf{y}$, for $\mathbf{y}$) to become

$$\mathbf{y}(0) = S^T M^{1/2}\mathbf{q}(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \frac{0.3}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving for y_1 proceeds by first computing the particular solution (see Section 1.4 with zero damping) by assuming that $y(t) = X\sin 3t$ in the modal equation for $y_1$to obtain

$$-9X\sin 3t + 2X\sin 3t = \frac{1}{\sqrt{2}}\sin 3t$$

Solving for $\mathbf{X}$ leads to the particular solution $y_{1\mathrm{p}}(t) = \left(-1/(7\sqrt{2})\right)\sin 3t$. The total solution (from Equation (1.21) with zero damping) is then

$$y_1(t) = a\sin\sqrt{2}t + b\cos\sqrt{2}t - \frac{1}{7\sqrt{2}}\sin 3t$$

Applying the modal initial conditions yields

$$y_1(0) = b = \frac{0.3}{\sqrt{2}}, \qquad \dot{y}_1(0) = \sqrt{2}a - \frac{3}{7\sqrt{2}} = 0 \Rightarrow a = \frac{3}{14}$$

Thus, the solution of the first modal equation is

$$y_1(t) = \frac{3}{14}\sin\sqrt{2}t + \frac{0.3}{\sqrt{2}}\cos\sqrt{2}t - \frac{1}{7\sqrt{2}}\sin 3t$$

Likewise, the solution to the second modal equation is

$$y_2(t) = \frac{0.3}{\sqrt{2}}\sin 2t - \frac{0.3}{\sqrt{2}}\cos 2t - \frac{0.2}{\sqrt{2}}\sin 3t$$

The solution in physical coordinates is then found from $\mathbf{q}(t) = M^{-1/2}S\mathbf{y}(t)$.

5.4 RESPONSE OF SYSTEMS WITH DAMPING

The key to using modal analysis to solve for the forced response of systems with velocity-dependent terms is whether or not the system can be decoupled. As in the case of the free response, discussed in Section 3.5, this will happen for symmetric systems if and only if the coefficient matrices commute, i.e., if $KM^{-1}D = DM^{-1}K$. Ahmadian and Inman (1984a) reviewed previous work on decoupling and extended the commutivity condition to systems with asymmetric coefficients. Inman (1982) and Ahmadian and Inman (1984b) used the decoupling condition to carry out modal analysis for general asymmetric systems with commuting coefficients. In each of these cases the process is the same, with an additional transformation into symmetric coordinates, as introduced in Section 4.9. Hence, only the symmetric case is illustrated here.

To this end, consider the problem of calculating the forced response of the nongyroscopic damped linear system given by

$$M\ddot{\mathbf{q}} + D\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}(t) \tag{5.36}$$

where M and K are symmetric and positive definite and D is symmetric and positive semidefinite. In addition, it is assumed that $KM^{-1}D = DM^{-1}K$. Let S_m be the modal matrix of K normalized with respect to M, as defined by Equations (3.69) and (3.70). Then, the commutivity of the coefficient matrices yields

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T D S_m &= \Lambda_D = \mathrm{diag}(2\zeta_i\omega_i) \\ S_m^T K S_m &= \Lambda_K = \mathrm{diag}(\omega_i^2) \end{aligned} \tag{5.37}$$

where Λ_D and Λ_K are diagonal matrices, as indicated. Making the substitution $\mathbf{q} = \mathbf{q}(t) = S_m\mathbf{y}(t)$ in Equation (5.36) and premultiplying by S_m^T as before yields

$$I\ddot{\mathbf{y}} + \Lambda_D\dot{\mathbf{y}} + \Lambda_K\mathbf{y} = S_m^T\mathbf{f}(t) \tag{5.38}$$

Equation (5.38) is diagonal and can be written as n decoupled equations of the form

$$\ddot{\mathbf{y}}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = f_i(t) \tag{5.39}$$

Here, $\zeta_i = \lambda_i(D)/2\omega_i$, where $\lambda_i(D)$ denotes the eigenvalues of matrix D. In this case these are the nonzero elements of Λ_D. This expression is the nonhomogeneous counterpart of Equation (3.71).

If it is assumed that $4\tilde{K} - \tilde{D}^2$ is positive definite, then $0 < \zeta_i < 1$, and the solution of Equation (5.39) (assuming all initial conditions are zero) is

$$y_i(t) = \frac{1}{\omega_{di}}\int_0^t e^{-\zeta_i\omega_i\tau} f_i(t-\tau)\sin(\omega_{di}\tau)\,d\tau, \qquad i = 1, 2, 3, \ldots, n \tag{5.40}$$

where $\omega_{di} = \omega_i\sqrt{1-\zeta_i^{(2)}}$. If $4\tilde{K} - \tilde{D}^2$ is not positive definite, other forms of $y_i(t)$ result, depending on the eigenvalues of the matrix $4\tilde{K} - \tilde{D}^2$, as discussed in Section 3.6.

In addition to the forced response given by Equation (5.40), there will be a transient response, or homogeneous solution, due to any nonzero initial conditions. If this response is denoted by y_i^H, then the total response of the system in the decoupled coordinate system is the sum $y_i(t) + y_i^H(t)$. This solution is related to the solution in the original coordinates by the modal matrix S_m and is given by

$$\mathbf{q}(t) = S_m[\mathbf{y}(t) + \mathbf{y}^H(t)] \tag{5.41}$$

For asymmetric systems, the procedure is similar, with the exception of computing a second transformation; this transforms the asymmetric system into an equivalent symmetric system, as done in Section 4.9.

For systems in which the coefficient matrices do not commute, i.e., for which $KM^{-1}D \neq DM^{-1}K$ in Equation (5.36), modal analysis of a sort is still possible without resorting to state space. To this end, consider the symmetric case given by the system

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}(t) \tag{5.42}$$

where M, D, and K are symmetric.

Let $\mathbf{u}_i$ be the eigenvectors of the lambda matrix

$$(M\lambda_i^2 + D\lambda_i + K)\mathbf{u}_i = \mathbf{0} \tag{5.43}$$

with associated eigenvalues λ_i. Let n be the number of degrees of freedom (so there are $2n$ eigenvalues), let $2s$ be the number of real eigenvalues, and let $2(n - s)$ be the number of complex eigenvalues. Assuming that $D_2(\lambda)$ is simple and that the $\mathbf{u}_i$ are normalized so that

$$\mathbf{u}_i^T(2M\lambda_i + D)\mathbf{u}_i = 1 \tag{5.44}$$

a particular solution of Equation (5.42) is given by Lancaster (1966) in terms of the generalized modes $\mathbf{u}_i$ to be

$$\mathbf{q}(t) = \sum_{k=1}^{2s} \mathbf{u}_k\mathbf{u}_k^T \int_0^t e^{-\lambda_k(t+\tau)}\mathbf{f}(\tau)\,d\tau + \sum_{k=2s+1}^{2n} \int_0^t \mathrm{Re}\{e^{\lambda_k(t-\tau)}\mathbf{u}_k\mathbf{u}_k^T\}\mathbf{f}(\tau)\,d\tau \tag{5.45}$$

This expression is more difficult to compute but does offer some insight into the form of the solution that is useful in modal testing, as will be illustrated in Chapter 8. Note that the eigenvalues indexed λ_1 through λ_{2s} are real, whereas those labeled λ_{2s+1} through λ_{2n} are complex. The complex eigenvalues λ_{2s+1} and $\lambda_{2(s+1)}$ are conjugates of each other. Also, note that the nature of the matrices D and K completely determines the value of s. In fact, if $4\tilde{K} - \tilde{D}^2$ is positive definite, $s = 0$ in Equation (5.44). Also note that, if λ_k is real, so is the corresponding $\mathbf{u}_k$. On the other hand, if λ_k is complex, the corresponding eigenvector $\mathbf{u}_k$ is real if and only if $KM^{-1}D = DM^{-1}K$; otherwise, the eigenvectors are complex valued.

The particular solution (5.45) has the advantage of being stated in the original, or physical, coordinate system. To obtain the total solution, the transient response developed in Section (3.4) must be added to Equation (5.45). This should be done unless steady state conditions prevail.

Example 5.4.1

Consider example 5.3.1 with a damping force applied of the form $D = 0.1K$ (proportional damping). In this case $4\tilde{K} - \tilde{D}$ is positive definite so that each mode will be underdamped. The alternative transformation used in example 5.3.1 is employed here to find the modal equations given in Equation (5.39). The equations of motion and initial conditions (assuming compatible units) are

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \dot{\mathbf{q}}(t) + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{q}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t, \quad \mathbf{q}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{q}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since the damping is proportional, the undamped transformation computed in examples 3.3.2 and 5.3.1 can be used to decouple the equations of motion. Using the transformation $\mathbf{y}(t) = S^T M^{1/2} \mathbf{q}(t)$ and premultiplying the equations of motion by $S^T M^{-1/2}$ yields the uncoupled modal equations

$$\ddot{y}_1(t) + 0.2\dot{y}_1(t) + 2y_1(t) = \frac{1}{\sqrt{2}} \sin 3t, \qquad y_1(0) = \frac{3}{\sqrt{2}}, \qquad \dot{y}_1(0) = 0$$

$$\ddot{y}_2(t) + 0.4\dot{y}_2(t) + 4y_2(t) = \frac{1}{\sqrt{2}} \sin 3t, \qquad y_2(0) = \frac{-3}{\sqrt{2}}, \qquad \dot{y}_2(0) = 0$$

From the modal equations the frequencies and damping ratios are evident:

$$\omega_1 = \sqrt{2} = 1.414\,\text{rad/s}, \qquad \zeta_1 = \frac{0.2}{2\sqrt{2}} = 0.071 < 1, \qquad \omega_{d_1} = \omega_1\sqrt{1-\zeta_1^2} = 1.41\,\text{rad/s}$$

$$\omega_2 = \sqrt{4} = 2\,\text{rad/s}, \qquad \zeta_2 = \frac{0.4}{(2)(2)} = 0.1 < 1, \qquad \omega_{d_2} = \omega_2\sqrt{1-\zeta_2^2} = 1.99\,\text{rad/s}$$

Solving the two modal equations using the approach of example 1.4.1 (y_2 is solved there) yields

$$y_1(t) = e^{-0.1t}(0.3651 \sin 1.41t + 2.1299 \cos 1.41t) - 0.1015 \sin 3t - 0.0087 \cos 3t$$
$$y_2(t) = -e^{-0.2t}(0.0084 \sin 1.99t + 2.0892 \cos 1.99t) - 0.1325 \sin 3t - 0.032 \cos 3t$$

This forms the solution in modal coordinates. To regain the solution in physical coordinates, use the transformation $\mathbf{q}(t) = M^{-1/2} S \mathbf{y}(t)$. Note that the transient term is multiplied by a decaying exponential in time and will decay off, leaving the steady state to persist.

5.5 BOUNDED-INPUT, BOUNDED-OUTPUT STABILITY

In the previous chapter, several types of stability for the free response of a system were defined and discussed in great detail. In this section the concept of stability as it applies to the forced response of a system is discussed. In particular, systems are examined in the state-space form given by Equation (5.1).

The stability of the forced response of a system is defined in terms of bounds of the response vector $\mathbf{x}(t)$. Hence, it is important to recall that a vector $\mathbf{x}(t)$ is bounded if

$$\|\mathbf{x}(t)\| = \sqrt{\mathbf{x}^T\mathbf{x}} < M \tag{5.46}$$

where M is some finite positive real number. The quantity $||\mathbf{x}||$ just defined is called the *norm* of $\mathbf{x}(t)$. The response $\mathbf{x}(t)$ is also referred to as the *output* of the system, hence the phrase bounded-output stability.

A fundamental classification of stability of forced systems is called *bounded-input, bounded-output* (BIBO) *stability*. The system described by Equation (5.1) is called BIBO stable if any bounded forcing function $\mathbf{f}(t)$, called the input, produces a bounded response $\mathbf{x}(t)$, i.e., a bounded output, regardless of the bounded initial condition $\mathbf{x}(0)$. An example of a system that is not BIBO stable is given by the single-degree-of-freedom oscillator

$$\ddot{y} + \omega_n^2 y = \sin \omega t \tag{5.47}$$

In state-space form this becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \mathbf{x} + \mathbf{f}(t) \tag{5.48}$$

where $\mathbf{x} = [y \quad \dot{y}]^T$ and $\mathbf{f}(t) = [0 \quad \sin \omega t]^T$. This system is not BIBO stable, since, for a bounded input $y(t)$, and hence $\mathbf{x}(t)$, blows up when $\omega = \omega_n$ (i.e., at resonance).

A second classification of stability is called *bounded stability*, or *Lagrange stability*, and is a little weaker than BIBO stability. The system described in Equation (5.1) is said to be *Lagrange stable* with respect to a *given* input $\mathbf{f}(t)$ if the response $\mathbf{x}(t)$ is bounded for any bounded initial condition $\mathbf{x}(0)$. Referring to the example of the previous paragraph, if $\omega \neq \omega_n$, then the system described by Equation (5.48) is bounded with respect to $\mathbf{f}(t) = [0 \quad \sin \omega t]^T$ because, when $\omega \neq \omega_n$, $\mathbf{x}(t)$ does not blow up. Note that, if a given system is BIBO stable, it will also be Lagrange stable. However, a system that is Lagrange stable may not be BIBO stable.

As an example of a system that is BIBO stable, consider adding damping to the preceding system. The result is a single-degree-of-freedom damped oscillator that has the state matrix

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{5.49}$$

Recall that the damping term prevents the solution $\mathbf{x}(t)$ from becoming unbounded at resonance. Hence, $y(t)$ and $\dot{y}(t)$ are bounded for any bounded input $\mathbf{f}(t)$, and the system is BIBO stable as well as Lagrange stable.

The difference in the two examples is due to the stability of the free response of each system. The undamped oscillator is stable but not asymptotically stable, and the forced response *is not* BIBO stable. On the other hand, the damped oscillator is asymptotically stable and *is* BIBO stable. To some extent this is true in general. Namely, it is shown by Müller and Schiehlen (1977) that, if the forcing function $\mathbf{f}(t)$ can be written as a constant matrix B times a vector $\mathbf{u}$, i.e., $\mathbf{f}(t) = B\mathbf{u}$, then, if

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & A^3B \cdots A^{2n-1}B \end{bmatrix} = 2n \tag{5.50}$$

where $2n$ is the dimension of matrix A, the system in Equation (5.1) is BIBO stable if and only if the free response is asymptotically stable. If $\mathbf{f}(t)$ does not have this form or does not satisfy the rank condition (5.50), then asymptotically stable systems are BIBO stable, and BIBO stable systems have a stable free response.

Another way to look at the difference between the above two examples is to consider the phenomenon of resonance. The undamped single-degree-of-freedom oscillator of Equation (5.48) experiences an infinite amplitude at $\omega = \omega_n$, the resonance condition, which is certainly unstable. However, the underdamped single-degree-of-freedom oscillator of Equation (5.49) is bounded at the resonance condition, as discussed in Section 1.4. Hence, the damping 'lowers' the peak response at resonance from infinity to some finite, or bounded, value, resulting in a system that is BIBO stable.

The obvious use of the preceding conditions is to use the stability results of Chapter 4 for the free response to guarantee BIBO stability or boundedness of the forced response, $\mathbf{x}(t)$. To this extent, other concepts of stability of systems subject to external forces are not developed. Instead, some specific bounds on the forced response of a system are examined in the next section.

5.6 RESPONSE BOUNDS

Given that a system is either BIBO stable or at least bounded, it is sometimes of interest to calculate bounds for the forced response of the system without actually calculating the response itself. A summary of early work on the calculation of bounds is given in review papers by Nicholson and Inman (1983) and Nicholson and Lin (1996). More recent work is given in Hu and Eberhard (1999). A majority of the work reported there examines bounds for systems in the physical coordinates $\mathbf{q}(t)$ in the form of Equation (5.36). In particular, if $DM^{-1}K = KM^{-1}D$ and if the forcing function or input is of the form (periodic)

$$\mathbf{f}(t) = \mathbf{f}_0 e^{j\omega t} \tag{5.51}$$

where $\mathbf{f}_0$ is an $n \times 1$ vector of constants, $j^2 = -1$, and ω is the driving frequency, then

$$\frac{||\mathbf{q}(t)||}{||\mathbf{f}_0||} \leqslant \max_j \begin{cases} \dfrac{1}{\lambda_j(K)} & \text{if } \dfrac{\lambda_j(K)}{\lambda_j(M)} < \dfrac{\lambda_j^2(D)}{2\lambda_j^2(M)} \\ \sqrt{\dfrac{\lambda_i(M)}{\lambda_i^2(D)\lambda_i(K)}} & \text{otherwise} \end{cases} \tag{5.52}$$

Here, $\lambda_i(M)$, $\lambda_i(D)$, and $\lambda_i(K)$ are used to denote the ordered eigenvalues of the matrices M, D, and K respectively. The first inequality in expression (5.52) is satisfied if the free system is overdamped, and the bound $[\lambda_i^2(D)\lambda_i(K)/\lambda_i(M)]^{-1/2}$ is applied for underdamped systems.

Bounds on the forced response are also available for systems that do not decouple, i.e., for systems with coefficient matrices such that $DM^{-1}K \neq KM^{-1}D$. One way to approach such systems is to write the system in the normal coordinates of the undamped system. Then the resulting damping matrix can be written as the sum of a diagonal matrix and an off-diagonal matrix, which clearly indicates the degree of decoupling.

Substituting $\mathbf{q}(t) = S_m\mathbf{x}$ into Equation (5.36) and premultiplying by S_m^T, where S_m is the modal matrix for K, yields

$$I\ddot{\mathbf{x}} + (\Lambda_D + D_1)\dot{\mathbf{x}} + \Lambda_K \mathbf{x} = \tilde{\mathbf{f}}(t) \tag{5.53}$$

The matrix Λ_D is the diagonal part of $S_m^T D S_m$, D_1 is the matrix of off-diagonal elements of $S_m^T D S_m$, Λ_K is the diagonal matrix of squared undamped natural frequencies, and $\tilde{\mathbf{f}} = S_m^T \mathbf{f}$.

The steady state response of Equation (5.53) with a sinusoidal input, i.e., $\tilde{\mathbf{f}} = \mathbf{f}_0 e^{j\omega t}$ and Equation (5.53) underdamped, is given by

$$\frac{||\mathbf{q}(t)||}{||\mathbf{f}_0||} < \frac{2}{\lambda_{\min}(\Lambda_D \Lambda_C)} e^{(\beta||D_1||/\lambda_{\min})} \tag{5.54}$$

Here, $\lambda_{\min}(\Lambda_D \Lambda_C)$ denotes the smallest eigenvalue of the matrix $\Lambda_D \Lambda_C$, where $\Lambda_C = (4\Lambda_K - \Lambda_D^2)^{1/2}$. Also, $\lambda_{\min}$ is the smallest eigenvalue of the matrix Λ_D, $||D_1||$ is the matrix norm defined by the maximum value of the square root of the largest eigenvalue of $D_1^T D_1 = D_1^2$, and β is defined by

$$\beta = \sqrt{\left\| I + (\Lambda_C^{-1} \Lambda_D)^2 \right\|} \tag{5.55}$$

Examination of the bound in Equation (5.53) shows that, the greater the coupling in the system characterized by $||D_1||$, the larger is the bound. Thus, for small values of $||D_1||$, i.e., small coupling, the bound is good, whereas for large values of $||D_1||$ or very highly coupled systems, the bound will be very large and too conservative to be of practical use. This is illustrated in example 5.6.1.

Example 5.6.1

Consider a system defined by Equation (5.36) with $M = I$:

$$K = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}, \qquad D = 0.5K + 0.5I + \xi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

subject to a sinusoidal driving force applied to the first mass so that $\mathbf{f}_0 = [1 \quad 0]^T$. The parameter ξ clearly determines the degree of proportionality or coupling in the system. The bounds are tabulated in Table 5.1 for various values of ξ, along with a comparison with the exact solution.

Examination of Table 5.1 clearly illustrates that, as the degree of coupling increases (larger ξ), the bound gets farther away from the actual response. Note that the value given in the 'exact' column is the largest value obtained by the exact response.

Table 5.1 Forced response bounds.

ξ	Exact solution	Bound
0	1.30	1.50
−0.1	1.56	2.09
−0.2	1.62	3.05
−0.3	1.70	4.69
−0.4	1.78	8.60

5.7 FREQUENCY RESPONSE METHODS

This section attempts to extend the concept of frequency response introduced in Sections 1.5 and 1.6 to multiple-degree-of-freedom systems. In so doing, the material in this section makes the connection between analytical modal analysis and experimental modal analysis discussed in Chapter 8. The development starts by considering the response of a structure to a harmonic or sinusoidal input, denoted by $\mathbf{f}(t) = \mathbf{f}_0 e^{j\omega t}$. The equations of motion in spatial or physical coordinates given by Equation (5.36) with no damping ($D = 0$) or Equation (5.23) are considered first. In this case, an oscillatory solution of Equation (5.23) of the form

$$\mathbf{q}(t) = \mathbf{u}e^{j\omega t} \tag{5.56}$$

is assumed. This is equivalent to the frequency response theorem stated in Section 1.5. That is, if a system is harmonically excited, the response will consist of a steady state term that oscillates at the driving frequency with different amplitude and phase.

Substitution of the assumed oscillatory solution into Equation (5.36) with $D = 0$ yields

$$\left(K - \omega^2 M\right)\mathbf{u}e^{j\omega t} = \mathbf{f}_0 \mathrm{e}^{j\omega t} \tag{5.57}$$

Dividing through by the nonzero scalar $e^{j\omega t}$ and solving for $\mathbf{u}$ yields

$$\mathbf{u} = (K - \omega^2 M)^{-1}\mathbf{f}_0 \tag{5.58}$$

Note that the matrix inverse of $(K - \omega^2 M)$ exists as long as ω is not one of the natural frequencies of the structure. This is consistent with the fact that, without damping, the system is Lagrange stable and not BIBO stable. The matrix coefficient of Equation (5.58) is defined as the *receptance matrix*, denoted by $\alpha(\omega)$, i.e.,

$$\alpha(\omega) = (K - \omega^2 M)^{-1} \tag{5.59}$$

Equation (5.58) can be thought of as the *response model* of the structure. Solution of Equation (5.58) yields the vector $\mathbf{u}$, which, coupled with Equation (5.56), yields the steady state response of the system to the input force $\mathbf{f}(t)$.

Each element of the response matrix can be related to a single-frequency response function by examining the definition of matrix multiplication. In particular, if all the elements of the vector $\mathbf{f}_0$, denoted by f_i, except the jth element are set equal to zero, then the ijth element of $\alpha(\omega)$ is just the receptance transfer function between u_i, the ith element of the response vector $\mathbf{u}$, and f_j. That is

$$\alpha_{ij}(\omega) = \frac{u_i}{f_j}, \qquad f_i = 0, \qquad i = 0, \ldots, n, \qquad i \neq j \tag{5.60}$$

Note that, since $\alpha(\omega)$ is symmetric, this interpretation implies that $u_i/f_j = u_j/f_i$. Hence, a force applied at position j yields the same response at point i as a force applied at i does at point j. This is called *reciprocity*.

An alternative to computing the inverse of the matrix $(K - \omega^2 M)$ is to use the modal decomposition of $\alpha(\omega)$. Recalling Equations (3.20) and (3.21) from Section 3.2, the matrices M and K can be rewritten as

$$M = S_m^{-T} S_m^{-1} \tag{5.61}$$

$$K = S_m^{-T} \mathrm{diag}(\omega_r^2) S_m^{-1} \tag{5.62}$$

where ω_i are the natural frequencies of the system and S_m is the matrix of modal vectors normalized with respect to the mass matrix. Substitution of these 'modal' expressions into Equation (5.59) yields

$$\begin{aligned}\alpha(\omega) &= \{S_m^{-T}[\mathrm{diag}(\omega_r^2) - \omega^2 I]S_m^{-T}\}^{-1} \\ &= S_m\{\mathrm{diag}[\omega_r^2 - \omega^2]^{-1}\}S_m^T\end{aligned} \tag{5.63}$$

Expression (5.63) can also be written in summation notation by considering the ijth element of $\alpha(\omega)$, recalling formula (2.6), and partitioning the matrix S_m into columns, denoted by $\mathbf{s}_r$. The vectors $\mathbf{s}_r$ are, of course, the eigenvectors of the matrix K normalized with respect to the mass matrix M. This yields

$$\alpha(\omega) = \sum_{r=1}^{n} (\omega_r^2 - \omega^2)^{-1} \mathbf{s}_r \mathbf{s}_r^T \tag{5.64}$$

The ijth element of the receptance matrix becomes

$$\alpha_{ij}(\omega) = \sum_{r=1}^{n} (\omega_r^2 - \omega^2)^{-1} [\mathbf{s}_r \mathbf{s}_r^T]_{ij} \tag{5.65}$$

where the matrix element $[\mathbf{s}_r \mathbf{s}_r^T]_{ij}$ is identified as the *modal constant* or *residue* for the rth mode, and the matrix $\mathbf{s}_r \mathbf{s}_r^T$ is called the *residue matrix*. Note that the right-hand side of Equation (5.65) can also be rationalized to form a single fraction consisting of the ratio of two polynomials in ω^2. Hence, $[\mathbf{s}_r \mathbf{s}_r^T]_{ij}$ can also be viewed as the matrix of constants in the partial fraction expansion of Equation (5.60).

Next, consider the same procedure applied to Equation (5.36) with nonzero damping. As always, consideration of damped systems results in two cases: those systems that decouple and those that do not.

First consider Equation (5.36) with damping such that $DM^{-1}K = KM^{-1}D$, so that the system decouples and the system eigenvectors are real. In this case the eigenvectors of the undamped system are also eigenvectors for the damped system, as was established in Section 3.5. The definition of the receptance matrix takes on a slightly different form to reflect the damping in the system. Under the additional assumption that the system is underdamped, i.e., that the matrix $4\tilde{K} - \tilde{D}^2$ is positive definite, the modal damping ratios ζ_r are all between 0 and 1. Equation (5.58) becomes

$$\mathbf{u} = (K + j\omega D - \omega^2 M)^{-1} \mathbf{f}_0 \tag{5.66}$$

Because the system decouples, matrix D can be written as

$$D = S_m^{-T}\mathrm{diag}(2\zeta_r\omega_r)S_m^{-1} \tag{5.67}$$

Substitution of Equations (5.61), (5.62), and (5.67) into Equation (5.66) yields

$$\mathbf{u} = S_m[\mathrm{diag}(\omega_r^2 + 2j\zeta_r\omega_r\omega - \omega^2)^{-1}]S_m^T\mathbf{f}_0 \tag{5.68}$$

This expression defines the complex receptance matrix given by

$$\alpha(\omega) = \sum_{r=1}^{n}(\omega_r^2 + 2j\zeta_r\omega_r\omega - \omega^2)^{-1}\mathbf{s}_r\mathbf{s}_r^T \tag{5.69}$$

Next, consider the general viscously damped case. In this case the eigenvectors $\mathbf{s}_r$ are complex and the receptance matrix is given (see Lancaster, 1966) as

$$\alpha(\omega) = \sum_{r=1}^{n}\left\{\frac{\mathbf{s}_r\mathbf{s}_r^T}{j\omega - \lambda_r} + \frac{\mathbf{s}_r^*\mathbf{s}_r^{*T}}{j\omega - \lambda_r^*}\right\} \tag{5.70}$$

Here, the asterisk denotes the conjugate, the λ_r are the complex system eigenvalues, and the $\mathbf{s}_r$ are the system eigenvectors.

The expressions for the receptance matrix and the interpretation of an element of the receptance matrix given by Equation (5.60) form the background for modal testing. In addition, the receptance matrix forms a response model for the system. Considering the most general case [Equation (5.70)], the phenomenon of resonance is evident. In fact, if the real part of λ_r is small, $j\omega - \lambda_r$ is potentially small, and the response will be dominated by the associated mode $\mathbf{s}_r$. The receptance matrix is a generalization of the frequency response function of Section 1.5. In addition, like the transition matrix of Section 5.2, the receptance matrix maps the input of the system into the output of the system.

5.8 NUMERICAL SIMULATION IN MATLAB

This section extends Section 3.9 to include simulation of systems subject to an applied force. The method is the same as that described in Section 3.9, with the exception that the forcing function is now included in the equations of motion. All the codes mentioned in Section 3.9 have the ability numerically to integrate the equations of motion including both the effects of the initial conditions and the effects of any applied forces. Numerical simulation provides an alternative to computing the time response by modal methods, as done in Equation (5.45). The approach is to perform numerical integration following the material in Sections 1.10 and 3.9 with the state-space model. For any class of second-order systems the equations of motion can be written in the state-space form, as by Equation (5.1),

subject to appropriate initial conditions on the position and velocity. While numerical solutions are a discrete time approximation, they are systematic and relatively easy to compute with modern high-level codes. The following example illustrates the procedure in MATLAB.

Example 5.8.1

Consider the system in physical coordinates defined by:

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t),$$

$$\mathbf{q}(0) = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \qquad \dot{\mathbf{q}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In order to use the Runge–Kutta numerical integration, first put the system into the state-space form. Computing the inverse of the mass matrix and defining the state vector $\mathbf{x}$ by

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{bmatrix}^T$$

the state equations become

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3/5 & 1/5 & -3/5 & 1/2 \\ 1 & -1 & 1/2 & -1/2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/5 \\ 1 \end{bmatrix}\sin 4t, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

The steps to solve this numerically in MATLAB follow those of example 3.9.1, with the additional term for the forcing function. The corresponding m-file is

```
function v=f581(t,x)
M=[5 0; 0 5]; D=[3 -0.5;-0.5 0.5]; K=[3 -1;-1 1];
A=[zeros(2) eye(2);-inv(M)* K -inv(M)* D]; b=[0;0;0.2;1];
v=A*x+b*sin(4*t);
```

This function must be saved under the name `f581.m.` Once this is saved, the following is typed in the command window:

```
EDU>clear all
EDU>xo=[0;0.1;1;0];
EDU>ts=[0 50];
EDU>[t,x]=ode45('f581',ts,xo);
EDU>plot(t,x(:,1),t,x(:,2),'--'),title('x1,x2 versus time')
```

This returns the plot shown in Figure 5.1. Note that the command `x(:,1)` pulls off the record for $x_1(t)$ and the command ode45 calls a fifth-order Runge–Kutta program. The command `ts=[0 50];` tells the code to integrate from 0 to 50 time units (seconds in this case).

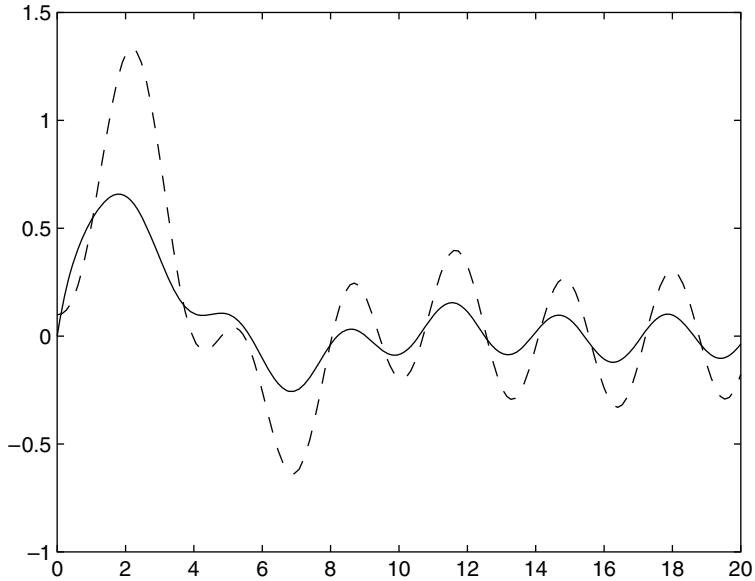

Figure 5.1 Displacement response to the initial conditions and forcing function of example 5.8.1

CHAPTER NOTES

The field of systems theory and control has advanced the idea of using matrix methods for solving large systems of differential equations (see Patel, Laub and Van Dooren, 1994). Thus, the material in Section 5.2 can be found in most introductory systems theory texts such as Chen (1998) or Kailath (1980). In addition, those texts contain material on modal decoupling of the state matrix, as covered in Section 5.3. Theoretical modal analysis (Section 5.3) is just a method of decoupling the equations of motion of a system into a set of simple-to-solve single-degree-of-freedom equations. This method is extended in Section 5.4 and generalized to equations that cannot be decoupled. For such systems, modal analysis of the solution is simply an expansion of the solution in terms of its eigenvectors. This material parallels the development of the free response in Section 3.5. The material of Section 5.6 on bounds is not widely used. However, it does provide some methodology for design work. The results presented in Section 5.6 are from Yae and Inman (1987). The material on frequency response methods presented in Section 5.7 is essential in understanding experimental modal analysis and testing and is detailed in Ewins (2000). Section 5.8 is a brief introduction to the important concept of numerical simulation of dynamic systems.

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PROBLEMS

5.1 Use the resolvent matrix to calculate the solution of

$$\ddot{x} + \dot{x} + 2x = \sin 3t$$

with zero initial conditions.

5.2 Calculate the transition matrix e^{At} for the system of problem 5.1.

5.3 Prove that $e^{-A} = (e^{A})^{-1}$ and show that $e^{A}e^{-A} = I$.

5.4 Compute e^{At}, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

5.5 Show that, if $\mathbf{z}(t) = ae^{-j\phi}e^{(\mu+j\omega)t}\mathbf{f}(t)$, where a and $\mathbf{f}(t)$ are real and $j^2 = -1$, then $\mathrm{Re}(\mathbf{z}) = a\mathbf{f}(t)\,e^{\mu t}\cos(\omega t - \phi)$.

5.6 Consider problem 3.6. Let $\mathbf{f}(t) = [\mu(t) \;\; 0 \;\; 0]^T$, where $\mu(t)$ is the unit step function, and calculate the response of that system with $\mathbf{f}(t)$ as the applied force and zero initial conditions.

5.7 Let $\mathbf{f}(t) = [\sin(t) \;\; 0]^T$ in problem 3.10 and solve for $\mathbf{x}(t)$.

5.8 Calculate a bound on the forced response of the system given in problem 5.7. Which was easier to calculate, the bound or the actual response?

5.9 Calculate the receptance matrix for the system of example 5.6.1 with $\zeta = -1$.

5.10 Discuss the similarities between the receptance matrix, the transition matrix, and the resolvent matrix.

5.11 Using the definition of the matrix exponential, prove each of the following:

(a) $(e^{At})^{-1} = e^{-At}$;
(b) $e^0 = I$;
(c) $e^A e^B = e^{(A+B)}$, if $AB = BA$.

5.12 Develop the formulation for modal analysis of symmetrizable systems by applying the transformations of Section 4.9 to the procedure following Equation (5.23).

5.13 Using standard methods of differential equations, solve Equation (5.39) to obtain Equation (5.40).

5.14 Plot the response of the system in example 5.6.1 along with the bound indicated in Equation (5.54) for the case $\zeta = -1$.

5.15 Derive the solution of Equation (5.39) for the case $\zeta_i > 1$.

5.16 Show that $\mathbf{f}_0 e^{j\omega t}$ is periodic.

5.17 Compute the modal equations for the system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 2t$$

subject to the initial conditions of zero initial velocity and an initial displacement of $\mathbf{x}(0) = [0 \;\; 1]^T$ mm.

5.18 Repeat problem 5.17 for the same system with the damping matrix defined by $C = 0.1K$.

5.19 Derive the relationship between the transformations S and S_m.

5.20 Consider example problem 5.4.1 and compute the total response in physical coordinates.

5.21 Consider example problem 5.4.1 and plot the response in physical coordinates.

5.22 Consider the problem of example 5.4.1 and use the method of numerical integration discussed in Section 5.8 to solve and plot the solution. Compare your results to the analytical solution found in problem 5.21.

5.23 Consider the following undamped system:

$$\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.23500 \\ 2.97922 \end{bmatrix} \sin(2.756556t)$$

(a) Compute the natural frequencies and mode shapes and discuss whether or not the system experiences resonance.
(b) Compute the modal equations.
(c) Simulate the response numerically.

5.24 For the system of example 5.4.1, plot the frequency response function over a range of frequencies from 0 to 8 rad/s.

5.25 Compute and plot the frequency response function for the system of example 5.4.1 for the damping matrix having the value $D = \alpha K$ for several different values of α ranging from 0.1 to 1. Discuss your results. What happens to the peaks?

6
Design Considerations

6.1 INTRODUCTION

The word *design* means many different things to different people. Here, design is used to denote an educated method of choosing and adjusting the physical parameters of a vibrating system in order to obtain a more favorable response. The contents of this chapter are somewhat chronological in the sense that the topics covered first, such as vibration absorbers, are classical vibration design techniques, whereas the later sections, such as the one on control, represent more contemporary methods of design. A section on controls serves as an introduction to Chapter 7, which is devoted entirely to control methods. A section on damping treatments introduces a commonly used method of vibration suppression. The chapter ends with a section on model reduction, which is not a design method but a technique commonly used to provide reasonable sized models to help in design analysis.

6.2 ISOLATORS AND ABSORBERS

Isolation of a vibrating mass refers to designing the connection of a mass (machine part or structure) to ground in such a way as to reduce unwanted effects or disturbances through that connection. Vibration absorption, on the other hand, refers to adding an additional degree of freedom (spring and mass) to the structure to *absorb* the unwanted disturbance. The typical model used in vibration isolation design is the simple single-degree-of-freedom system of Figure 1.1(a) without damping, or Figure 1.4(a) with damping. The idea here is twofold. First, if a harmonic force is applied to the mass through movement of the ground (i.e., as the result of a nearby rotating machine, for instance), the values of c and k should be chosen to minimize the resulting response of the mass. The design *isolates* the mass from the effects of ground motion. The springs on an automobile serve this purpose.

A second use of the concept of isolation is that in which the mass represents the mass of a machine, causing an unwanted harmonic disturbance. In this case the values of m, c, and k are chosen so that the disturbance force passing through the spring and dashpot to ground is minimized. This isolates the ground from the effects of the machine. The motor mounts in an automobile are examples of this type of isolation.

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

In either case, the details of the governing equations for the isolation problem consist of analyzing the steady state forced harmonic response of equations of form (1.17). For instance, if it is desired to isolate the mass of Figure 1.8 from the effects of a disturbance $\mathbf{F}_0 \sin(\omega t)$, then the magnification curves of Figure 1.9 indicate how to choose the damping ζ and the isolator frequency ω_n so that the amplitude of the resulting vibration is as small as possible. Curves similar to the magnification curves, called transmissibility curves, are usually used in isolation problems.

The ratio of the amplitude of the force transmitted through the connection between the ground and the mass to the amplitude of the driving force is called the *transmissibility*. For the system of Figure 1.8, the force transmitted to ground is transmitted through the spring, k, and the damper, c. From Equation (1.21), these forces at steady state are

$$F_k = kx_{ss}(t) = kX\sin(\omega t - \phi) \tag{6.1}$$

and

$$F_c = c\dot{x}_{ss}(t) = c\omega X\cos(\omega t - \phi) \tag{6.2}$$

Here, F_k and F_c denote the force in the spring and the force in the damper respectively, and X is the magnitude of the steady state response as given in Section 1.4. The magnitude of the transmitted force is the magnitude of the vector sum of these two forces, denoted by F_T, and is given by

$$F_T^2 = |kx_{ss} + c\dot{x}_{ss}|^2 = [(kX)^2 + (c\omega X)^2] \tag{6.3}$$

Thus, the magnitude of transmitted force becomes

$$F_T = kX\left[1 + \left(\frac{c\omega}{k}\right)^2\right]^{1/2} \tag{6.4}$$

The amplitude of the applied force is just F_0, so that the transmissibility ratio, denoted by TR, becomes

$$TR = \frac{F_T}{F_0} = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}} \tag{6.5}$$

Plots of expression (6.5) versus the frequency ratio ω/ω_n for various values of ζ are called *transmissibility curves*. One such curve is illustrated in Figure 6.1. This curve indicates that, for values of $\omega/\omega_n > \sqrt{2}$ (that is, $TR < 1$), vibration isolation occurs, whereas for values of $\omega/\omega_n < \sqrt{2}$ ($TR > 1$) an amplification of vibration occurs. Of course, the largest increase in amplitude occurs at resonance.

If the physical parameters of a system are constrained such that isolation is not feasible, a vibration absorber may be included in the design. A vibration absorber consists of an attached second mass, spring, and damper, forming a two-degree-of-freedom system. The second spring–mass system is then 'tuned' to resonate and hence absorb all the vibrational energy of the system.

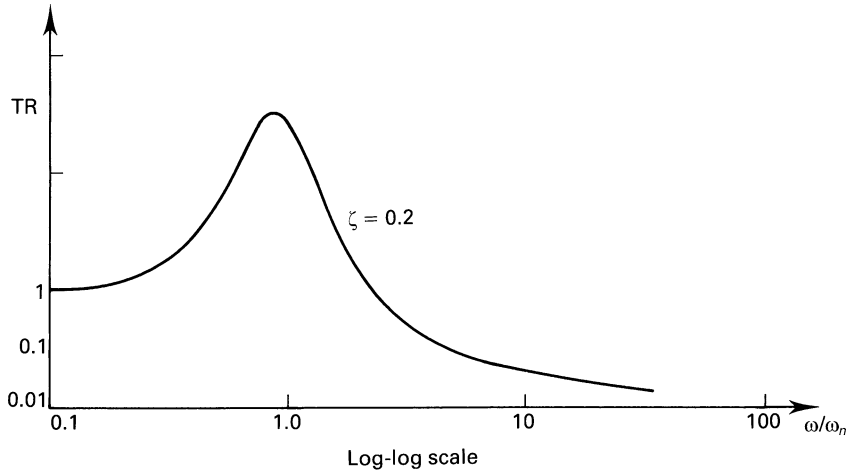

Figure 6.1 Transmissibility curve used in determining frequency values for vibration isolation.

The basic method of designing a vibration absorber is illustrated here by examining the simple case with no damping. To this end, consider the two-degree-of-freedom system of Figure 2.4 with $c_1 = c_2 = f_2 = 0$, $m_2 = m_a$, the absorber mass, $m_1 = m$, the primary mass, $k_1 = k$, the primary stiffness, and $k_2 = k_a$, the absorber spring constant. In addition, let $x_1 = x$, the displacement of the primary mass, and $x_2 = x_a$, the displacement of the absorber. Also, let the driving force $F_0 \sin(\omega t)$ be applied to the primary mass, m. The absorber is designed to the steady state response of this mass by choosing the values of m_a and k_a. Recall that the steady state response of a harmonically excited system is found by assuming a solution that is proportional to a harmonic term of the same frequency as the driving frequency.

From Equation (2.25) the equations of motion of the two-mass absorber system are

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{6.6}$$

Assuming that in the steady state the solution of Equation (6.6) will be of the form

$$\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t \tag{6.7}$$

and substituting into Equation (6.6) yields

$$\begin{bmatrix} k + k_a - m\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{6.8}$$

Solving for the magnitudes $\mathbf{X}$ and $\mathbf{X}_a$ yields

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{1}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \begin{bmatrix} (k_a - m_a\omega^2) & k_a \\ k_a & (k + k_a - m\omega^2) \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \tag{6.9}$$

or

$$X = \frac{(k_a - m_a\omega^2)F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{6.10}$$

and

$$X_a = \frac{k_a F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{6.11}$$

As can be seen by examining Equation (6.10), if k_a and m_a are chosen such that $k_a - m_a\omega^2 = 0$, i.e., such that $\sqrt{k_a/m_a} = \omega$, then the magnitude of the steady state response of the primary m is zero, i.e., $X = 0$. Hence, if the added absorber mass, m_a, is 'tuned' to the driving frequency ω, then the amplitude of the steady state vibration of the primary mass, X, is zero and the absorber mass effectively absorbs the energy in the system.

The addition of damping into the absorber-mass system provides two more parameters to be adjusted for improving the response of the mass m. However, with damping, the magnitude X cannot be made exactly zero. The next section illustrates methods for choosing the design parameters to make X as small as possible in the damped case.

6.3 OPTIMIZATION METHODS

Optimization methods (see, for instance, Gill, Murray, and Wright, 1981) can be used to obtain the 'best' choice of the physical parameters m_a, c_a, and k_a in the design of a vibration absorber or, for that matter, any degree-of-freedom vibration problem (Vakakis and Paipetis, 1986). The basic optimization problem is described in the following and then applied to the damped vibration absorber problem mentioned in the preceding section.

The general form for standard nonlinear programming problems is to minimize some scalar function of the vector of design variables $\mathbf{y}$, denoted by $J(\mathbf{y})$, subject to p inequality constraints and q equality constraints, denoted by

$$g_s(\mathbf{y}) < 0, \qquad s = 1, 2, \ldots, p \tag{6.12}$$

$$h_r(\mathbf{y}) = 0, \qquad r = 1, 2, \ldots, q \tag{6.13}$$

respectively. The function $J(\mathbf{y})$ is referred to as the *objective function*, or *cost function*. The process is an extension of the constrained minimization problems studied in beginning calculus.

There are many methods available to solve such optimization problems. The intention of this section is not to present these various methods of design optimization but rather to introduce the use of optimization techniques as a vibration design method. The reader should consult one or more of the many texts on optimization for details of the various methods.

A common method for solving optimization problems with equality constraints is to use the method of *Lagrange multipliers*. This method defines a new vector $\theta = [\theta_1\ \theta_2 \cdots \theta_q]^T$ called the vector of Lagrange multipliers (in optimization literature they are sometimes

denoted by λ_i), and the constraints are added directly to the objective function by using the scalar term $\theta^T\mathbf{h}$. The *new cost function* becomes $J' = J(\mathbf{y}) + \theta^T\mathbf{h}(\mathbf{y})$, which is then minimized as a function of y_i and θ_i. This is illustrated in the following example.

Example 6.3.1

Suppose it is desired to find the smallest value of the damping ratio ζ and the frequency ratio $r = \omega/\omega_n$ such that the transmissibility ratio is 0.1. The problem can be formulated as follows. Since TR = 0.1, then $TR^2 = 0.01$ or $(TR)^2 - 0.01 = 0$, which is the constraint $h_1(\mathbf{y})$ in this example. The vector $\mathbf{y}$ becomes $\mathbf{y} = [\zeta \quad r]^T$, and the vector θ is reduced to the scalar θ. The cost function J' then becomes

$$\begin{aligned} J' &= \zeta^2 + r^2 + \theta(TR^2 - 0.01) \\ &= \zeta^2 + r^2 + \theta[0.99 + 0.02r^2 - 0.01r^4 + 3.96\zeta^2 r^2] \end{aligned}$$

The necessary (but not sufficient) conditions for a minimum are that the first derivatives of the cost function with respect to the design variables must vanish. This yields

$$\begin{aligned} J'_\zeta &= 2\zeta + \theta(7.92\zeta r^2) = 0 \\ J'_r &= 2r + \theta[0.04r - 0.04r^3 + 7.92\zeta^2 r] = 0 \\ J'_\theta &= 0.99 + 0.02r^2 - 0.01r^4 + 3.96\zeta^2 r^2 = 0 \end{aligned}$$

where the subscripts of J' denote partial differentiation with respect to the given variable. These three nonlinear algebraic equations in the three unknowns ζ, r, and θ can be solved numerically to yield $\zeta = 0.037$, $r = 3.956$, and $\theta = -0.016$.

The question arises as to how to pick the objective function $J(\mathbf{y})$. The choice is arbitrary, but the function should be chosen to have a single global minimum. Hence, the choice of the quadratic form $(\zeta^2 + r^2)$ in the previous example. The following discussion on absorbers indicates how the choice of the cost function affects the result of the design optimization. The actual minimization process can follow several formulations; the results presented next follow Fox (1971).

Soom and Lee (1983) examined several possible choices of the cost function $J(\mathbf{y})$ for the absorber problem and provided a complete analysis of the absorber problem (for a two-degree-of-freedom system) given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_a \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} c_1 + c_a & -c_a \\ -c_a & c_a \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_a \end{bmatrix} + \begin{bmatrix} k_1 + k_a & -k_a \\ -k_a & k_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_a \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \cos \omega t \tag{6.14}$$

These equations are first nondimensionalized by defining the following new variables and constants:

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}, \qquad \zeta_1 = \frac{c_1}{2\sqrt{m_1 k_1}}$$
$$\omega = \frac{\omega}{\omega_1}, \qquad \zeta_2 = \frac{c_a}{2\sqrt{m_1 k_1}}$$
$$\tau = \omega_1, \qquad P = \frac{f}{k_1 L}, \text{ where } L \text{ is the static deflection of } x_1$$
$$\mu = \frac{m_a}{m_1}, \qquad z_1 = \frac{x_1}{L}$$
$$k = \frac{k_a}{k_1}, \qquad z_2 = \frac{x_a}{L}$$

Substitution of these into Equation (6.14) and dividing by $k_1 L$ yields the dimensionless equations

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} + \begin{bmatrix} 2(\zeta_1 + \zeta_2) & -2\zeta_2 \\ -2\zeta_2 & 2\zeta_2 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 1+k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix} \cos \omega\tau \tag{6.15}$$

where the overdots now indicate differentiation with respect to τ. As before, the steady state responses of the two masses are assumed to be of the form

$$z_1 = |A_1| \cos(\omega\tau + \phi_1)$$
$$z_2 = |A_2| \cos(\omega\tau + \phi_2) \tag{6.16}$$

Substitution of Equations (6.16) into Equation (6.15) and solving for the amplitudes $|A_1|$ and $|A_2|$ yields

$$|A_2| = \sqrt{(a/q)^2 + (b/q)^2}\ |A_1| \tag{6.17}$$

$$|A_1| = \frac{P}{\sqrt{(1 - \omega^2 - r/q)^2 + (2\zeta_1\omega + s/q)^2}} \tag{6.18}$$

where the constants a, b, q, r, and s are defined by

$$a = k^2 + 4\zeta_2^2\omega^2 - \mu k\omega^2$$
$$b = -2\zeta_2\mu\omega^3$$
$$q = (k - \mu\omega^2)^2 + 4\zeta_2^2\omega^2$$
$$r = \mu k^2\omega^2 - \mu^2 k\omega^4 + 4\zeta_2^2\omega^4$$
$$s = 2\zeta_2\mu^2\omega^5$$

Note that Equations (6.17) and (6.18) are similar in form to Equations (6.10) and (6.11) for the undamped case. However, the tuning condition is no longer obvious, and there are

many possible design choices to make. This marks the difference between an absorber with damping and one without. The damped case is, of course, a much more realistic model of the absorber dynamics.

Optimization methods are used to make the best design choice among all the physical parameters. The optimization is carried out using the design variable defined by the design vector. Here, α is the tuning condition defined by

$$\alpha = \sqrt{\frac{k}{\mu}}$$

and ζ_2' is a damping ratio defined by

$$\zeta_2' = \frac{\zeta_2}{\sqrt{\mu k}}$$

The quantity ζ_2' is the damping ratio of the 'added-on' absorber system of mass m_a. The tuning condition α is the ratio of the two undamped natural frequencies of the two masses.

The designer has the choice of making up objective functions. In this sense, the optimization produces an arbitrary best design. Choosing the objective function is the art of optimal design. However, several cost or objective functions can be used, and the results of each optimization compared. Soom and Lee (1983) considered several different objective functions:

J_1 = the maximum value of $|A_1|$, the magnitude of the displacement response in the frequency domain;
$J_2 = \sum(|A_1| - 1)^2$ for frequencies where $|A_1| > 1$ and where the sum runs over a number of discrete points on the displacement response curves of mass m_1;
J_3 = maximum $(\omega|A_1|)$, the maximum velocity of m_1;
$J_4 = \sum |A_1|^2$, the mean squared displacement response;
$J_5 = \sum(\omega|A_1|)^2$, the mean squared velocity response.

These objective functions were all formed by taking 100 equally spaced points in the frequency range from $\omega = 0$ to $\omega = 2$. The only constraints imposed were that the stiffness and damping coefficients be positive.

Solutions to the various optimizations yields the following interesting design conclusions:

1. From minimizing J_1, the plot of J_1 versus ζ_1 for various mass ratios is given in Figure 6.2 and shows that one would not consider using a dynamic absorber for a system with a damping ratio much greater than $\zeta_1 = 0.2$. The plots clearly show that not much reduction in magnitude can be expected for systems with large damping in the main system.
2. For large values of damping, $\zeta_1 = 0.3$, the different objective functions lead to different amounts of damping in the absorber mass, ζ_2, and the tuning ratio, α. Thus, the choice of the objective function changes the optimum point. This is illustrated in Figure 6.3.
3. The peak, or maximum value, of $z_1(t)$ at resonance also varies somewhat, depending on the choice of the cost function. The lowest reduction in amplitude occurs with objective function J_1, as expected, which is 30% lower than the value for J_4.

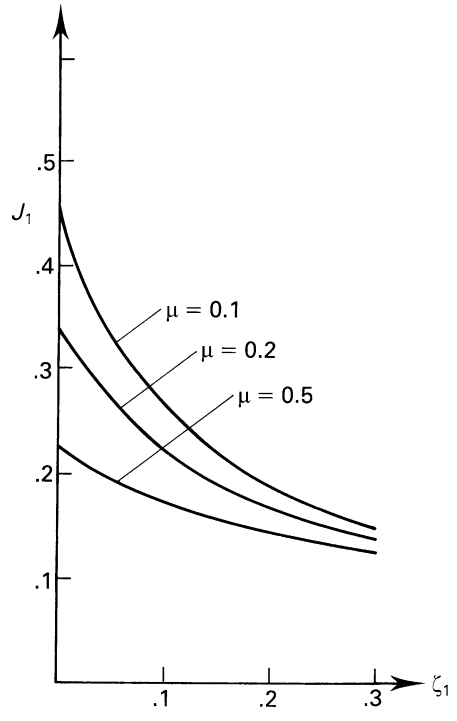

Figure 6.2 Plot of the cost function J_1, versus the damping ratio ζ_1 for various mass ratios.

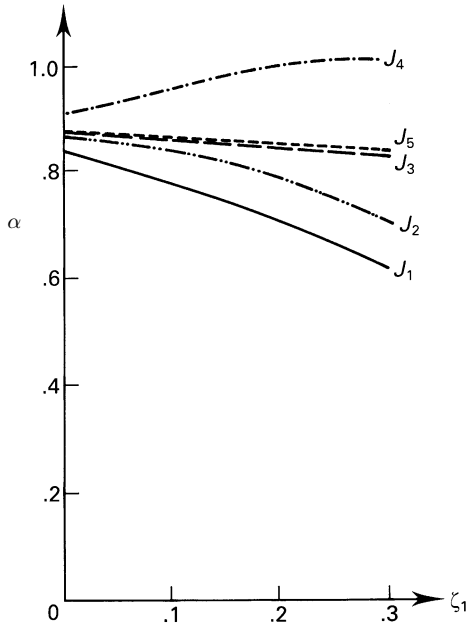

Figure 6.3 Tuning parameter α versus the first mode damping ratio ζ_1 for each cost function J, indicating a broad range of optimal values of α.

It was concluded from this study that, while optimal design can certainly improve the performance of a device, a certain amount of ambiguity exists in an optimal design based on the choice of the cost function. Thus, the cost function must be chosen with some understanding about the design objectives as well as physical insight.

6.4 DAMPING DESIGN

This section illustrates a method of adjusting the individual mass, damping, and stiffness parameters of a structure in order to produce a desired damping ratio. Often in the design of systems, damping is introduced to achieve a reduced level of vibrations, or to perform *vibration suppression*. Consider a symmetric system of the form

$$M\ddot{\mathbf{x}} + D\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{0} \tag{6.19}$$

where M, D, and K are the usual symmetric, positive definite mass, damping, and stiffness matrices, to be adjusted so that the modal damping ratios, ζ_i, have desired values. This in turn provides insight into how to adjust or design the individual elements m_i, c_i, and k_i such that the desired damping ratios are achieved.

Often in the design of a mechanical part, the damping in the structure is specified in terms of either a value for the loss factor or a percentage of critical damping, i.e., the damping ratio. This is mainly true because these are easily understood concepts for a single-degree-of-freedom model of a system. However, in many cases, of course, the behavior of a given structure may not be satisfactorily modeled by a single modal parameter. Hence, the question of how to interpret the damping ratio for a multiple-degree-of-freedom system such as the symmetric positive definite system of Equation (6.19) arises.

An n-degree-of-freedom system has n damping ratios, ζ_i. These damping ratios are, in fact, defined by Equation (5.40) for the normal mode case (i.e., under the assumption that $DM^{-1}K$ is symmetric). Recall that, if the equations of motion decouple, then each mode has a damping ratio ζ_i defined by

$$\zeta_i = \frac{\lambda_i(D)}{2\omega_i} \tag{6.20}$$

where ω_i is the ith undamped natural frequency of the system and $\lambda_i(D)$ denotes the ith eigenvalue of matrix D.

To formalize this definition and to examine the nonnormal mode case ($DM^{-1}K \neq KM^{-1}D$), the *damping ratio matrix*, denoted by Z, is defined in terms of the critical damping matrix D_{cr} of Section 3.6. The damping ratio matrix is defined by

$$Z = D_{cr}^{-1/2}\tilde{D}D_{cr}^{-1/2} \tag{6.21}$$

where $\tilde{D}$ is the mass normalized damping matrix of the structure. Furthermore, define the matrix Z' to be the diagonal matrix of eigenvalues of matrix Z, i.e.,

$$Z' = \mathrm{diag}\,[\lambda_i(Z)] = \mathrm{diag}\,[\zeta_i^*] \tag{6.22}$$

Here, the ζ_i^* are damping ratios in that, if $0 < \zeta_i^* < 1$, the system is underdamped. Note, of course, that, if $DM^{-1}K = KM^{-1}D$, then $Z = Z'$.

By following the definitions of underdamped and critically damped systems of Section 3.6, it can easily be shown that the definiteness of the matrix $I - Z'$ determines whether a given system oscillates.

Example 6.4.1

As an example, consider Equation (6.19) with the following numerical values for the coefficient matrices:

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 10 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix}$$

In this case, D_{cr} is calculated to be

$$D_{cr} = 2\tilde{K}^{1/2} = \begin{bmatrix} 4.4272 & -0.6325 \\ -0.6325 & 1.8974 \end{bmatrix}$$

where $\tilde{K} = M^{-1/2}KM^{-1/2}$.

From Equation (6.21) the damping ratio matrix becomes

$$Z = \begin{bmatrix} 0.3592 & -0.2205 \\ -0.2205 & 0.4660 \end{bmatrix}$$

It is clear that the matrix $[I - Z]$ is positive definite, so that each mode in this case should be underdamped. That is

$$(I - Z) = \begin{bmatrix} 0.3592 & 0.2205 \\ 0.2205 & 0.5340 \end{bmatrix}$$

so that the principle minors become $0.3592 > 0$ and $\det(I - Z) = 0.1432 > 0$. Hence, the matrix $[I - Z]$ is positive definite.

Calculating the eigenvalues of the matrix Z yields $\lambda_1(Z) = 0.7906$ and $\lambda_2(Z) = 0.3162$, so that $0 < \lambda_1(Z) < 1$ and $0 < \lambda_2(Z) < 1$, again predicting that the system is underdamped in each mode, since each ζ_i^* is between 0 and 1.

To illustrate the validity of these results for this example, the latent roots of the system can be calculated. They are

$$\lambda_{1,2} = -0.337 \pm 0.8326\, j, \qquad \lambda_{3,4} = -1.66 \pm 1.481\, j$$

where $j = \sqrt{-1}$. Thus, each mode is, in fact, underdamped as predicted by both the damping ratio matrix Z and the modal damping ratio matrix Z'.

It would be a useful design technique to be able to use this defined damping ratio matrix to assign damping ratios to each mode and back-calculate from the matrix Z' to obtain the required damping matrix D. Unfortunately, although the eigenvalues of matrix Z' specify the qualitative behavior of the system, they do not correspond to the actual modal damping

ratios unless the matrix $DM^{-1}K$ is symmetric. However, if the damping is proportional, then Equation (6.21) can be used to calculate the desired damping matrix in terms of the specified damping ratios, i.e.,

$$\tilde{D} = D_{cr}^{1/2} Z D_{cr}^{1/2}$$

This damping matrix would then yield a system with modal damping ratios exactly as specified.

This section is a prelude to active control where one specifies the desired eigenvalues of a system (i.e., damping ratios and natural frequencies) and then computes a control law to achieve these values. The pole placement method introduced in Section 6.6 is such a method. The hardware concerns for achieving the desired damping rates are discussed in Nashif, Jones, and Henderson (1985).

6.5 DESIGN SENSITIVITY AND REDESIGN

Design sensitivity analysis usually refers to the study of the effect of parameter changes on the result of an optimization procedure or an eigenvalue–eigenvector computation. For instance, in the optimization procedure presented in Section 6.3, the nonabsorber damping ratio ζ_1 was not included as a parameter in the optimization. How the resulting optimum changes as ζ_1 changes is the topic of sensitivity analysis for the absorber problem. The eigenvalue and eigenvector perturbation analysis of Section 3.7 is an example of design sensitivity for the eigenvalue problem. This can, on the other hand, be interpreted as the *redesign* problem, which poses the question as to how much the eigenvalue and eigenvector solution changes as a specified physical parameter changes because of some other design process. In particular, if a design change causes a system parameter to change, the eigensolution can be computed without having to recalculate the entire eigenvalue/eigenvector set. This is also referred to as a *reanalysis* procedure and sometimes falls under the heading of *structural modification*. These methods are all fundamentally similar to the perturbation methods introduced in Section 3.7. This section develops the equations for discussing the sensitivity of natural frequencies and mode shapes for conservative systems.

The motivation for studying such methods comes from examining the large-order dynamical systems often used in current vibration technology. Making changes in large systems is part of the design process. However, large amounts of computer time are required to find the solution of the redesigned system. It makes sense, then, to develop efficient methods to update existing solutions when small design changes are made in order to avoid a complete reanalysis. In addition, this approach can provide insight into the design process.

Several approaches are available for performing a sensitivity analysis. The one presented here is based on parameterizing the eigenvalue problem. Consider a conservative n-degree-of-freedom system defined by

$$M(\alpha)\ddot{\mathbf{q}}(t) + K(\alpha)\mathbf{q}(t) = \mathbf{0} \tag{6.23}$$

where the dependence of the coefficient matrices on the design parameter α is indicated. The parameter α is considered to represent a change in the matrix M and/or the matrix K. The related eigenvalue problem is

$$M^{-1}(\alpha)K(\alpha)\mathbf{u}_i(\alpha) = \lambda_i(\alpha)\mathbf{u}_i(\alpha) \tag{6.24}$$

Here, the eigenvalue $\lambda_i(\alpha)$ and the eigenvector $\mathbf{u}_i(\alpha)$ will also depend on the parameter α. The mathematical dependence is discussed in detail by Whitesell (1980). It is assumed that the dependence is such that M, K, λ_i, and $\mathbf{u}_i$ are all twice differentiable with respect to the parameter α.

Proceeding, if $\mathbf{u}_i$ is normalized with respect to the mass matrix, differentiation of Equation (6.24) with respect to the parameter α yields

$$\frac{d}{d\alpha}(\lambda_i) = \mathbf{u}_i^T \left[\frac{d}{d\alpha}(K) - \lambda_i \frac{d}{d\alpha}(M) \right] \mathbf{u}_i \tag{6.25}$$

Here, the dependence of α has been suppressed for notational convenience. The second derivative of λ_i can also be calculated as

$$\begin{aligned} \frac{d^2}{d\alpha^2}\lambda_i = {} & 2\mathbf{u}_i'^T \left[\frac{d}{d\alpha}(K) - \lambda_i \frac{d}{d\alpha}(M) \right] \mathbf{u}_i' \\ & + \mathbf{u}_i^T \left[\frac{d^2}{d\alpha^2}(K) - \frac{d}{d\alpha}(\lambda_i)\frac{d}{d\alpha}(M) - \lambda_i \frac{d^2}{d\alpha^2}(M) \right] \mathbf{u}_i \end{aligned} \tag{6.26}$$

The notation $\mathbf{u}'$ denotes the derivative of the eigenvector with respect to α. The expression for the second derivative of λ_i requires the existence and computation of the derivative of the corresponding eigenvector. For the special case where M is a constant, and with some manipulation (see Whitesell, 1980), the eigenvector derivative can be calculated from the related problem for the eigenvector $\mathbf{v}_i$ from the formula

$$\frac{d}{d\alpha}(\mathbf{v}_i) = \sum_{k=1}^{n} c_k(\mathrm{i}, \alpha)\mathbf{v}_k \tag{6.27}$$

where the vectors $\mathbf{v}_k$ are related to $\mathbf{u}_k$ by the mass transformation $\mathbf{v}_k = M^{1/2}\mathbf{u}_k$. The coefficients $c_k(\mathrm{i}, \alpha)$ in this expansion are given by

$$c_k(\mathrm{i}, \alpha) = \begin{cases} 0 & i = k \\ \dfrac{1}{\lambda_i - \lambda_k}\mathbf{u}_k^T \dfrac{dA}{d\alpha}\mathbf{u}_i, & i \neq k \end{cases} \tag{6.28}$$

where the matrix A is the symmetric matrix $M^{-1/2}KM^{-1/2}$ depending on α.

Equations (6.25) and (6.27) yield the sensitivity of the eigenvalues and eigenvectors of a conservative system to changes in the stiffness matrix. More general and computationally efficient methods for computing these sensitivities are available in the literature. Adhikari and Friswell (2001) give formulae for damped systems and reference to additional methods.

Example 6.5.1

Consider the system discussed previously in example 3.3.2. Here, take $M = I$, and $\tilde{K}$ becomes

$$\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} = K$$

The eigenvalues of the matrix are $\lambda_{1,2} = 2, 4$, and the normalized eigenvectors are $\mathbf{u}_1 = \mathbf{v}_1 = (1/\sqrt{2})\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $\mathbf{u}_2 = \mathbf{v}_2 = (1/\sqrt{2})\begin{bmatrix}-1 & 1\end{bmatrix}^T$. It is desired to compute the sensitivity of the natural frequencies and mode shapes of this system as a result of a parameter change in the stiffness of the spring attached to ground. To this end, suppose the new design results in a new stiffness matrix of the form

$$K(\alpha) = \begin{bmatrix} 3+\alpha & -1 \\ -1 & 3 \end{bmatrix}$$

Then

$$\frac{d}{d\alpha}(M) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \frac{d}{d\alpha}(K) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Following Equations (6.25) and (6.27), the derivatives of the eigenvalues and eigenvectors become

$$\frac{d\lambda_1}{d\alpha} = 0.5, \qquad \frac{d\lambda_2}{d\alpha} = 0.5, \qquad \frac{d\mathbf{u}_1}{d\alpha} = \frac{1}{4\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \frac{d\mathbf{u}_2}{d\alpha} = \frac{-1}{4\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These quantities are an indication of the sensitivity of the eigensolution to changes in the matrix K. To see this, substitute the preceding expressions into the expansions for $\lambda(\alpha)$ and $\mathbf{u}_i(\alpha)$ given by Equations (3.98) and (3.99). This yields

$$\lambda_1(\alpha) = 2 + 0.5\alpha, \qquad \lambda_2(\alpha) = 4 + 0.5\alpha$$
$$\mathbf{u}_1(\alpha) = 0.707\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.177\alpha\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2(\alpha) = 0.707\begin{bmatrix} -1 \\ 1 \end{bmatrix} - 0.177\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This last set of expressions allows the eigenvalues and eigenvectors to be evaluated for any given parameter change α without having to resolve the eigenvalue problem. These formulae constitute an approximate reanalysis of the system.

It is interesting to note this sensitivity in terms of a percentage. Define the percentage change in λ_1 by

$$\frac{\lambda_1(\alpha) - \lambda_1}{\lambda_1} 100\% = \frac{(2 + 0.5\alpha) - 2}{2} 100\% = (25\%)\alpha$$

If the change in the system is small, say $\alpha = 0.1$, then the eigenvalue λ_1 changes by only 2.5%, and the eigenvalue λ_2 changes by 1.25%. On the other hand, the change in the elements of the eigenvector $\boldsymbol{u}_2$ is 2.5%. Hence, in this case the eigenvector is more sensitive to parameter changes than the eigenvalue is.

By computing higher-order derivatives of λ_i and $\mathbf{u}_i$, more terms of the expansion can be used, and greater accuracy in predicting the eigensolution of the new system results. By using the appropriate matrix computations, the subsequent evaluations of the eigenvalues and eigenvectors as the design is modified can be carried out with substantially less computational effort (reportedly of the order of n^2 multiplications). The sort of calculation provided by eigenvalue and eigenvector derivatives can provide an indication of how changes to an initial design will affect the response of the system. In the example, the shift in value of the first spring is translated into a percentage change in the eigenvalues and hence in the natural frequencies. If the design of the system is concerned with avoiding resonance, then knowing how the frequencies shift with stiffness is critical.

6.6 PASSIVE AND ACTIVE CONTROL

In the redesign approach discussed in the previous section, the added structural modification α can be thought of as a passive control. If α represents added stiffness chosen to improve the vibrational response of the system, then it can be thought of as a passive control procedure. As mentioned in Section 1.8, passive control is distinguished from active control by the use of added power or energy in the form of an actuator, required in active control.

The material on isolators and absorbers of Section 6.2 represents two possible methods of passive control. Indeed, the most common passive control device is the vibration absorber. Much of the other work in passive control consists of added layers of damping material applied to various structures to increase the damping ratios of troublesome modes. Adding mass and changing stiffness values are also methods of passive control used to adjust a frequency away from resonance. Damping treatments increase the rate of decay of vibrations, so they are often more popular for vibration suppression.

Active control methods have been introduced in Sections 1.8, 2.3, and 4.10. Here we examine active control as a design method for improving the response of a vibrating system. This section introduces the method of *eigenvalue placement* (often called *pole placement*), which is useful in improving the free response of a vibrating system by shifting natural frequencies and damping ratios to desired values. The method of Section 6.4 is a primitive version of placing the eigenvalues by adjusting the damping matrix. The next chapter is devoted to formalizing and expanding this method (Section 7.3), as well as introducing some of the other techniques of control theory.

There are many different methods of approaching the eigenvalue placement problem. Indeed, it is the topic of ongoing research. The approach taken here is simple. The characteristic equation of the structure is written. Then a feedback law is introduced with undetermined gain coefficients of the form given by Equations (4.24) through (4.26). The characteristic equation of the closed-loop system is then written and compared with the characteristic equation of the open-loop system. Equating coefficients of the powers of λ in the two characteristic equations yields algebraic equations in the gain parameters, which are then solved. This yields the control law, which causes the system to have the desired eigenvalues. The procedure is illustrated in the following example.

Example 6.6.1

Consider the undamped conservative system of example 2.4.4 with $M = I$, $D = 0$, $k_1 = 2$, and $k_2 = 1$. The characteristic equation of the system becomes

$$\lambda^2 - 4\lambda + 2 = 0$$

This has roots $\lambda_1 = 2 - \sqrt{2}$ and $\lambda_2 = 2 + \sqrt{2}$. The natural frequencies of the system are then $\sqrt{2 - \sqrt{2}}$ and $\sqrt{2 + \sqrt{2}}$. Suppose now that it is desired to raise the natural frequencies of this system to be $\sqrt{2}$ and $\sqrt{3}$ respectively. Furthermore, assume that the values of k_i and m_i cannot be adjusted, i.e., that passive control is not a design option in this case.

First, consider the control and observation matrices of Section 4.10 and the solution to Problem 4.7. The obvious choice would be to measure the positions $q_1(t)$ and $q_2(t)$, so that $C_v = 0$ and $C_p = I$, and apply forces proportional to their displacements, so that

$$G_f = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

with the actuators placed at x_1 and x_2 respectively. In this case, the matrix B_f becomes $B_f = I$. Then, the closed-loop system of Equation (4.27) has the characteristic equation

$$\lambda^2 - (4 + g_1 + g_2)\lambda + 2 + g_1 + 3g_2 + g_1 g_2 = 0 \tag{6.29}$$

If it is desired that the natural frequencies of the closed-loop system be $\sqrt{2}$ and $\sqrt{3}$, then the eigenvalues must be changed to 2 and 3, which means the desired characteristic equation is

$$(\lambda - 3)(\lambda - 2) = \lambda^2 - 5\lambda + 6 = 0 \tag{6.30}$$

By comparing the coefficients of λ and λ^0 (constant) terms of Equations (6.29) and (6.30), it can be seen that the gains g_1 and g_2 must satisfy

$$5 = (4 + g_1 + g_2)$$
$$6 = 2 + g_1 + 3g_2 + g_1 g_2$$

which has no real solutions.

From Equation (6.29) it is apparent that, in order to achieve the goal of placing the eigenvalues, and hence the natural frequencies, the gains must appear in some different order in the coefficients of Equation (6.29). This condition can be met by reexamining the matrix B_f. In fact, if B_f is chosen to be

$$B_f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

the characteristic equation for the closed-loop system becomes

$$\lambda^2 - (4 + g_2)\lambda + 2 + 3g_2 + g_1 = 0 \tag{6.31}$$

Comparison of the coefficients of λ in Equations (6.30) and (6.31) yields values for the gains of $g_1 = 1$ and $g_2 = 1$.

The eigenvalues with these gains can be easily computed as a check to see that the scheme works. They are in fact $\lambda = 2$ and $\lambda = 3$, resulting in the desired natural frequencies.

As illustrated by the preceding example, the procedure is easy to calculate but does not always yield real values or even realistic values of the gains. The way in which G_f, B_f, and C_p are chosen and, in fact, whether or not such matrices even exist are topics of the next chapter. Note that the ability to choose these matrices is the result of the use of feedback and illustrates the versatility gained by using active control as against passive control. In passive control, g_1 and g_2 have to correspond to changes in mass or stiffness. In active control,

g_1 and g_2 are often electronic settings and hence are easily adjustable within certain bounds (but at other costs).

The use of pole placement assumes that the designer understands, or knows, what eigenvalues are desirable. This knowledge comes from realizing the effect that damping ratios and frequencies, and hence the eigenvalues, have on the system response. Often these are interpreted from, or even stated in terms of, design specifications. This is the topic of the next section.

6.7 DESIGN SPECIFICATIONS

The actual design of a mechanism starts and ends with a list of performance objectives or criteria. These qualitative criteria are eventually stated in terms of quantitative design specifications. Sample specifications form the topic of this section. Three performance criteria are considered in this section: speed of response, relative stability, and resonance.

The *speed of response* addresses the length of time required before steady state is reached. In classical control this is measured in terms of rise time, settling time, and bandwidth, as discussed in Section 1.4. In vibration analysis, speed of response is measured in terms of a decay rate or logarithmic decrement. Speed of response essentially indicates the length of time for which a structure or machine experiences transient vibrations. Hence, it is the time elapsed before the steady state response dominates. If just a single output is of concern, then the definitions of these quantities for multiple-degree-of-freedom systems are similar to those for the single-degree-of-freedom systems of Chapter 1.

For instance, for an n-degree-of-freedom system with position vector $\mathbf{q} = [q_1(t) \quad q_2(t) \ldots q_n(t)]^T$, if one force is applied, say at position m_1, and one displacement is of concern, say $q_8(t)$, then specifications for the speed of response of $q_8(t)$ can be defined as follows. The *settling time* is the time required for the response $q_8(t)$ to remain within $\pm\alpha$ percent of the steady state value of $q_8(t)$. Here, α is usually 2, 3, or 5. The *rise time* is the time required for the response $q_8(t)$ to go from 10 to 90% of its steady state value. The *log decrement* discussed in Equation (1.35) can be used as a measure of the decay rate of the system. All these specifications pertain to the transient response of a single-input, single-output (SISO) configuration.

On the other hand, if interest is in the total response of the system, i.e., the vector $\mathbf{q}$, then the response bounds of Section 5.6 yield a method of quantifying the decay rate for the system. In particular, the constant β, called a decay rate, may be specified such that

$$\|\mathbf{q}(t)\| < M\mathrm{e}^{-\beta t}$$

is satisfied for all $t > 0$. This can also be specified in terms of the *time constant* defined by the time, t, required for $\beta t = 1$. Thus, the time constant is $t = 1/\beta$.

Example 6.7.1

Consider the system of example 5.2.1. The response norm of the position is the first component of the vector $\mathbf{x}(t)$ so that $q(t) = (1 - e^{-t})e^{-t}$ and its norm is $|e^{-t} - e^{-2t}| < |e^{-t}| = e^{-t}$. Hence $\beta = 1$, and the decay rate is also 1.

Some situations may demand that the relative stability of a system be quantified. In particular, requiring that a system be designed to be stable or asymptotically stable may not be enough. This is especially true if some of the parameters in the system may change over a period of time or change owing to manufacturing tolerances or if the system is under active control. Often the concept of a stability margin is used to quantify relative stability.

In Chapter 4 several systems are illustrated that can become unstable as one or more parameters in the system change. For systems in which a single parameter can be used to characterize the stability behavior of the system, the *stability margin*, denoted by *sm*, of the system can be defined as the ratio of the maximum stable value of the parameter to the actual value for a given design configuration. The following example illustrates this concept.

Example 6.7.2

Consider the system defined in example 4.6.1 with $\gamma = 1$, $c_1 = 6$, and $c_2 = 2$ and calculate the stability margin of the system as the parameter changes. Here, η is being considered as a design parameter. As the design parameter η increases, the system approaches an unstable state. Suppose the operating value of η, denoted by η_{op}, is 0.1. Then, the stiffness matrix becomes semidefinite for $\eta = 1$ and indefinite for $\eta > 1$, and the maximum stable value of η is $\eta_{max} = 1$. Hence, the stability margin is

$$sm = \frac{\eta_{max}}{\eta_{op}} = \frac{1}{0.1} = 10$$

If the design of the structure is such that $\eta_{op} = 0.5$, then $sm = 2$. Thus, all other factors being equal, the design with $\eta_{op} = 0.1$ is 'more stable' than the same design with $\eta_{op} = 0.5$, because $\eta_{op} = 0.1$ has a larger stability margin.

The resonance properties, or modal properties, of a system are obvious design criteria in the sense that in most circumstances resonance is to be avoided. The natural frequencies, mode shapes, and modal damping ratios are often specified in design work. Methods of designing a system to have particular modal properties have been discussed briefly in this chapter in terms of passive and active control. Since these specifications can be related to the eigenvalue problem of the system, the question of designing a system to have specified modal properties is answered by the pole placement methods and eigenstructure assignment methods of control theory discussed in Section 7.3.

6.8 MODEL REDUCTION

A difficulty with many design and control methods is that they work best for systems with a small number of degrees of freedom. Unfortunately, many interesting problems have a large number of degrees of freedom. One approach to this dilemma is to reduce the size of the original model by essentially removing those parts of the model that affect its dynamic response of interest the least. This process is called *model reduction*, or *reduced-order modeling*.

Quite often the mass matrix of a system may be singular or nearly singular owing to some elements being much smaller than others. In fact, in the case of finite element modeling (discussed in Section 13.3), the mass matrix may contain zeros along a portion of the diagonal (called an *inconsistent mass matrix*). Coordinates associated with zero, or relatively small mass, are likely candidates for being removed from the model.

Another set of coordinates that are likely choices for removal from the model are those that do not respond when the structure is excited. Stated another way, some coordinates may have more significant responses than others. The distinction between significant and insignificant coordinates leads to a convenient formulation of the model reduction problem due to Guyan (1965).

Consider the undamped forced vibration problem of Equation (5.22) and partition the mass and stiffness matrices according to significant displacements, denoted by $\mathbf{q}_1$, and insignificant displacements, denoted by $\mathbf{q}_2$. This yields

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} \tag{6.32}$$

Note that the coordinates have been rearranged so that those having the least significant displacements associated with them appear last in the partitioned displacement vector $\mathbf{q}^T = \begin{bmatrix} \mathbf{q}_1^T & \mathbf{q}_2^T \end{bmatrix}$.

Next consider the potential energy of the system defined by the scalar $V_e = (1/2)\mathbf{q}^T K \mathbf{q}$ or, in partitioned form,

$$V_e = \frac{1}{2} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix}^T \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \tag{6.33}$$

Likewise, the kinetic energy of the system can be written as the scalar $T_e = (1/2)\dot{\mathbf{q}}^T M \dot{\mathbf{q}}$, which becomes

$$T_e = \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix}^T \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_1 \\ \dot{\mathbf{q}}_2 \end{bmatrix} \tag{6.34}$$

in partitioned form. Since each coordinate $\mathbf{q}_i$ is acted upon by a force $\mathbf{f}_i$, the condition that there is no force in the direction of the insignificant coordinates, $\mathbf{q}_2$, requires that $\mathbf{f}_2 = \mathbf{0}$ and that $\partial V_e / \partial \mathbf{q}_2 = 0$. This yields

$$\frac{\partial}{\partial \mathbf{q}_2} (\mathbf{q}_1^T K_{11} \mathbf{q}_1 + \mathbf{q}_1^T K_{12} \mathbf{q}_2 + \mathbf{q}_2^T K_{21} \mathbf{q}_1 + \mathbf{q}_2^T K_{22} \mathbf{q}_2) = 0 \tag{6.35}$$

Solving Equation (6.35) yields a constraint relation between $\mathbf{q}_1$ and $\mathbf{q}_2$ which (since $K_{12} = K_{21}^T$) is as follows:

$$\boldsymbol{q}_2 = -K_{22}^{-1} K_{21} \mathbf{q}_1 \tag{6.36}$$

This last expression suggests a coordinate transformation (which is *not* a similarity transformation) from the full coordinate system $\mathbf{q}$ to the reduced coordinate system $\mathbf{q}_1$. If the transformation matrix P is defined by

$$P = \begin{bmatrix} I \\ -K_{22}^{-1} \quad K_{21} \end{bmatrix} \tag{6.37}$$

then, if $\mathbf{q} = P\mathbf{q}_1$ is substituted into Equation (6.32) and this expression is premultiplied by P^T, a new reduced-order system of the form

$$P^T MP\ddot{\mathbf{q}}_1 + P^T KP\mathbf{q}_1 = P^T \mathbf{f}_1 \tag{6.38}$$

results. The vector $P^{\mathrm{T}}\mathbf{f}_1$ now has the dimension of $\mathbf{q}_1$. Equation (6.38) represents the reduced-order form of Equation (6.32), where

$$P^T MP = M_{11} - K_{21}^T K_{22}^{-1} M_{21} - M_{12} K_{22}^{-1} K_{21} + K_{21}^T K_{22}^{-1} M_{22} K_{22}^{-1} K_{21} \tag{6.39}$$

and

$$P^T KP = K_{11} - K_{12} K_{22}^{-1} K_{21} \tag{6.40}$$

These last expressions are commonly used to reduce the order of vibration problems in a consistent manner in the case where some of the coordinates (represented by $\mathbf{q}_2$) are thought to be inactive in the system response. This can greatly simplify design and analysis problems in some cases.

If some of the masses in the system are negligible or zero, then the preceding formulae can be used to reduce the order of the vibration problem by setting $M_{22} = 0$ in Equation (6.39). This is essentially the method referred to as *mass condensation* (used in finite element analysis).

Example 6.8.1

Consider a four-degree-of-freedom system with the mass matrix

$$M = \frac{1}{420}\begin{bmatrix} 312 & 54 & 0 & -13 \\ 54 & 156 & 12 & -22 \\ 0 & 13 & 8 & -3 \\ -13 & -22 & -3 & 4 \end{bmatrix}$$

and the stiffness matrix

$$K = \begin{bmatrix} 24 & -12 & 0 & 6 \\ -12 & 12 & -6 & -6 \\ 0 & -6 & 2 & 4 \\ 6 & -6 & 4 & 4 \end{bmatrix}$$

Remove the effect of the last two coordinates. The submatrices of Equation (6.32) are easily identified:

$$M_{11} = \frac{1}{420}\begin{bmatrix} 312 & 54 \\ 54 & 156 \end{bmatrix}, \qquad M_{12} = \frac{1}{420}\begin{bmatrix} 0 & -13 \\ 13 & -22 \end{bmatrix} = M_{21}^T$$

$$M_{22} = \frac{1}{420}\begin{bmatrix} 8 & -3 \\ -3 & 4 \end{bmatrix}, \qquad K_{22} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

$$K_{11} = \begin{bmatrix} 24 & -12 \\ -12 & 12 \end{bmatrix}, \qquad K_{12} = \begin{bmatrix} 0 & 6 \\ -6 & -6 \end{bmatrix} = K_{21}^T$$

Using Equations (6.39) and (6.40) yields

$$P^{\mathrm{T}}MP = \begin{bmatrix} 1.021 & 0.198 \\ 0.198 & 0.236 \end{bmatrix}$$
$$P^{\mathrm{T}}KP = \begin{bmatrix} 9 & 3 \\ 3 & 3 \end{bmatrix}$$

These last two matrices form the resulting reduced-order model of the structure.

It is interesting to compare the eigenvalues (frequencies squared) of the full-order system with those of the reduced-order system, remembering that the transformation P used to perform the reduction is *not* a similarity transformation and subsequently does not preserve eigenvalues. The eigenvalues of the reduced system and full-order systems are

$$\lambda_1^{\mathrm{rom}} = 6.981, \qquad \lambda_2^{\mathrm{rom}} = 12.916$$
$$\lambda_1 = 6.965, \qquad \lambda_2 = 12.196$$
$$\lambda_3 = 230.934, \qquad \lambda_4 = 3.833 \times 10^3$$

where the superscript 'rom' refers to the eigenvalues of the reduced-order model. Note that in this case the reduced-order model captures the nature of the first two eigenvalues very well. This is not always the case because the matrix P defined in Guyan reduction, unlike the matrix P from modal analysis, does not preserve the system eigenvalues. More sophisticated model reduction algorithms exist, and some are presented in Section 7.7.

CHAPTER NOTES

A vast amount of literature is available on methods of vibration isolation and absorption. In particular, the books by Balandin, Bolotnik, and Pilkey (2001), Rivin (2003), and by Korenev and Reznikov (1993) should be consulted to augment the information of Section 6.2. The absorber optimization problem discussed in Section 6.3 is directly from the paper of Soom and Lee (1983). Haug and Arora (1976) provide an excellent account of optimal design methods. Example 6.4.1 is from Inman and Jiang (1987). More on the use of damping materials can be found in the book by Nashif, Jones, and Henderson (1985). The material of Section 6.5 comes from Whitesell (1980), which was motivated by the work of Fox and Kapoor (1968). More advanced approaches to eigensystem derivatives can be found in Adhikari and Friswell (2001). The pole placement approach to control can be found in almost any text on control, such as Kuo and Golnaraghi (2003), and is considered in more detail in Section 7.3. The section on design specification (Section 6.7), is an attempt to quantify some of the terminology often used by control and structure researchers in discussing the response of a system. An excellent treatment of reduction of order is given by Meirovitch (1980) and by Antoulas (2005) who provides a mathematical approach. A more advanced treatment of model reduction is given in Section 7.7 from the controls perspective. An excellent summary of model reduction methods, including

damped systems, is given by Qu (2004), which contains an extensive bibliography of model reduction papers.

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PROBLEMS

6.1 Calculate the value of the damping ratio required in a vibration isolation design to yield a transmissibility ratio of 0.1 given that the frequency ratio ω/ω_n is fixed at 6.

6.2 A single-degree-of-freedom system has a mass of 200 kg and is connected to its base by a simple spring. The system is being disturbed harmonically at 2 rad/s. Choose the spring stiffness so that the transmissibility ratio is less than 1.

6.3 A spring–mass system consisting of a 10 kg mass supported by a 2000 N m spring is driven harmonically by a force of 20 N at 4 rad/s. Design a vibration absorber for this system and compute the response of the absorber mass.

6.4 Find the minimum and maximum points of the function

$$J(\mathbf{y}) = y_1^3 + 3y_1y_2^2 - 3y_1^2 - 3y_2^2 + 4$$

Which points are actually minimum?

6.5 Calculate the minimum of the cost function

$$J(\mathbf{y}) = y_1 + 2y_2^2 + y_3^2 + y_4^2$$

subject to the equality constraints

$$h_1(\mathbf{y}) = y_1 + 3y_2 - y_3 + y_4 - 2 = 0$$
$$h_2(\mathbf{y}) = 2y_1 + y_2 - y_3 + 2y_4 - 2 = 0$$

6.6 Derive Equations (6.17) and (6.18).

6.7 Derive Equation (6.25). (Hint: first multiply Equation (6.24) by M, then differentiate.)

6.8 Consider Example 6.5.1. Calculate the change in the eigenvalues of this system if the mass, m_1, is changed an unknown amount rather than the stiffness (refer to example 3.3.2).

6.9 Consider example 2.4.4 with $M = I$, $c_1 = 2$, $c_2 = 1$, $k_1 = 4$, and $k_2 = 1$. Calculate a control law causing the closed-loop system to have eigenvalues $\lambda_{1,2} = -1 \pm j$ and $\lambda_{3,4} = -2 \pm j$, using the approach of Section 6.6.

6.10 By using the results of Section 3.6, show that the damping ratio matrix Z' determines whether the modes of a nonproportionally damped system are underdamped or critically damped.

6.11 Consider the system of example 6.4.1 with the damping matrix D set to zero. Calculate a new damping matrix D such that the new system has modal damping ratios $\zeta_1 = 0.1$ and $\zeta_2 = 0.01$.

6.12 Consider the cost function $J(\mathbf{y})$. The partial derivative J with respect to the elements of the vector $\mathbf{y}$ yield only necessary conditions for a minimum. The second-order condition and sufficient condition is that the matrix of second partial derivatives $[J_{ik}]$ be positive definite. Here, J_{ij} denotes the second partial derivative with respect to y_i and y_k. Apply this second condition to problem 6.2 and verify this result for that particular example.

6.13 Derive second-order conditions (see problem 6.10) for example 6.3.1 using a symbolic manipulation program.

6.14 Show that the reduction transformation P of Section 6.8 is not a similarity transformation. Are eigenvalues invariant under P?

6.15 Show that $K_{21} = K_{12}^T$ and hence derive Equation (6.36). In addition, show that $P^T MP$ and $P^T KP$ are both symmetric.

6.16 Calculate a reduced-order model of the following system by removing the last two coordinates:

$$\begin{bmatrix} 312 & 0 & 54 & -6.5 \\ 0 & 2 & 6.5 & -0.75 \\ 54 & 6.5 & 156 & -11 \\ -6.5 & -0.75 & -11 & 1 \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & \frac{1}{2} \\ -12 & -3 & 12 & -3 \\ 3 & \frac{1}{2} & -3 & 1 \end{bmatrix} \mathbf{q}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} f(t)$$

Then, calculate the natural frequencies of both the reduced-order and the full-order system (using a code such as MATLAB) and compare them. Also, plot the response of each system to the initial conditions $\mathbf{q} = [1 \quad 0 \quad 0 \quad 0]^T$ and $\dot{\mathbf{q}} = \mathbf{0}$ and compare the results.

6.17 The characteristic equation of a given system is

$$\lambda^3 + 5\lambda^2 + 6\lambda + \eta = 0$$

where η is a design parameter. Calculate the stability margin of this system for $\eta_{op} = 15.1$.

6.18 Compare the time response of the coordinates $q_1(t)$ and $q_2(t)$ of the full-order system of example 6.8.1 with the same coordinates in the reduced-order system for an initial displacement of $q_1(0) = 1$ and all other initial conditions set to zero.

6.19 Consider the system of example 6.4.1 with the damping matrix set to zero. Use the pole placement approach of Section 6.6 to compute a control law that will cause the closed-loop system to have frequencies of 2 and 3 rad/s.

6.20 Consider the vibration absorber designed in problem 6.3. Use numerical simulation to plot the response of the system to an initial 0.01 m displacement disturbance of m_1 (zero initial velocity). Discuss your results.

7
Control of Vibrations

7.1 INTRODUCTION

This chapter formalizes the aspects of control theory introduced in previous chapters and applies the theory to vibration suppression of structures. This topic is usually called *structural control* and has become increasingly important, as the design of mechanisms and structures has become more precise and less tolerant of transient vibrations. Many structures, such as tall buildings, robotic manipulator arms, and flexible spacecraft, have been designed using active vibration suppression as part of the total design. Active control provides an important tool for the vibration engineer.

Control technology of linear systems is a mature discipline with many excellent texts and journals devoted to the topic. Control methods can be split into three categories: single-input, single-output frequency domain methods (classical control), state-space methods which allow multiple-input, multiple-output (MIMO) control (focused on time domain control), and modern control theory, which looks at MIMO control in the frequency domain. Like design methods, most classical control depends on being able to use low-order models of the structure (also called the plant). On the other hand, state-space control theory uses matrix theory that is compatible with the vector differential equation commonly used to describe the vibrations of structures. Hence, in this chapter, more emphasis is placed on time domain methods relying on matrix techniques. The concepts of frequency response function and other frequency domain topics common to classical control are, however, very useful in vibrations. In particular, Chapter 8, on modal testing, uses many frequency domain ideas to aid in measuring vibration properties of structures.

The words *structure* and *plant* are used interchangeably to describe the vibrating mechanical part or system of interest. The phrase *control system* refers to an actuator (or group of actuators), which is a force-generating device used to apply control forces to the structure, the sensors used to measure the response of the structure (also called the output), and the rule or algorithm that determines how the force is applied. The structure is often called the *open-loop system*, and the structure along with the control system is called the *closed-loop system*. The topic of control has been briefly introduced in Sections 1.8, 2.3, 4.10, and 6.6. The notation of these sections is summarized here as an introduction to the philosophy of active control.

Feedback control of a vibrating structure or machine requires measurements of the response (by using sensing transducers) and the application of a force to the system (by using force

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

transducers) on the basis of these measurements. The mathematical representation of the method of computing how the force is applied on the basis of the measurements is called the *control law*. As is often the case in modeling physical systems, there are a variety of mathematical representations of feedback control systems. Equation (2.17) represents a method of modeling the use of control forces proportional to position (denoted by $K_p\mathbf{q}$) and velocity (denoted by $K_v\dot{\mathbf{q}}$) used to shape the response of the structure. In the notation of Section 2.3, the closed-loop system is modeled by

$$M\ddot{\mathbf{q}} + (D+G)\dot{\mathbf{q}} + (K+H)\mathbf{q} = -K_p\mathbf{q} - K_v\dot{\mathbf{q}} + \mathbf{f}$$

as given in Equation (2.17) and represents *state variable feedback* (or position and velocity feedback).

Another form of feedback, called *output feedback*, is discussed in Section 4.10 and results if Equation (4.24) is substituted into Equation (4.23) to yield the closed-loop system (with $\mathbf{f} = \mathbf{0}$)

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u}$$

In this case, the control vector $\mathbf{u}$ is a function of the response coordinates of interest, denoted by the vector $\mathbf{y}$, i.e., $\mathbf{u}(t) = -G_f\mathbf{y}$. This form of control is called output feedback. The vector $\mathbf{y}$ can be any combination of state variables (i.e., position and velocities), as denoted by the output equation [Equation (4.25)], which is

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}}$$

The matrices C_p and C_v denote the locations of and the electronic gains associated with the transducers used to measure the various state variables.

Each of these two mathematical formulations can also be expressed in state-space form, as indicated in Equation (2.20) and repeated here. The difference between output feedback and state variable feedback is discussed in Section 7.2. The important point is that each of these various mathematical models is used to determine how to design a system with improved vibration performance using active feedback control, which provides an alternative to passive design methods.

Active control is most often formulated in the state space by

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

as given by Equation (2.18), with the output equation

$$\mathbf{y} = C\mathbf{x}$$

as used in the following section. The relationships between the physical coordinates (M, A_2, K, and B_f) and the state-space representation (A, B, and C) are given in Equation (2.20). Most control results are described in the state-space coordinate system. The symbols $\mathbf{y}$ and $\mathbf{u}$ in both the physical coordinate system and the state-space coordinate system are the same because they represent different mathematical models of the same physical control devices. The various relationships between the measurement or output $\mathbf{y}$ and the control input $\mathbf{u}$ determine the various types of control law, some of which are discussed in the following sections.

7.2 CONTROLLABILITY AND OBSERVABILITY

As pointed out in Section 6.6 using pole placement, it is not always possible to find a control law of a given form that causes the eigenvalues of the closed-loop system to have desired values. This inability to find a suitable control law raises the concept of *controllability*. A closed-loop system, meaning a structure and the applied control system, is said to be completely *controllable*, or state controllable, if every state variable (i.e., all positions and velocities) can be affected in such a way as to cause it to reach a particular value within a finite amount of time by some unconstrained (unbounded) control, $\mathbf{u}(t)$. If one state variable cannot be affected in this way, the system is said to be *uncontrollable*. Figures 7.1(a) and (b) illustrate two mechanical oscillators subject to the same control force $u(t)$ acting on the mass m_2. System (a) of the figure is uncontrollable because m_1 remains unaffected for any choice of $u(t)$. On the other hand, system (b) is controllable, since any nonzero choice of $u(t)$ affects both masses. Note that, if a second control force is applied to m_1 in Figure 7.1(a), then that system becomes controllable too. Hence, controllability is a function of both the system dynamics and of where and how many control actuators are applied. For instance, the system in Figure 7.1(b) is controllable with a single actuator, while that in Figure 7.1(a) requires two actuators to be controllable.

The formal definition of controllability for linear time-invariant systems is given in state space rather than in physical coordinates. In particular, consider the first-order system defined as before by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{7.1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{7.2}$$

Recall that $\mathbf{x}(t)$ is the $2n \times 1$ state vector, $\mathbf{u}(t)$ is an $r \times 1$ input vector, $\mathbf{y}(t)$ is a $p \times 1$ output vector, A is the $2n \times 2n$ state matrix, B is a $2n \times r$ input coefficient matrix, and C is a $p \times 2n$ output coefficient matrix. The control influence matrix B is determined by the position of control devices (actuators) on the structure. The number of outputs is p, which is the same as $2s$, where s is defined in Equation (4.25) in physical coordinates as the number of sensors. In state space the state vector includes velocity and position coordinates and hence has twice the size ($p = 2s$) since the velocities can only be measured at the same locations as the position coordinates. The state, $\mathbf{x}(t)$, is said to be controllable at $t = t_0$ if there exists a piecewise continuous bounded input $\mathbf{u}(t)$ that causes the state vector to move

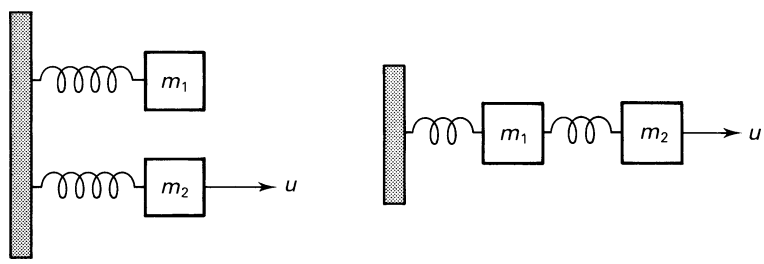

Figure 7.1 Example of (a) an uncontrollable mechanical system and (b) a controllable mechanical system.

to any final value $\mathbf{x}(t_f)$ in a finite time $t_f > t_0$. If each state $\mathbf{x}(t_0)$ is controllable, the system is said to be *completely state controllable*, which is generally what is implied when a system is said to be controllable (see, for instance, Kuo and Gholoaraghi, 2003, or Kailath, 1980).

The standard check for the controllability of a system is a rank test of a certain matrix, similar to the stability conditions used earlier for the asymptotic stability of systems with semidefinite damping. That is, the system of Equation (7.1) is completely state controllable if and only if the $2n \times 2nr$ matrix R, defined by

$$R = [B \; AB \; A^2B \cdots A^{2n-1}B] \tag{7.3}$$

has rank $2n$. In this case the pair of matrices $[A, B]$ is said to be controllable. The matrix R is called the *controllability matrix* for the matrix pair $[A, B]$.

Example 7.2.1

It is easily seen that a damped single-degree-of-freedom system is controllable. In this case, $n = 1, r = 1$. Then

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ f/m \end{bmatrix}$$

so that the controllability matrix becomes

$$R = \begin{bmatrix} 0 & f/m \\ f/m & -cf/m^2 \end{bmatrix}$$

which has rank $2 = 2n$. Thus, this system is controllable (even without damping, i.e., even if $c = 0$).

Example 7.2.2

As a second simple example, consider the state matrix of example 7.2.1 with $k/m = 1, c/m = 2$, and a control force applied in such a way as to cause $B = [1 \quad -1]^T$. Then, the controllability matrix R becomes

$$R = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which has rank $1 \neq 2n$ and the system is not controllable. Fortunately, this choice of B is not an obvious physical choice for a control law for this system. In fact, this choice of B causes the applied control force to cancel.

A similar concept to controllability is the idea that every state variable in the system has some effect on the output of the system (response) and is called *observability*. A system is observable if examination of the response (system output) determines information about *each* of the state variables. The linear time-invariant system of Equations (7.1) and (7.2) is completely observable if, for each initial state $\mathbf{x}(t_0)$, there exists a finite time $t_f > t_0$ such that knowledge of $\mathbf{u}(t), A, B, C$, and the output $\mathbf{y}(t)$ is sufficient to determine $\mathbf{x}(t_0)$ for any

(unbounded) input $\mathbf{u}(t)$. The test for observability is very similar to that for controllability. The system described by Equations (7.1) and (7.2) is completely observable if and only if the $2np \times 2n$ matrix O defined by

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{2n-1} \end{bmatrix} \tag{7.4}$$

has rank $2n$. The matrix O is called the *observability matrix*, and the pair of matrices $[A, C]$ are said to be observable if the rank of the matrix O is $2n$. The concept of observability is also important to vibration measurement and is discussed in Chapter 8.

Example 7.2.3

Consider again the single-degree-of-freedom system of example 7.2.1. If just the position is measured, the matrix C reduces to the row vector $[1\ 0]$ and $\mathbf{y}(t)$ becomes a scalar. The observability matrix becomes

$$O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has rank $2(=2n)$, and this system is observable. This condition indicates that measurement of the displacement $x(t)$ (the output in this case) allows determination of both the displacement and the velocity of the system, i.e., both states are observable.

The amount of effort required to check controllability and observability can be substantially reduced by taking advantage of the physical configuration rather than using the state-space formulation of Equations (7.1) and (7.2). For example, Hughes and Skelton (1980) have examined the controllability and observability of conservative systems

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}_f(t) \tag{7.5}$$

with observations defined by

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}} \tag{7.6}$$

Here, $\mathbf{f}_f(t) = B_f\mathbf{u}(t)$ and $\mathbf{u}(t) = -G_f\mathbf{y}(t)$ defines the input as specified in Section 4.10. In this case it is convenient to assign $G_f = I$. Then, the system of Equation (7.5) is controllable if and only if the $n \times 2n$ matrix R_n defined by

$$R_n = [\tilde{B}_f \quad \Lambda_K\tilde{B}_f \cdots \Lambda_K^{n-1}\tilde{B}_f] \tag{7.7}$$

has rank n, where $\tilde{B}_f = S_m^T B_f$ and $\Lambda_K = S_m^T K S_m$. Here, S_m is the modal matrix of Equation (7.5), and Λ_K is the diagonal matrix of eigenvalues of Equation (7.5). Thus, controllability for a conservative system can be reduced to checking the rank of a smaller-order matrix than the $2n \times 2nr$ matrix R of Equation (7.3).

This condition for controllability can be further reduced to the simple statement that system (7.5) is controllable if and only if the rank of each matrix B_q is n_q, where B_q are the partitions of the matrix $\tilde{B}_f$ according to the multiplicities of the eigenvalues of K. Here, $d < n$ denotes the number of distinct eigenvalues of the stiffness matrix, and $q = 1, 2, \ldots, d$. The integer n_q refers to the order of a given multiple eigenvalue. The matrix $\tilde{B}_f$ is partitioned into n_q rows. For example, if the first eigenvalue is repeated ($\lambda_1 = \lambda_2$), then $n_1 = 2$ and B_1 consists of the first two rows of the matrix $\tilde{B}_f$. If the stiffness matrix has distinct eigenvalues, the partitions B_q are just the rows of $\tilde{B}_f$. Thus, in particular, if the eigenvalues of K are distinct, then the system is controllable if and only if each row of $\tilde{B}_f$ has at least one nonzero entry.

For systems with repeated roots, this last result can be used to determine the minimum number of actuators required to control the response. Let d denote the number of distinct eigenvalues of K, and let n_q denote the multiplicities of the repeated roots so that $n_1 + n_2 + \cdots + n_q = n$, the number of degrees of freedom of the system, which corresponds to the partitions $\tilde{B}_q$ of $\tilde{B}_f$. Then the minimum number of actuators for the system to be controllable must be greater than or equal to the maximum of the set $\{n_1, n_2, \ldots, n_d\}$. Note that, in the case of distinct roots, this test indicates that the system could be controllable with one actuator. Similar results for general asymmetric systems can also be stated. These are discussed by Ahmadian (1985).

As in the rank conditions for stability, if the controllability or observability matrix is square, then the rank check consists of determining if the determinant is nonzero. The usual numerical question then arises concerning how to interpret the determinant having a very small value of, say, 10^{-6}. This situation raises the concept of 'degree of controllability' and 'degree of observability.' One approach to measuring the degree of controllability is to define a *controllability norm*, denoted by C_q, of

$$C_q = [\det(B_q B_q^T)]^{1/(2n_q)} \tag{7.8}$$

where q again denotes the partitioning of the control matrix $\tilde{B}_f$ according to the repeated eigenvalues of K. According to Equation (7.8), the system is controllable if and only if $C_q > 0$ for all q. In particular, the larger the value of C_q, the more controllable are the modes associated with the qth natural frequency. Unfortunately, the definition in Equation (7.8) is dependent on the choice of coordinate systems. Another more reliable measure of controllability is given later in Section 7.7, and a modal approach is given in Section 7.9.

Example 7.2.4

Consider the two-degree-of-freedom system of Figure 7.2. If k_1 is chosen to be unity and $k_2 = 0.01$, two orders of magnitude smaller, then the mass m_2 is weakly coupled to the mass m_1. Because of the increased number of forces acting on the system, a control system that acts on both m_1 and m_2 should be much more controllable than a control system acting just on the mass m_1. The following calculation, based on the controllability norm of Equation (7.8), verifies this notion. For simplicity, the masses are set at unity, i.e., $m_1 = m_2 = 1$. The equation of motion for the system of Figure 7.2 becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1 & -0.01 \\ -0.01 & 1.01 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

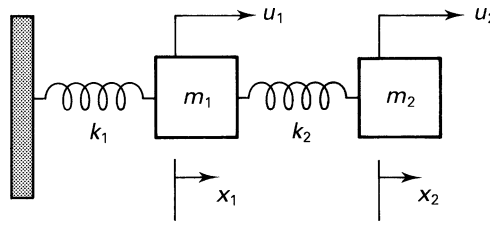

Figure 7.2 Two-degree-of-freedom structure with two control forces acting on it.

so that $\mathbf{u} = [u_1 \ u_2]^T$. In this case, $B_f = \mathbf{I}$, so that $\tilde{B}_f = S_m^T$, the transpose of the normalized modal matrix of the stiffness matrix K. Matrix K has distinct eigenvalues, so that Equation (7.8) yields $C_1 = C_2 = 1$, and both modes are controllable in agreement with the physical notion that $\mathbf{u}$ affects both x_1 and x_2. Next, consider a second control configuration with a single actuator acting on mass m_1 only. In this case, $u_2 = 0$, and B_f becomes the vector $[1 \ 0]^T$, since the vector $\mathbf{u}$ collapses to the scalar $u = u_1$. Alternatively, u could still be considered to be a vector, i.e., $\mathbf{u} = [u_1 \ 0]^T$, and B_f could then be the matrix

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Using either model, calculation of $\tilde{B}_f$ yields $C_1 = 0.8507$ and $C_2 = 0.5207$. Both of these numbers are smaller than 1, so the controllability measure has decreased from the two-actuator case. In addition, the second mode measure is smaller than the first mode measure ($C_2 < C_1$), so that the second mode is not as controllable as the first with the actuator placed at m_1. In addition, neither mode is as controllable as the two-actuator case. This numerical measure provides quantification of the controllability notion that, for the weakly coupled system of this example, it would be more difficult to control the response of m_2 (x_2) by applying a control force at $m_1 (C_2 = 0.5207)$. The system is still controllable, but not as easily so. Again, this is in agreement with the physical notion that pushing on m_1 will affect x_2, but not as easily as pushing on m_2 directly.

Complete controllability results if (but not only if) complete state feedback is used. Complete state feedback results if each of the state variables is used in the feedback law. In the physical coordinates of Equation (2.17), this use of *full state feedback* amounts to nonzero choices of C_p and C_v. In state space, state feedback is obtained by controls of the form

$$\mathbf{u} = -K_f \mathbf{x} \tag{7.9}$$

where K_f is a feedback gain matrix of appropriate dimension. If the control $\mathbf{u}$ is a scalar, then K_f is a row vector given by

$$\mathbf{k}_f = \left[g_1 \ g_2 \cdots g_{2n} \right] \tag{7.10}$$

and u is a scalar with the form

$$u = g_1 x_1 + g_2 x_2 + \cdots + g_{2n} \dot{x}_{2n} \tag{7.11}$$

The state equation [Equation (7.1)] becomes

$$\dot{\mathbf{x}} = (A - \mathbf{b}\mathbf{k}_f)\mathbf{x} \tag{7.12}$$

where B is reduced to a column vector $\mathbf{b}$. The product $\mathbf{b}\mathbf{k}_f$ is then a matrix (a column vector times a row vector). For complete state feedback, each gain g_i is nonzero, and the gains can be chosen such that the closed-loop system defined by the matrix $(A + \mathbf{b}\mathbf{k}_f)$ has the desired behavior (see problem 7.6).

By contrast, *output feedback* is defined for the system of Equation (7.1) by

$$\mathbf{u}(t) = -G_f\mathbf{y} \tag{7.13}$$

where G_f is the output feedback gain matrix of dimensions $r \times p$. Note that, since $\mathbf{y} = C\mathbf{x}$, where C indicates which states are measured, Equation (7.13) becomes

$$\mathbf{u}(t) = -G_f C\mathbf{x} \tag{7.14}$$

This expression appears to be similar to state feedback. The difference between state feedback and output feedback is that, unless $G_f C = K_f$, of full rank, output feedback does not use information about each state directly. On the other hand, use of the complete state variable feedback implies that a measurement of each state variable is available and is used in designing the control law. In output feedback, the output $\mathbf{y}$ is used to determine the control law, whereas in state feedback the state vector $\mathbf{x}$ is used. In general, the vector $\mathbf{y}$ is of lower order than $\mathbf{x}$. Thus, in state feedback there are more 'gains' that can be manipulated to produce the desired effect than there are in output feedback (recall example 6.6.1). Any control performance achievable by output feedback can also be achieved by complete state feedback, but the converse is not necessarily true.

In the next section, pole placement by output feedback is considered. Problem 7.6 illustrates that this task is easier with complete state feedback. Obviously, complete state feedback is the more versatile approach. However, complete state feedback requires knowledge (or measurement) of each state, which is not always possible. In addition, the hardware required to perform full state feedback is much more extensive than that required for output feedback. Section 7.5 discusses how state observers can be used to mimic state feedback when output feedback is not satisfactory for a given application, or when hardware issues do not allow for measurement of all of the states.

7.3 EIGENSTRUCTURE ASSIGNMENT

Section 6.6 points out a simple method for designing a feedback control system that causes the resulting closed-loop system to have eigenvalues (poles) specified by the designer. In this section, the concept of placing eigenvalues is improved and extended to placing mode shapes as well as natural frequencies. From an examination of the modal expansion for the forced response, it is seen that the mode shapes as well as the eigenvalues have a substantial impact on the form of the response. Hence, by placing both the eigenvalues and eigenvectors, the response of a vibrating system may be more precisely shaped.

First, the procedure of Section 6.6 is formalized into matrix form. To this end, let M_0 and K_0 define a desired mass and stiffness matrix resulting from a redesign process. Therefore, the desired system

$$\ddot{\mathbf{q}} + M_0^{-1}K_0\mathbf{q} = \mathbf{0} \tag{7.15}$$

has the *eigenstructure*, i.e., eigenvalues and eigenvectors, that are desired by design. Next, consider the closed-loop system with the existing structure (M and K) as the plant and only position feedback (so that $C_v = 0$). The system is

$$M\ddot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u} \tag{7.16}$$

$$\mathbf{y} = C_p\mathbf{q} \tag{7.17}$$

where the various vectors and matrices have the dimensions and definitions stated for Equations (4.24) and (4.25). Recall that the constant matrix C_p represents the placement and instrument gains associated with measurement of the positions, and B_f denotes the position of the force actuators.

The class of control problems considered here uses output feedback. Output feedback uses only the output vector $\mathbf{y}$ rather than the state vector $\mathbf{x}$ in computing the gain and is defined as calculating the matrix G_f such that the control law

$$\mathbf{u}(t) = -G_f\mathbf{y}(t) \tag{7.18}$$

yields the desired response. In this case, it is desired to calculate the gain matrix G_f such that the closed-loop system has the form of Equation (7.15), which has the required eigenvalues and eigenvectors. This procedure is a form of mechanical design, as discussed in Chapter 6.

Proceeding, the closed-loop system (7.16) under output feedback becomes

$$\ddot{\mathbf{q}} + M^{-1}K\mathbf{q} = -M^{-1}B_fG_fC_p\mathbf{q}$$

or

$$\ddot{\mathbf{q}} + M^{-1}(K + B_fG_fC_p)\mathbf{q} = \mathbf{0} \tag{7.19}$$

which has the same form as Equation (4.27) without damping or velocity feedback. If Equation (7.19) is to have the same eigenstructure as the design choice given by the matrices M_0 and K_0, then comparison of Equations (7.19) and (7.15) indicates that G_f must satisfy

$$M^{-1}K + M^{-1}B_fG_fC_p = M_0^{-1}K_0 \tag{7.20}$$

Solving this expression for G_f following the rules of matrix algebra yields

$$B_fG_fC_p = M(M_0^{-1}K_0 - M^{-1}K) \tag{7.21}$$

In general, the matrices B_f and C_p are not square matrices (unless each mass has an actuator and sensor attached), so the inverses of these matrices, required to solve for G_f, do not exist. However, a *left generalized inverse*, defined by

$$B_f^l = (B_f^TB_f)^{-1}B_f^T \tag{7.22}$$

and a *right generalized inverse*, defined by

$$C_p^I = C_p^T (C_p C_p^T)^{-1} \tag{7.23}$$

can be used to 'solve' Equation (7.21). The matrices C_p^I and B_f^I are called generalized inverses in the sense that

$$B_f^I B_f = (B_f^T B_f)^{-1} B_f^T B_f = I_{r\times r} \tag{7.24}$$

and

$$C_p C_p^I = C_p C_p^T (C_p C_p^T)^{-1} = I_{s\times s} \tag{7.25}$$

where the subscripts on the identity matrices indicate their size. Note that the calculation of the generalized inverses given by Equations (7.22) and (7.23) requires that the matrices $B_f^T B_f$ and $C_p C_p^T$ both be nonsingular. Other solutions can still be found using a variety of methods (see, for instance, Golub and Van Loan, 1996). Generalized inverses are briefly discussed in Appendix B.

Premultiplying Equation (7.21) by Equation (7.22) and postmultiplying Equation (7.21) by Equation (7.23) yields a solution for the $m \times s$ gain matrix G_f to be

$$G_f = (B_f^T B_f)^{-1} B_f^T M (M_0^{-1} K_0 - M^{-1} K) C_p^T (C_p C_p^T)^{-1} \tag{7.26}$$

If this value of the gain matrix G_f is implemented, the resulting closed-loop system will have the eigenstructure and response *approximately* equal to that dictated by the design set M_0 and K_0. Note, if the system is very close to the desired system, the matrix difference $(M_0^{-1} K_0 - M^{-1} K)$ will be small and G_f will be small. However, the matrix G_f depends on where the measurements are made because it is a function of C_p and also depends on the position of the actuators because it is a function of B_f.

Example 7.3.1

Consider the two-degree-of-freedom system with original design defined by $M = I$ and

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Suppose it is desired to build a control for this system so that the resulting closed-loop system has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ and eigenvectors given by $\mathbf{v}_1 = [1 \quad -1]^T/\sqrt{2}$ and $\mathbf{v}_2 = [1 \quad 1]^T/\sqrt{2}$, which are normalized. A system with such eigenvalues and eigenvectors can be calculated from Equation (3.21) or

$$M_0^{-1/2} K_0 M_0^{-1/2} = S \operatorname{diag}\begin{bmatrix} 2 & 4 \end{bmatrix} S^T$$

This expression can be further simplified if M_0 is taken to be the identity matrix I. Then

$$K_0 = S \operatorname{diag}\begin{bmatrix} 2 & 4 \end{bmatrix} S^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The pair (I, K_0) then represents the desired system.

Next, based on knowledge of controllability and observability, the matrices C_p and B_f are each chosen to be identity matrices, i.e., each position is measured and each mass has a force applied to it (i.e., an actuator attached to it). This condition ensures that the system is completely controllable and observable and that the controllability and observability norms are large enough. Equation (7.26) for the gain matrix G_f becomes

$$G_f = \left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

which causes the original system with closed-loop control [i.e., Equation (7.16)] to have the desired eigenstructure. Note that the eigenvalues of K are 0.382 and 2.618 and those of $G_f + K$ are 2 and 4, as desired. In addition, the eigenvectors of $G_f + K$ are computed to be as desired:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Although not obvious from the introductory material just presented, Wonham (1967) has shown that all the eigenvalues can be placed if and only if the system is controllable. In case more than one actuator is used, i.e., in the multi-input case, the calculated feedback gain matrix G_f is not uniquely determined by assigning just the eigenvalues (see Moore, 1976). Hence, the remaining freedom in the choice of G_f can also be used to place the mode shapes, as was the case in the preceding example. However, only mode shapes that satisfy certain criteria can be placed. These issues are discussed in detail by Andry, Shapiro, and Chung (1983), who also extended the process to damped and asymmetric systems.

7.4 OPTIMAL CONTROL

One of the most commonly used methods of modern control theory is called optimal control. Like optimal design methods, optimal control involves choosing a cost function or performance index to minimize. Although this method again raises the issue of how to choose the cost function, optimal control remains a powerful method of obtaining a desirable vibration response. Optimal control formulations also allow a more natural consideration of constraints on the state variables as well as consideration for reducing the amount of time, or final time, required for the control to bring the response to a desired level.

Consider again the control system and structural model given by Equations (4.23) and (4.24) in Section 4.10 and its state-space representation given in Equations (7.1) and (7.2). The *optimal control problem* is to calculate the control $\mathbf{u}(t)$ that minimizes some specified performance index, denoted by $J = J(\mathbf{q}, \dot{\mathbf{q}}, t, \mathbf{u})$, subject to the constraint that

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u}$$

is satisfied, and subject to the given initial conditions $\mathbf{q}(t_0)$ and $\dot{\mathbf{q}}(t_0)$. This last expression is called a differential constraint and is usually written in state-space form. The cost function is usually stated in terms of an integral. The design process in optimal control consists of

the judicious choice of the function J. The function J must be stated in such a way as to reflect a desired performance. The best, or optimal, $\mathbf{u}$, denoted by $\mathbf{u}^*$, has the property that

$$J(\mathbf{u}^*) < J(\mathbf{u}) \tag{7.27}$$

for any other choice of $\mathbf{u}$. Solving optimal control problems extends the concepts of maximum and minimum from calculus to functionals J. Designing a vibration control system using optimal control involves deciding on the performance index J. Once J is chosen, the procedure is systematic.

Before proceeding with the details of calculating an optimal control, $\mathbf{u}^*$, several examples of common choices of the cost function J, corresponding to various design goals, will be given. The *minimum time problem* consists of defining the cost function by

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

which indicates that the state equations take the system from the initial state at time t_0 [i.e., $\mathbf{x}(t_0)$] to some final state $\mathbf{x}(t_f)$ at time t_f, in a minimum amount of time.

Another common optimal control problem is called the *linear regulator problem*. This problem has specific application in vibration suppression. In particular, the design objective is to return the response (actually, all the states) from the initial state value $\mathbf{x}(t_0)$ to the system equilibrium position (which is usually $\mathbf{x}_e = \mathbf{0}$ in the case of structural vibrations). The performance index for the linear regulator problem is defined as

$$J = \frac{1}{2}\int_{t_0}^{t_f} (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})\, dt \tag{7.28}$$

where Q and R are symmetric positive definite *weighting matrices*. The larger the matrix Q, the more emphasis is placed by optimal control on returning the system to zero, since the value of $\mathbf{x}$ corresponding to the minimum of the quadratic form $\mathbf{x}^T Q\mathbf{x}$ is $\mathbf{x} = \mathbf{0}$. On the other hand, increasing R has the effect of reducing the amount, or magnitude, of the control effort allowed. Note that positive quadratic forms are chosen so that the functional being minimized has a clear minimum. Using both nonzero Q and R represents a compromise in the sense that, based on a physical argument, making $\mathbf{x}(t_f)$ zero requires $\mathbf{u}(t)$ to be large. The linear regulator problem is an appropriate cost function for control systems that seek to eliminate, or minimize, transient vibrations in a structure. The need to weight the control effort (R) results from the fact that no solution exists to the variational problem when constraining the control effort. That is, the problem of minimizing J with the inequality constraint $\|\mathbf{u}(t)\| < c$, where c is a constant, is not solved. Instead, R is adjusted in the cost function until the control is limited enough to satisfy $\|\mathbf{u}(t)\| < c$.

On the other hand, if the goal of the vibration design is to achieve a certain value of the state response, denoted by the state vector $\mathbf{x}_d(t)$, then an appropriate cost function would be

$$J = \int_0^{t_f} (\mathbf{x} - \mathbf{x}_d)^T Q(\mathbf{x} - \mathbf{x}_d)\, dt \tag{7.29}$$

where Q is again symmetric and positive definite. This problem is referred to as the *tracking problem*, since it forces the state vector to follow, or track, the vector $\mathbf{x}_d(t)$.

In general, the optimal control problem is difficult to solve and lends itself very few closed-form solutions. With the availability of high-speed computing, the resulting numerical solutions do not present much of a drawback. The following illustrates the problem by analyzing the linear regulator problem.

Consider the linear regulator problem for the state-space description of a structure given by Equations (7.1) and (7.2). That is, consider calculating $\mathbf{u}$ such that $J(\mathbf{u})$ given by Equation (7.28) is a minimum subject to the constraint that Equation (7.1) is satisfied. A rigorous derivation of the solution is available in most optimal control texts (see, for instance, Kirk, 1970). Proceeding less formally, assume that the form of the desired optimal control law will be

$$\mathbf{u}^*(t) = -R^{-1}B^T S(t)\mathbf{x}^*(t) \tag{7.30}$$

where $\mathbf{x}^*(t)$ is the solution of the state equation with optimal control $\mathbf{u}^*$ as input, and $S(t)$ is a symmetric time-varying $2n \times 2n$ matrix to be determined (not to be confused with the orthogonal matrix of eigenvectors S). Equation (7.30) can be viewed as a statement that the desired optimal control be in the form of state feedback. With some manipulation (see, for instance, Kirk, 1970), $S(t)$ can be shown to satisfy what is called the *matrix Riccati equation* given by

$$Q - S(t)BR^{-1}B^T S(t) + A^T S(t) + S(t)A + \frac{dS(t)}{dt} = 0 \tag{7.31}$$

subject to the final condition $S(t_f) = 0$. This calculation is a backward-in-time matrix differential equation for the unknown time-varying matrix $S(t)$. The solution for $S(t)$ in turn gives the optimal linear regulator control law (7.28), causing $J(\mathbf{u})$ to be a minimum. Unfortunately, this calculation requires the solution of $2n(2n+1)/2$ nonlinear ordinary differential equations simultaneously, backward in time (which forms a difficult numerical problem).

In most practical problems – indeed, even for very simple examples – the Riccati equation must be solved numerically for $S(t)$, which then yields the optimal control law via Equation (7.30).

The Riccati equation, and hence the optimal control problem, becomes simplified if one is interested only in controlling the steady state vibrational response and controlling the structure over a long time interval. In this case, the final time in the cost function $J(\mathbf{u})$ is set to infinity and the Riccati matrix $S(t)$ is constant for completely controllable, time-invariant systems (see, for instance, Kirk, 1970). Then, $dS(t)/dt$ is zero and the Riccati equation simplifies to

$$Q - SBR^{-1}B^T S + A^T S + SA = 0 \tag{7.32}$$

which is now a nonlinear *algebraic* equation in the constant matrix S. The effect of this method on the vibration response of a simple system is illustrated in example 7.4.1.

Example 7.4.1

This example calculates the optimal control for the infinite time linear quadratic regulator problem for a single-degree-of-freedom oscillator of the form

$$\ddot{x}(t) = 4x(t) = f(t)$$

In this case, the cost function is of the form

$$J = \int_0^\infty (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})\, dt$$

which is Equation (7.28) with $t_f = \infty$. A control (state feedback) is sought of the form

$$\mathbf{u} = -R^{-1}B^T S\mathbf{x}(t) = -K_f\mathbf{x}(t)$$

The state equations are

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}$$

Two cases are considered to illustrate the effect of the arbitrary (but positive definite) weighting matrices Q and R. The system is subject to the initial condition $\mathbf{x}(0) = [1\ 1]^T$. In the first case, let $Q = R = I$, and the optimal control is calculated from Equation (7.32) using MATLAB (Moler *et al.*, 1985) to be

$$K_f = \begin{bmatrix} 0 & 0 \\ -0.1231 & -1.1163 \end{bmatrix}$$

The resulting response and control effort are plotted in Figures 7.3 through 7.5. Figure 7.5 is the control law, $\mathbf{u}^*$, calculated by using Equation (7.30); Figures 7.3 and 7.4 illustrate the resulting position and velocity response to initial conditions under the action of the control system.

In case 2, the same problem is solved again, with the control weighting matrix set at $R = (10)I$. The result is that the new control law is given by

$$G_f = \begin{bmatrix} 0 & 0 \\ -0.0125 & -0.3535 \end{bmatrix}$$

which is 'smaller' than the first case. The resulting position response, velocity response, and control effort are plotted in Figures 7.6 through 7.8 respectively.

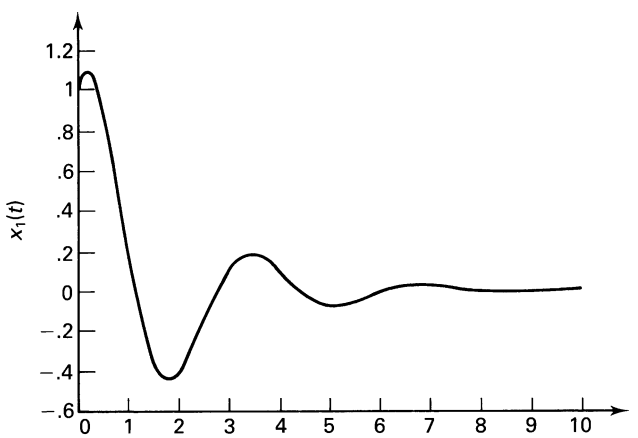

Figure 7.3 Position, $x_1(t)$, versus time for the case $Q = R = I$.

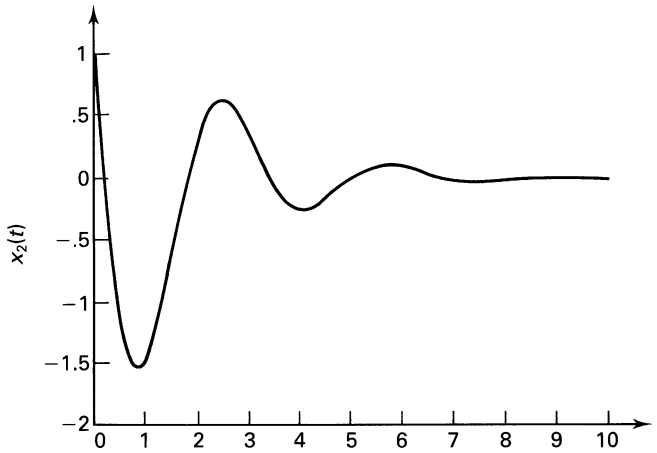

Figure 7.4 Velocity, $x_2(t)$, versus time for the case $Q = R = I$.

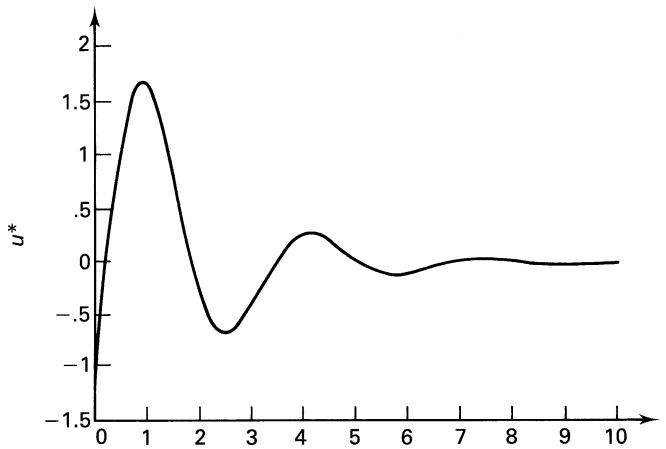

Figure 7.5 Control, $u^*(t)$, versus time for the case $Q = R = I$.

In the second case, with larger values for R, the control effort is initially much more limited, i.e., a maximum value of 0.7 units as opposed to 1.75 units for the first case. In addition, the resulting response is brought to zero faster in the case with more control effort (i.e., Figures 7.3, 7.4, and 7.6). These examples illustrate the effect that the weighting matrices have on the response. Note, a designer may have to restrict the control magnitude (hence, use large relative values of R) because the amount of control energy available for a given application is usually limited even though a better response (shorter time to settle) is obtained with larger values of control effort.

Optimal control has a stabilizing effect on the closed-loop system. Using a quadratic form for the cost function guarantees stability (bounded output) of the closed-loop system. This guaranty can be seen by considering the uncontrolled system subject to Equation (7.29)

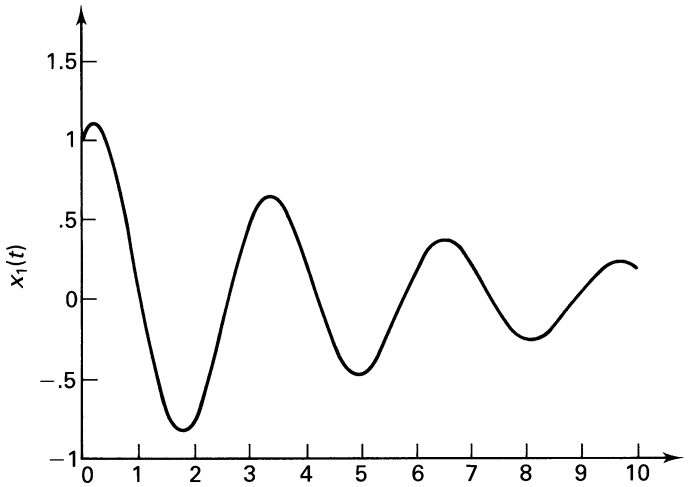

Figure 7.6 Position, $x_1(t)$, versus time for the case $Q = I, R = 10I$.

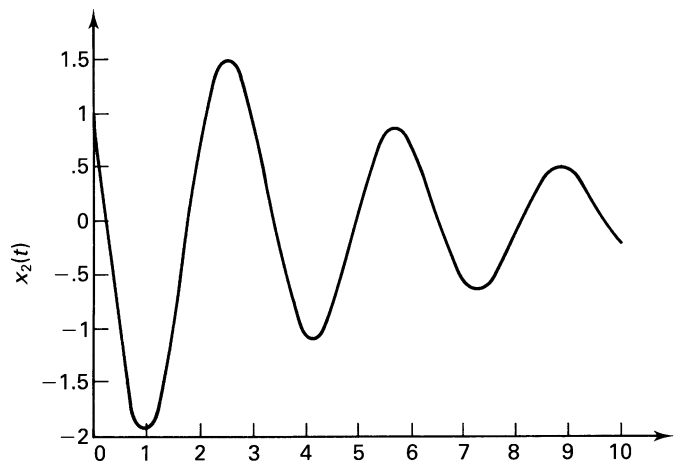

Figure 7.7 Velocity, $x_2(t)$, versus time for the case $Q = I, R = 10I$.

with $\mathbf{x}_d$ taken as the origin. Asymptotic stability requires that $\mathbf{x}$ approaches zero as time approaches infinity so that the integral

$$J = \int_0^\infty \mathbf{x}^T Q \mathbf{x}\, dt$$

converges as long as Q is positive semidefinite. Define the quadratic from $V(t) = \mathbf{x}^T P \mathbf{x}$, where P is the positive definite solution of the Lyapunov equation [Equation (4.29)]

$$PA + A^T P = -Q$$

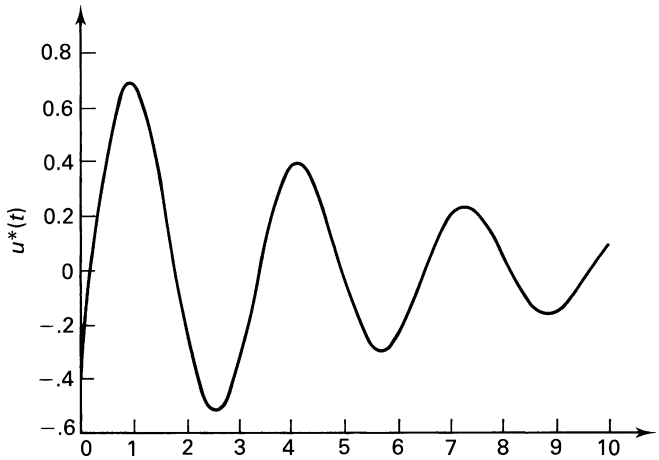

Figure 7.8 Control, $u^*(t)$, versus time for the case $Q = I, R = 10I$.

Here, A is the state matrix satisfying Equation (7.1) with $B = 0$. Computing the time derivative of $V(t)$ yields

$$\dot{V}(t) = \frac{d}{dt}(\mathbf{x}^T P\mathbf{x}) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} = \mathbf{x}^T(A^T P + PA)\mathbf{x} = -\mathbf{x}^T Q\mathbf{x} < 0$$

Hence, $V(t)$ is positive definite with a negative definite time derivative and is thus a Lyapunov function of the system of Equation (7.1). As a result, the homogeneous system is asymptotically stable (recall Section 4.10). Since the homogeneous system is asymptotically stable, the closed-loop system (forced response) will be bounded-input, bounded-output stable, as discussed in Section 5.5. Also, note that, if $\mathbf{x}_0$ is the initial state, integrating the cost function yields

$$J = \int_0^\infty \mathbf{x}^T Q\mathbf{x}\,\mathrm{d}t = \int_0^\infty \frac{d}{dt}(-\mathbf{x}^T P\mathbf{x})\,dt = \mathbf{x}_0^T P\mathbf{x}_0$$

since asymptotic stability requires that $\mathbf{x}$ approach zero at the upper limit. This calculation indicates that the value of the cost function depends on the initial conditions.

This section is not intended to provide the reader with a working knowledge of optimal control methods. It is intended to illustrate the use of optimal control as an alternative vibration suppression technique and to encourage the reader to pursue the use of optimal control through one of the references.

7.5 OBSERVERS (ESTIMATORS)

In designing controllers for vibration suppression, often not all of the velocities and displacements can be conveniently measured. However, if the structure is known and is observable, i.e., if the state matrix A is known and if the measurement matrix C is such that O has full rank, one can design a subsystem, called an *observer* (or *estimator*, in the stochastic

case), from measurements of the input and of the response of the system. The observer then provides an approximation of the missing measurements.

Consider the state-space system given by Equations (7.1) and (7.2) with output feedback as defined by Equation (7.13). To simplify this discussion, consider the case where the control $u(t)$ and the output $y(t)$ are both scalars. This is the *single-input, single-output* (SISO) case. In this situation the matrix B becomes a column vector, denoted by $\mathbf{b}$, and the matrix C becomes a row vector, denoted by $\mathbf{c}^T$.

The output $y(t)$ is proportional to a state variable or linear combination of state variables. Sometimes recovering the state vector from the measurement of $y(t)$ is trivial. For instance, if each state is measured (multiple output), C is square and nonsingular, so that $\mathbf{x} = C^{-1}\mathbf{y}$. In this section it is assumed that the state vector is not directly measured and is not easily determined from the scalar measurement $y(t)$. However, since the quantities $A, \mathbf{b}$, and $\mathbf{c}^T$, as well as measurements of the input $u(t)$ and the output $y(t)$, are known, the desired state vector can be *estimated*. Constructing this estimated state vector, denoted by $\mathbf{x}_e$, is the topic of this section. State observers can be constructed if the system is completely observable (see, for instance, Chen, 1998).

The simplest observer to implement is the *open-loop estimator*. An open-loop estimator is simply the solution of the state equations with the same initial conditions as the system under consideration. Let $\mathbf{x}_e$ denote the estimated state vector. The integration of

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{b}u \tag{7.33}$$

yields the desired estimated state vector. The estimated state vector can then be used to perform state feedback or output feedback. Note that integration of Equation (7.33) requires knowledge of the initial condition, $\mathbf{x}_e(0)$, which is not always available.

Unfortunately, the open-loop estimator does not work well if the original system is unstable (or almost unstable) or if the initial conditions of the unmeasured states are not known precisely. In most situations, the initial state is not known. A better observer can be obtained by taking advantage of the output of the system, $y(t)$, as well as the input. For instance, consider using the difference between the output $y(t)$ of the actual system and the output $y_e(t)$ of the estimator as a correction term in the observer of Equation (7.33). The observer then becomes

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{r}_e(y - y_e) + \mathbf{b}u \tag{7.34}$$

where $\mathbf{r}_e$ is the gain vector of the observer and is yet to be determined. Equation (7.34) is called an *asymptotic state estimator*, which is designed by choosing the gain vector $\mathbf{r}_e$. The error between the actual state vector $\mathbf{x}$ and the estimated state vector $\mathbf{x}_e$ must satisfy the difference between Equation (7.34) and Equation (7.1). This difference yields

$$(\dot{\mathbf{x}} - \dot{\mathbf{x}}_e) = A(\mathbf{x} - \mathbf{x}_e) + \mathbf{r}_e(y_e - y) \tag{7.35}$$

Since $y_e - y = \mathbf{c}^T(\mathbf{x}_e - \mathbf{x})$, this last expression becomes

$$\dot{\boldsymbol{e}} = (A - \mathbf{r}_e\mathbf{c}^T)\mathbf{e} \tag{7.36}$$

where the error vector $\mathbf{e}$ is defined to be the difference vector $\mathbf{e} = \mathbf{x} - \mathbf{x}_e$. This expression is the dynamic equation that determines the error between the actual state vector and the

estimated state vector. Equation (7.36) describes how the difference between the actual initial condition $\mathbf{x}(0)$ and the assumed condition $\mathbf{x}_e(0)$ evolves in time.

The idea here is to design the observer, i.e., to choose $\mathbf{r}_e$, so that the solution of Equation (7.36) remains as close to zero as possible. For instance, if the eigenvalues of $(A - \mathbf{r}_e\mathbf{c}^T)$ are chosen to have negative real parts that are large enough in absolute value, the vector $\mathbf{e}$ goes to zero quickly, and $\mathbf{x}_e$ approaches the real state $\mathbf{x}$. Thus, any difference between the actual initial conditions and the assumed initial conditions for the observer dies out with time instead of increasing with time, as could be the case with the open-loop observer.

Obviously, there is some difference between using the actual state vector $\mathbf{x}$ and the estimated state vector $\mathbf{x}_e$ in calculating a control law. This difference usually shows up as increased control effort; that is, a feedback control based on $\mathbf{x}_e$ has to exert more energy than one based on the actual state variables $\mathbf{x}$. However, it can be shown that, as far as placing eigenvalues are concerned, there is no difference in state feedback between using the actual state and the estimated state (Chen, 1970). Furthermore, the design of the observer and the control can be shown to be equivalent to performing the separate design of a control, assuming that the exact states are available, and the subsequent design of the observer (called the *separation theorem*).

To solve the control problem with state estimation, the state equation with the estimated state vector used as feedback ($u = gy = g\mathbf{c}^T\mathbf{x}_e$, recalling that $\mathbf{c}^T$ is a row vector) must be solved simultaneously with the state estimation equation [Equation (7.36)]. This solution can be achieved by rewriting the state equation as

$$\dot{\mathbf{x}} = A\mathbf{x} + g\mathbf{b}\mathbf{c}^T\mathbf{x}_e \tag{7.37}$$

and substituting the value for $\mathbf{x}_e$. Then

$$\dot{\mathbf{x}} = A\mathbf{x} + g\mathbf{b}\mathbf{c}^T(\mathbf{x} - \mathbf{e}) \tag{7.38}$$

or upon rearranging

$$\dot{\mathbf{x}} = (A + g\mathbf{b}\mathbf{c}^T)\mathbf{x} - g\mathbf{b}\mathbf{c}^T\mathbf{e} \tag{7.39}$$

Combining Equations (7.36) and (7.39) yields

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} A + g\mathbf{b}\mathbf{c}^T & -g\mathbf{b}\mathbf{c}^T \\ 0 & A - \mathbf{r}_e\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{7.40}$$

Here, the zero in the state matrix is a $2n \times 2n$ matrix of zeros. Expression (7.40) is subject to the actual initial conditions of the original state equation augmented by the assumed initial conditions of the estimator. These estimator initial conditions are usually set at zero, so that $[\mathbf{x}^T(0) \quad \mathbf{e}^T(0)]^T = [\mathbf{x}^T(0) \quad \mathbf{0}]^T$. Solution of Equation (7.40) yields the solution to the state feedback problem with the states estimated, rather than directly measured.

The following example illustrates the computation of a state observer as well as the difference between using a state observer and using the actual state in a feedback control problem.

Example 7.5.1

Consider a single-degree-of-freedom oscillator ($\omega_n^2 = 4$, $\zeta = 0.25$) with displacement as the measured output. The state-space formulation for a single-input, single-output system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x} + \mathbf{b}u, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7.41}$$

$$y = [1 \quad 0]\,\mathbf{x}$$

If output feedback is used, then $u = gy = g\mathbf{c}^T\mathbf{x}$, and the system becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x} + g \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} \tag{7.42}$$

Combining yields

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ g-4 & -1 \end{bmatrix} \mathbf{x} \tag{7.43}$$

The asymptotic estimator in this case is given by

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{r}_e(y - \mathbf{c}^T\mathbf{x}_e) + \mathbf{b}u \tag{7.44}$$

where $\mathbf{r}_e$ is chosen to cause the eigenvalues of the matrix $(A - \mathbf{r}_e\mathbf{c}^T)$ to have negative real parts that are large in absolute value. As mentioned previously, in this case $\mathbf{c}^T = [1\ 0]$ is a row vector, so that the product $\mathbf{r}_e\mathbf{c}^T$ is a matrix. Here, the eigenvalues of $(A - \mathbf{r}_e\mathbf{c}^T)$ are chosen to be -6 and -5 (chosen only because they cause the solution of Equation (7.36) to die out quickly; other values could be chosen). These equations are equivalent to requiring the characteristic equation of $(A - \mathbf{r}_e\mathbf{c}^T)$ to be $(\lambda + 6)(\lambda + 5)$, i.e.,

$$\begin{aligned} \det(\lambda I - A + \mathbf{r}_e\mathbf{c}^T) &= \lambda^2 + (r_1 + 1)\lambda + (r_1 + r_2 + 4) \\ &= \lambda^2 + 11\lambda + 30 \end{aligned} \tag{7.45}$$

Equating coefficients of λ in this expression yields the desired values for $\mathbf{r}_e = [r_1\ r_2]^T$ to be $r_1 = 10$ and $r_2 = 16$. Thus, the estimated state is taken as the solution of

$$\dot{\mathbf{x}}_e = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \mathbf{x}_e + \begin{bmatrix} 10 \\ 16 \end{bmatrix} (y - \mathbf{c}^T\mathbf{x}_e) + \mathbf{b}u \tag{7.46}$$

The solution $\mathbf{x}_e$ can now be used as feedback in the original control problem coupled with the state estimation equation. This solution yields [from Equation (7.40) with the control taken as $g = -1$]

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -1 & 1 & 0 \\ 0 & 0 & -10 & 1 \\ 0 & 0 & -20 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{7.47}$$

The plots in Figure 7.9 show a comparison between using the estimated state and the same control with the actual state used. In the figure, the control is fixed to be $g = -1$ and the actual initial conditions are taken to be $\mathbf{x}(0) = [1\ 1]^T$. The response for various different initial conditions for the observer $\mathbf{x}_e(0)$ are also plotted.

Note that in both cases the error in the initial conditions of the estimator disappears by about 1.2 time units, lasting just a little longer in Figure 7.10 than in Figure 7.9. This results because the assumed initial conditions for the observer are farther away from the actual initial condition in Figure 7.10 than in Figure 7.9. Comparison of the actual response in Figure 7.9 and Figure 7.10 with that of Figure 7.11 shows that the control law calculated using estimated state feedback is only slightly worse (takes slightly longer to decay) than those calculated using actual state feedback for this example.

In the preceding presentation of state observers, the scenario is that the state vector $\mathbf{x}$ is not available for use in output feedback control. Thus, estimated output feedback control is used instead, i.e., $u(t) = g\mathbf{c}^T\mathbf{x}_e$. A practical alternative use of an observer is to use the estimated state vector $\mathbf{x}_e$ to change a problem that is output feedback control ($g\mathbf{c}^T\mathbf{x}$) because

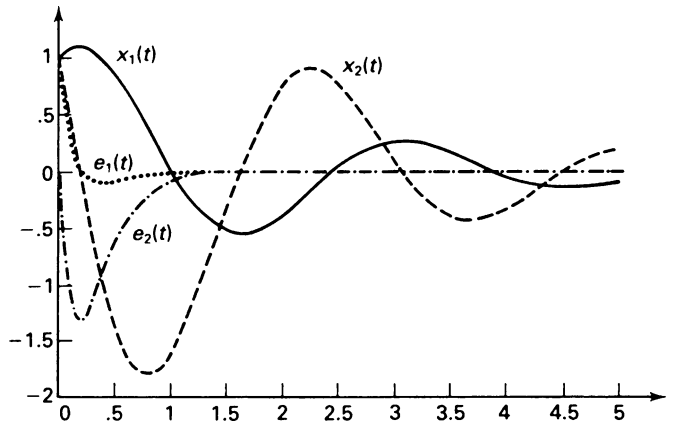

Figure 7.9 Comparison of the error vector $\mathbf{e}(t)$ versus time and the components of the state vector $\mathbf{x}(t)$ versus time for the initial condition $\mathbf{e}(0) = [1 \quad 0]^T$.

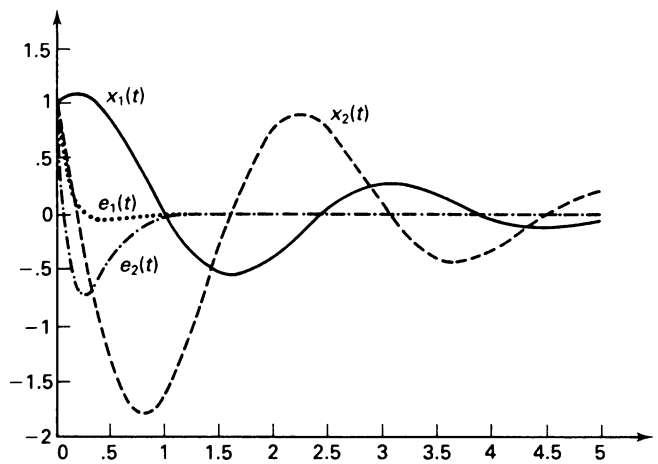

Figure 7.10 Components of the error vector $\mathbf{e}(t)$ versus time and the components of the state vector $\mathbf{x}(t)$ versus time for the initial condition $\mathbf{e}(0) = [1 \quad 1]^T$.

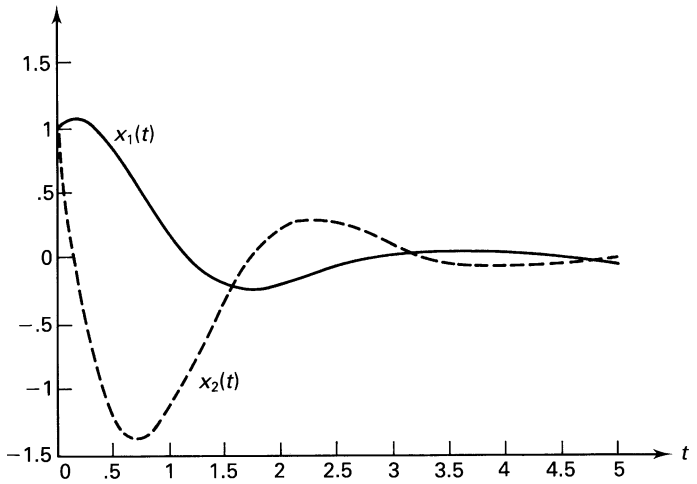

Figure 7.11 Components of the state vector $\mathbf{x}(t)$ versus time for the case of complete state feedback.

of hardware limitations to one that is augmented by the observer to use complete state variable feedback, i.e., $u(t) = gc^T\mathbf{x} + \mathbf{k}^T\mathbf{x}_e$. Here, the vector $\mathbf{k}$ makes use of each state variable, as opposed to $\boldsymbol{c}$, which uses only some state variables.

If multiple inputs and outputs are used, this analysis can be extended. The resulting observer is usually called a *Luenberger observer*. If, in addition, a noise signal is added as input to Equation (7.1), the estimation equation can still be developed. In this case they are referred to as *Kalman filters* (see, for instance, Anderson and Moore, 1979).

7.6 REALIZATION

In the preceding section, the state matrix A (and hence the coefficient matrices M, D, and K) and also the input matrix B and the output matrix C are all assumed to be known. In this section, however, the problem is to determine A, B, and C from the transfer function of the system. This problem was first introduced in Section 1.6, called plant identification, where the scalar coefficients m, c, and k were determined from Bode plots. Determining the matrices A, B, and C from the transfer function of a system is called *system realization*.

Consider again the SISO version of Equations (7.1) and (7.2). Assuming that the initial conditions are all zero, taking the Laplace transform of these two expressions yields

$$s\mathbf{X}(s) = A\mathbf{X}(s) + \mathbf{b}U(s) \tag{7.48}$$

$$y(s) = \mathbf{c}^T\mathbf{X}(s) \tag{7.49}$$

Solving Equation (7.48) for $\mathbf{X}(s)$ and substituting the result into Equation (7.49) yields

$$Y(s) = \mathbf{c}^T(sI - A)^{-1}\mathbf{b}U(s) \tag{7.50}$$

Since y and u are scalars here, the transfer function of the system is just

$$\frac{Y(s)}{U(s)} = G(s) = \mathbf{c}^T(sI - A)^{-1}\mathbf{b} \tag{7.51}$$

Here and in the following, $G(s)$ is assumed to be a rational function of s. If, in addition, $G(s)$ is such that

$$\lim_{s\to\infty} G(s) = 0 \tag{7.52}$$

then $G(s)$ is called a *proper rational function*. These two conditions are always satisfied for the physical models presented in Chapter 2. The triple A, $\mathbf{b}$, and $\mathbf{c}$ is called a *realization* of $G(s)$ if Equation (7.51) holds. The function $G(s)$ can be shown to have a realization if and only if $G(s)$ is a proper rational function. The triple $(A, \mathbf{b}, \mathbf{c})$ of minimum order that satisfies Equation (7.51) is called an *irreducible realization*, or a *minimal realization*. The triple $(A, \mathbf{b}, \mathbf{c})$ can be shown to be an irreducible realization if and only if it is both controllable and observable (see, for instance, Chen, 1998).

A transfer function $G(s)$ is said to be irreducible if and only if the numerator and denominator of G have no common factor. This statement is true if and only if the denominator of $G(s)$ is equal to the characteristic polynomial of the matrix A, and if and only if the degree of the denominator of $G(s)$ is equal to $2n$, the order of the system. While straightforward conditions are available for ensuring the existence of an irreducible realization, the realization is not unique. In fact, if $(A, \mathbf{b}, \mathbf{c})$ is an irreducible realization of the transfer function $G(s)$, then $(A', \mathbf{b}', \mathbf{c}')$ is an irreducible realization if and only if A is similar to A', i.e., there exits a nonsingular matrix P such that $A = PA'P^{-1}$, $\mathbf{b} = P\mathbf{b}'$, and $\mathbf{c} = \mathbf{c}'P^{-1}$. Hence, a given transfer function has an infinite number of realizations. This result is very important to remember when studying modal testing in Chapter 8.

There are several ways to calculate a realization of a given transfer function. The easiest method is to recall from differential equations the method of writing a $2n$-order differential equation as $2n$ first-order equations. Then, if the transfer function is given as

$$\frac{y(s)}{u(s)} = G(s) = \frac{\beta}{s^{2n} + \alpha_1 s^{2n-1} + \alpha_2 s^{2n-2} + \cdots + \alpha_{2n-1}s + \alpha_{2n}} \tag{7.53}$$

the time domain equivalent is simply obtained by multiplying this out to yield

$$s^{2n}y(s) + \alpha_1 s^{2n-1}y(s) + \cdots = \beta u(s) \tag{7.54}$$

and taking the inverse Laplace transform of Equation (7.54) to obtain

$$y^{2n} + \alpha_1 y^{(2n-1)} + \cdots + \alpha_{2n}y(t) = \beta u(t) \tag{7.55}$$

Here, $y^{(2n)}$ denotes the $2n$th time derivative of $y(t)$. Next, define the state variables by the scheme

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y^{(1)}(t) \\ &\vdots \\ x_{2n}(t) &= y^{(2n-1)}(t) \end{aligned}$$

The state equations for Equation (7.55) then become

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{2n} & -\alpha_{2n-1} & -\alpha_{2n-2} & \cdots & -\alpha_1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix} u(t) \tag{7.56}$$

and

$$y(t) = [1\ 0\ 0\ \cdots]\mathbf{x}(t) \tag{7.57}$$

The triple $(A, \mathbf{b}, \mathbf{c})$ defined by Equations (7.56) and (7.57) constitutes an irreducible realization of the transfer function given by Equation (7.53).

Realization procedures are also available for *multiple-input, multiple-output* (MIMO) systems (see, for instance, Ho and Kalman, 1965). In Chapter 8, realization methods are used to determine the natural frequencies, damping ratios, and mode shapes of a vibrating structure by measuring the transfer function and using it to construct a realization of the test structure.

Example 7.6.1

Consider the transfer function of a simple oscillator, i.e.,

$$G(s) = \frac{1}{s^2 + 2\zeta\omega s + \omega^2}$$

Following Equation (7.56), a state-space realization of this transfer function is given by

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

7.7 REDUCED-ORDER MODELING

Most control methods work best for structures with a small number of degrees of freedom. Many modeling techniques produce structural models of a large order. Hence, it is often necessary to reduce the order of a model before performing a control analysis and designing a control law. This topic was first introduced in Section 6.8. In that section the method of model reduction was based on knowledge that certain coordinates, such as those corresponding to a bending mode in a longitudinally excited beam, will not contribute to the response. These coordinates are then removed, producing a model of lower order. In this section, the coordinates to be removed are calculated in a more formal way as part of the reduction process, rather than specified through experience as in the Guyan reduction of Section 6.8.

The approach taken in this section is to reduce the order of a given model on the basis of deleting those coordinates, or modes, that are the least controllable and observable. The idea here is that controllability and observability of a state (coordinate) are indications of the contribution of that state (coordinate) to the response of the structure, as well as the ability of that coordinate to be excited by an external disturbance.

To implement this idea, a measure of the degree of controllability and observability is needed. One such measure of controllability is given by the controllability norm of Equation (7.8). However, an alternative, more useful measure is provided for asymptotically stable systems of the form given by Equations (7.1) and (7.2) by defining the *controllability grammian*, denoted by W_C, as

$$W_C^2 = \int_0^\infty e^{At} B B^T e^{A^T t}\, dt \tag{7.58}$$

and the *observability grammian*, denoted by W_O, as

$$W_O^2 = \int_0^\infty e^{A^T t} C^T C\, e^{At}\, dt \tag{7.59}$$

Here, the matrices A, B, and C are defined as in Equations (7.1) and (7.2). The properties of these two matrices provide useful information about the controllability and observability of the closed-loop system. If the system is controllable (or observable), the matrix W_C (or W_O) is nonsingular. These grammians characterize the degree of controllability and observability by quantifying just how far away from being singular the matrices W_C and W_O are. This is equivalent to quantifying rank deficiency. The most reliable way to quantify the rank of a matrix is to examine the singular values of the matrix, which is discussed next.

For any real $m \times n$ matrix A there exist orthogonal matrices $U_{m\times m}$ and $V_{n\times n}$ such that

$$U^T A V = \operatorname{diag}\left[\sigma_1\ \sigma_2 \cdots \sigma_p\right], \qquad p = \min(m, n) \tag{7.60}$$

where the σ_i are real and ordered via

$$\sigma_1 > \sigma_2 > \cdots > \sigma_p > 0 \tag{7.61}$$

(see, for instance, Golub and Van Loan, 1996). The numbers σ_i are called the *singular values* of matrix A. The singular values of matrix A are the nonnegative square roots of the eigenvalues of the symmetric positive definite matrix $A^T A$. The vectors $\mathbf{u}_i$ consisting of the columns of matrix U are called the *left singular vectors* of A. Likewise, the columns of V, denoted by $\mathbf{v}_i$, are called the *right singular vectors* of A. The process of calculating U, V,

and diag$[\sigma_1 \cdots \sigma_p]$ is called the *singular value decomposition* (denoted by SVD) of matrix A. Note that the singular values and vectors satisfy

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i, \qquad i = 1, \ldots, p \tag{7.62}$$

$$A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i, \qquad i = 1, \ldots, p \tag{7.63}$$

Note that if A is a square symmetric positive definite matrix, then $U = V$, and the $\mathbf{u}_i$ are the eigenvectors of matrix A and the singular values of A are identical to the eigenvalues of A^2.

The real square symmetric semipositive definite matrix A is, of course, singular, or rank deficient, if and only if it has a zero eigenvalue (or singular value). This statement leads naturally to the idea that the size of the singular values of a matrix quantify how close the matrix is to being singular. If the smallest singular value, σ_n, is well away from zero, then the matrix is of full rank and 'far away' from being singular (note: unlike frequencies, we order singular values from largest to smallest).

Applying the idea of singular values as a measure of rank deficiency to the controllability and observability grammians yields a systematic model reduction method. The matrices W_{O} and W_{C} are symmetric and hence are similar to a diagonal matrix. Moore (1981) showed that there always exists an equivalent system for which these two grammians are both equal and diagonal. Such a system is then called *balanced.*

In addition, Moore showed that W_{C} and W_{O} must satisfy the two Liapunov-type equations

$$\begin{aligned} AW_{\mathrm{C}}^2 + W_{\mathrm{C}}^2 A^T &= -BB^T \\ A^T W_{\mathrm{O}}^2 + W_{\mathrm{O}}^2 A &= -C^T C \end{aligned} \tag{7.64}$$

for asymptotically stable systems.

Let the matrix P denote a linear similarity transformation, which when applied to Equations (7.1) and (7.2) yields the equivalent system

$$\begin{aligned} \dot{\mathbf{x}}' &= A'\mathbf{x}' + B'\mathbf{u} \\ \mathbf{y} &= C'\mathbf{x}' \end{aligned} \tag{7.65}$$

These two equivalent systems are related by

$$\mathbf{x} = P\mathbf{x}' \tag{7.66}$$

$$A' = P^{-1}AP \tag{7.67}$$

$$B' = P^{-1}B \tag{7.68}$$

$$C' = CP \tag{7.69}$$

Here, matrix P can be chosen such that the new grammians defined by

$$W'_{\mathrm{C}} = P^{-1} W_{\mathrm{C}} P \tag{7.70}$$

and

$$W'_{\mathrm{O}} = P^{-1} W_{\mathrm{O}} P \tag{7.71}$$

are equal and diagonal. That is,

$$W'_{\mathrm{C}} = W'_{\mathrm{O}} = \Lambda_{\mathbf{W}} = \mathrm{diag}[\sigma_1 \ \sigma_2 \cdots \sigma_{2n}] \tag{7.72}$$

where the numbers σ_i are the singular values of the grammians and are ordered such that

$$\sigma_i > \sigma_{i+1}, \qquad i = 1, 2, \ldots, 2n-1 \tag{7.73}$$

Under these circumstances, i.e., when Equations (7.72) and (7.73) hold, the system given by Equations (7.65) is said to be *internally balanced*.

Next, let the state variables in the balanced system be partitioned into the form

$$\begin{bmatrix} \dot{\mathbf{x}}'_1 \\ \dot{\mathbf{x}}'_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_1 \\ \mathbf{x}'_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u} \tag{7.74}$$

$$\mathbf{y} = [C_1 \quad C_2] \begin{bmatrix} \mathbf{x}'_2 \\ \mathbf{x}'_1 \end{bmatrix} \tag{7.75}$$

where A_{11} is a $k \times k$ matrix and $\mathbf{x}'_2$ is the vector containing those states corresponding to the $(2n-k)$ smallest singular values of W_{C}. It can be shown (Moore, 1981) that the $\mathbf{x}'_2$ part of the state vector for Equations (7.74) and (7.75) affects the output much less than $\mathbf{x}'_1$ does. Thus, if σ_k is much greater than σ_{k+1}, i.e., $\sigma_k >> \sigma_{k+1}$, the $\mathbf{x}'_2$ part of the state vector does not affect the input–output behavior of the system as much as $\mathbf{x}'_1$ does.

The preceding comments suggest that a suitable low-order model of the system of Equations (7.1) and (7.2) is the subsystem given by

$$\dot{\mathbf{x}}'_1 = A_{11}\mathbf{x}'_1 + B_1\mathbf{u} \tag{7.76}$$

$$\mathbf{y} = C_1\mathbf{x}'_1 \tag{7.77}$$

This subsystem is referred to as a *reduced-order model* (often referred to as ROM). Note that, as pointed out by Moore (1981), a realization of Equations (7.1) and (7.2) should yield the reduced model of equations (7.76) and (7.77).

The reduced-order model can be calculated by first calculating an intermediate transformation matrix P_1 based on the controllability grammian. Solving the eigenvalue problem for W_{C} yields

$$W_{\mathrm{C}} = V_{\mathrm{C}} \Sigma_{\mathrm{C}}^2 V_{\mathrm{C}}^T \tag{7.78}$$

where V_C is the matrix of normalized eigenvectors of W_C and $\mathbf{\Sigma}_C^2$ is the diagonal matrix of eigenvalues of W_C. The square on $\mathbf{\Sigma}_C$ is a reminder that W_C is positive definite. Based on this decomposition, a nonsingular transformation P_1 is defined by

$$P_1 = V_C \mathbf{\Sigma}_C \tag{7.79}$$

Application of the transformation P_1 to the state equations yields the intermediate state equations defined by

$$A'' = P_1^{-1} A P_1 \tag{7.80}$$

$$B'' = P_1^{-1} B \tag{7.81}$$

$$C'' = C P_1 \tag{7.82}$$

To complete the balancing algorithm, these intermediate equations are balanced with respect to W_O''. That is, the eigenvalue problem for W_O'' yields the matrices V_O'' and $\mathbf{\Sigma}_O''^{\,2}$ such that

$$W_O'' = V_O'' \mathbf{\Sigma}_O''^{\,2} V_O''^{\,T} \tag{7.83}$$

These two matrices are used to define the second part of the balancing transformation, i.e.,

$$P_2 = V_O'' \mathbf{\Sigma}_O''^{\,-1/2} \tag{7.84}$$

The balanced version of the original state equations is then given by the product transformation $P = P_1 P_2$ in Equation (7.65). They are

$$A' = P_2^{-1} P_1^{-1} A P_1 P_2 \tag{7.85}$$

$$B' = P_2^{-1} P_1^{-1} B \tag{7.86}$$

$$C' = C P_1 P_2 \tag{7.87}$$

The balanced system is then used to define the reduced-order model of Equations (7.76) and (7.77) by determining the value of k such that $\sigma_k >> \sigma_{k+1}$. The following example illustrates an internally balanced reduced-order model.

Example 7.7.1

Consider the two-degree-of-freedom system of Figure 2.4 with $m_1 = m_2 = 1$, $c_1 = 0.2$, $c_2 = 0.1$, and $k_1 = k_2 = 1$. Let an impulse force be applied to m_2 and assume a position measurement of m_2 is available. The state matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.3 & 0.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix}$$

In addition, B becomes the vector $\mathbf{b} = [0\ 0\ 0\ 1]^T$, and the output matrix C becomes the vector $\mathbf{c}^T = [0\ 1\ 0\ 0]$.

The controllability and observability grammians can be calculated (for $t \to \infty$) from Equation (7.64) to be

$$W_C = \begin{bmatrix} 4.0569 & 6.3523 & 0.0000 & -0.3114 \\ 6.3523 & 10.5338 & 0.3114 & 0.0000 \\ 0.0000 & 0.3114 & 1.7927 & 2.2642 \\ -0.3114 & 0.0000 & 2.2642 & 4.1504 \end{bmatrix}$$

$$W_O = \begin{bmatrix} 1.8198 & 2.2290 & 0.5819 & 1.1637 \\ 2.2290 & 4.2315 & -0.8919 & 0.4181 \\ 0.5819 & -0.0819 & 4.0569 & 6.3523 \\ 1.1637 & 0.4181 & 6.3523 & 10.5338 \end{bmatrix}$$

Calculation of the matrix P that diagonalizes the two grammians yields

$$P = \begin{bmatrix} 0.4451 & -0.4975 & 0.4962 & -0.4437 \\ 0.7821 & -0.7510 & -0.2369 & 0.3223 \\ 0.2895 & 0.2895 & 0.6827 & 0.8753 \\ 0.4419 & 0.5112 & -0.4632 & -0.4985 \end{bmatrix}$$

The singular values of W_O and W_C are then $\sigma_1 = 9.3836$, $\sigma_2 = 8.4310$, $\sigma_3 = 0.2724$, and $\sigma_4 = 0.2250$. From examination of these singular values, it appears that the coordinates x_1 and x_2 associated with A' of Equation (7.85) are likely candidates for a reduced-order model (i.e., $k = 1$ in this case). Using Equations (7.85) through (7.87) yields the balanced system given by

$$A' = \begin{bmatrix} -0.0326 & 0.6166 & 0.0192 & -0.0275 \\ -0.6166 & -0.0334 & -0.0218 & 0.0280 \\ 0.0192 & 0.0218 & -0.1030 & 1.6102 \\ 0.0275 & 0.0280 & 1.6102 & -0.2309 \end{bmatrix}$$

$$B'^{\mathrm{T}} = \begin{bmatrix} 0.7821 & 0.7510 & -0.2369 & -0.03223 \end{bmatrix}^T$$

$$C' = \begin{bmatrix} 0.7821 & -0.7510 & -0.2369 & 0.3223 \end{bmatrix}$$

Given that $k = 2$ (from examination of the singular values), the coefficients in Equations (7.76) and (7.77) for the reduced-order model become

$$A_{11} = \begin{bmatrix} -0.0326 & 0.6166 \\ -0.6166 & -0.0334 \end{bmatrix}$$

$$B_1 = [0.7821 \quad 0.7510]^T$$

$$C_1 = [0.7821 \quad -0.7510]$$

Plots of the response of the full-order model and the balanced model are given in Figures 7.12 and 7.13 respectively. Note in Figure 7.13 that the two coordinates (x_3' and x_4') neglected in the reduced-order model do not contribute as much to the response and, in fact, die out after about 15 time units. However, all the coordinates in Figure 7.12 and x_1' and x_2' are still vibrating after 15 time units.

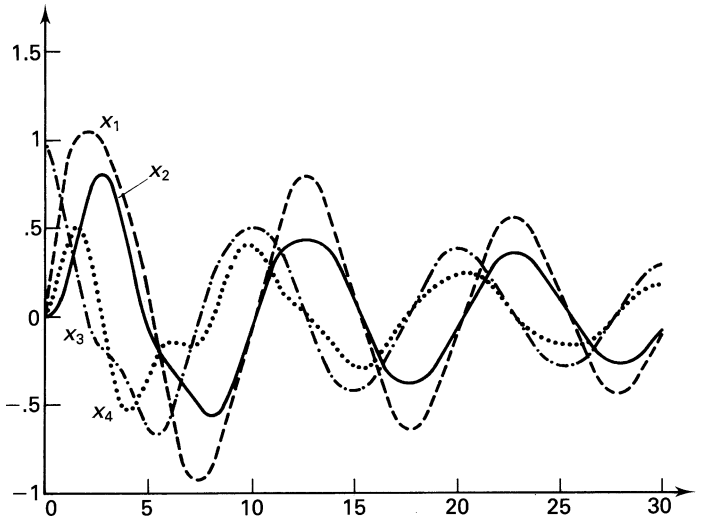

Figure 7.12 Response of the original state variables.

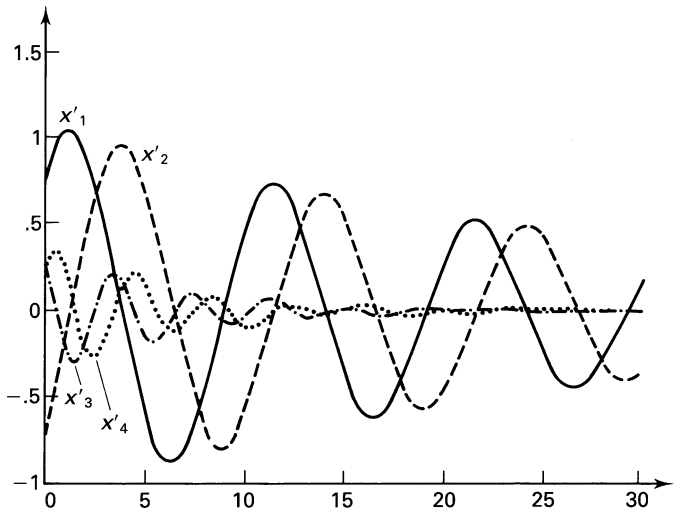

Figure 7.13 Response of the balanced state variables.

Note that the balanced reduced-order model will change if different inputs and outputs are considered, as the B and C matrices would change, altering the reduction scheme.

7.8 MODAL CONTROL IN STATE SPACE

In general, modal control refers to the procedure of decomposing the dynamic equations of a structure into modal coordinates, such as Equations (5.29) and (5.37), and designing

the control system in this modal coordinate system. In broad terms, any control design that employs a modal description of the structure is called a modal control method. Modal control works well for systems in which certain (just a few) modes of the structure dominate the response. That is, modal control works well for systems in which only a few of the modal participation factors (see Section 3.3) are large and the rest are relatively small. Modal control can be examined in either state space, via Equations (7.1) and (7.2), or in physical space, via Equations (4.23) through (4.25). This section examines modal control in the state-space coordinate system, and the following section examines modal control in the physical coordinate system.

Consider the state-space description of Equations (7.1) and (7.2). If matrix A has a diagonal Jordan form, then – following Section 5.3 – there exists a nonsingular matrix U such that

$$U^{-1}AU = \Lambda$$

where Λ is a diagonal matrix of the eigenvalues of the state matrix A. Note that the diagonal elements of Λ will be complex if the system is underdamped. Substituting $\mathbf{x} = U\mathbf{z}$ into Equation (7.1) and premultiplying by the inverse of the nonsingular matrix U yields the diagonal system

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + U^{-1}B\mathbf{u} \tag{7.88}$$

Here, the vector $\mathbf{z}$ is referred to as the *modal coordinate system*. In this form, the controllability problem and the pole placement problem become more obvious.

Consider first the controllability question for the case of a single input (i.e., $\mathbf{u}$ becomes a scalar). Then B is a vector $\mathbf{b}$ and $U^{-1}\mathbf{b}$ is a vector consisting of $2n$ elements denoted by b_i. Clearly, this system is controllable if and only if each $b_i \neq 0$, $i = 1, 2, \ldots, 2n$. If, on the other hand, b_i should happen to be zero for some index i, then the system is not controllable. With $b_i = 0$ the ith mode is not controllable, as no feedback law could possibly affect the ith mode. Thus, the vector $U^{-1}\mathbf{b}$ indicates the controllability of the system by inspection.

The form of Equation (7.88) can also be used to perform a quick model reduction (see problem 7.28). Suppose it is desired to control just the fastest modes or modes in a certain frequency range. Then these modes of interest can be taken as the reduced-order model and the others can be neglected.

Next, consider the pole placement problem. In modal form, it seems to be a trivial matter to choose the individual modal control gains b_i to place the eigenvalues of the system. For instance, suppose output feedback is used. Then $u = \mathbf{c}^T U\mathbf{z}$, and Equation (7.88) becomes

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + U^{-1}\mathbf{b}\mathbf{c}^T U\mathbf{z} = (\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U)\mathbf{z} \tag{7.89}$$

where $\mathbf{c}^T$ is a row vector. The closed-loop system then has poles determined by the matrix $(\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U)$. Suppose that the matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ is also diagonal. In this case the controls are also decoupled, and Equation (7.89) becomes the $2n$ decoupled equations

$$\dot{z}_i = (\lambda_i + u_i)z_i, \qquad i = 1, 2, \ldots, 2n \tag{7.90}$$

Here, u_i denotes the diagonal elements of the diagonal matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$. Note that the vector $U^{-1}\mathbf{b}\mathbf{c}^T U\mathbf{z}$ in Equation (7.89) is identical to the vector $U^{-1}\mathbf{f}$ in Equation (5.20). The

difference between the two equations is simply that in this section the forcing function is the result of a control force manipulated by a designer to achieve a desired response. In the development of Section 5.3, the forcing term represents some external disturbance.

The matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ in Equation (7.89) may not be diagonal. In this case, Equation (7.89) is not decoupled. The controls in this situation reintroduce coupling into the system. As a result, if it is desired to change a particular modal coordinate, the control force chosen to change this mode will also change the eigenvalues of some of the other modes. Following the arguments of Section 3.5, a necessary and sufficient condition for $U^{-1}\mathbf{b}\mathbf{c}^T U$ to be diagonal is for the matrix $\mathbf{b}\mathbf{c}^T$ to commute with the state matrix A. The remainder of this section is devoted to discussing the coupling introduced by control laws and measurement points in the case where it is desired to control independently a small number of modes of a given structure.

Suppose, then, that it is desired to control independently a small number of modes. For example, it may be desired to place a small number of troublesome poles while leaving the remaining poles unaffected. Let k denote the number of modes that are to be controlled independently of the remaining $2n-k$ modes. Furthermore, assume that it is the first k modes that are of interest, i.e., the desired modes are the lowest k. Then, partition the modal equations with state feedback into the k modal coordinates that are to be controlled and the $2n-k$ modal coordinates that are to be left undisturbed. This yields

$$\begin{bmatrix} \dot{\mathbf{z}}_k \\ \dot{\mathbf{z}}_{2n-k} \end{bmatrix} = \left(\begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_{2n-k} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_k \mathbf{c}_k^T & \mathbf{b}_k \mathbf{c}_{2n-k}^T \\ \mathbf{b}_{2n-k} \mathbf{c}_k^T & \mathbf{b}_{2n-k} \mathbf{c}_{2n-k}^T \end{bmatrix} \right) \begin{bmatrix} \mathbf{z}_k \\ \mathbf{z}_{2n-k} \end{bmatrix} \tag{7.91}$$

Here, the matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ is partitioned into blocks defined by

$$U^{-1}\mathbf{b} = \begin{bmatrix} \mathbf{b}_k \\ \mathbf{b}_{2n-k} \end{bmatrix}$$

$$\mathbf{c}^T U = [\mathbf{c}_k^T \quad \mathbf{c}_{2n-k}^T]$$

where $\mathbf{b}_k$ denotes the first k elements of the vector $\mathbf{b}$, $\mathbf{b}_{2n-k}$ denotes the last $2n-k$ elements, and so on. Likewise, the matrices Λ_k and Λ_{2n-k} denote diagonal matrices of the first k eigenvalues and the last $2n-k$ eigenvalues respectively. Let b_i denote the elements of the vector $U^{-1}\mathbf{b}$ and c_i denote the elements of the vector $\mathbf{c}^T U$. Then, $\mathbf{b}_k \mathbf{c}_k^T$ is the $k \times k$ matrix

$$\mathbf{b}_k \mathbf{c}_k^T = \begin{bmatrix} b_1 c_1 & b_1 c_2 & \cdots & b_1 c_k \\ b_2 c_1 & b_2 c_2 & \cdots & b_2 c_k \\ \vdots & \vdots & & \vdots \\ b_k c_1 & b_k c_2 & \cdots & b_k c_k \end{bmatrix} \tag{7.92}$$

Examination of Equation (7.91) illustrates that the first k modes of the system can be controlled independently of the last $(2n-k)$ modes if and only if the two vectors $\mathbf{b}_{2n-k}$ and $\mathbf{c}_{2n-k}$ are both zero. Furthermore, the first k modes can be controlled independently of each other only if Equation (7.92) is diagonal. This, of course, cannot happen, as clearly indicated by setting the off-diagonal terms of Equation (7.92) to zero. Takahashi *et al.* (1968) discussed decoupled or ideal control in detail. In general, it is difficult to control modes independently, unless a large number of actuators and sensors are used. The vector $\mathbf{c}_{2n-k}^T$ indicates the coupling introduced into the system by measurement, and the vector $\mathbf{b}_{2n-k}$

indicates coupling due to control action. This phenomenon is known as *observation spillover* and *control spillover* (Balas, 1978).

Example 7.8.1

Consider an overdamped two-degree-of-freedom structure with the state matrix given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & -9 & 4 \\ 1 & -1 & 4 & -4 \end{bmatrix}$$

Solving the eigenvalue problem for A yields the matrices Λ, U, and U^{-1} given by

$$\Lambda = \begin{bmatrix} -10.9074 & 0 & 0 & 0 \\ 0 & -1.2941 & 0 & 0 \\ 0 & 0 & -0.5323 & 0 \\ 0 & 0 & 0 & -0.2662 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.0917 & -0.4629 & 0.7491 & -0.0957 \\ 0.0512 & -0.7727 & 1.0000 & 1.0000 \\ 1.0000 & 0.5990 & -0.3988 & 0.0255 \\ -0.5584 & 1.0000 & 0.5323 & -0.2662 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} 0.2560 & -0.1121 & 0.7846 & -0.4381 \\ 0.7382 & 0.3741 & 0.7180 & 1.1986 \\ 1.6791 & 0.3378 & 0.5368 & 0.7167 \\ -1.1218 & 0.9549 & -0.0222 & 0.2320 \end{bmatrix}$$

which constitutes a modal decomposition of the state matrix (note that $A = U\Lambda U^{-1}$). Suppose next that $\mathbf{b} = [0\ 0\ 1\ 0]^T$ and $\mathbf{c}^T = [0\ 0\ 1\ 1]$ are the control and measurement vectors respectively. In the decoupled coordinate system, the control and measurement vectors become

$$U^{-1}\mathbf{b} = \begin{bmatrix} 0.7846 & 0.7180 & 0.5368 & -0.0222 \end{bmatrix}^T$$

and

$$\mathbf{c}^T U = \begin{bmatrix} 0.44161 & 0.5990 & -0.9311 & -0.2407 \end{bmatrix}$$

Notice that these vectors are fully populated with nonzero elements. This leads to the fully coupled closed-loop system given by Equation (7.89), since the term $U^{-1}\mathbf{b}\mathbf{c}^T U$ becomes

$$U^{-1}\mathbf{b}\mathbf{c}^T U = \begin{bmatrix} 0.3465 & 1.2545 & -0.7305 & -0.1888 \\ 0.3171 & 1.1481 & -0.6685 & -0.1728 \\ 0.2371 & 0.8584 & -0.4999 & -0.1292 \\ 0.0098 & -0.0355 & 0.0207 & 0.0053 \end{bmatrix}$$

This last expression illustrates the recoupling effect caused by the control and measurement locations. Note that the matrix $\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U$ of Equation (7.89) is not diagonal but fully populated, recoupling the dynamics.

7.9 MODAL CONTROL IN PHYSICAL SPACE

As indicated in Section 3.4, the left-hand side of Equation (4.23) can be decoupled into modal coordinates if and only if $DM^{-1}K = KM^{-1}D$. This was used in Section 5.4 to decouple Equation (5.35) into modal coordinates to solve for the forced response. A similar approach is taken here, except that the input force $S_m^T\mathbf{f}$ of Equation (5.37) becomes the control force $S_m^T B_f \mathbf{u}$. As in the state space, the difficulty lies in the fact that the transformation, S_m, of the equations of motion into modal coordinates, $\mathbf{z}$, does not necessarily decouple the control input $S_m^T B_f \mathbf{u}$. (Note that in this section $\mathbf{z}$ is an $n \times 1$ vector of modal positions, whereas in the previous section $\mathbf{z}$ is a $2n \times 1$ vector of modal state variables.) The control problem in second-order physical coordinates transformed into modal coordinates is, in the notation of Sections 4.10 and 5.4,

$$I\ddot{\mathbf{z}} + \Lambda_D \dot{\mathbf{z}} + \Lambda_K \mathbf{z} = S_m^T B_f \mathbf{u} \tag{7.93}$$

$$\mathbf{y} = C_p S_m \mathbf{z} + C_v S_m \dot{\mathbf{z}} \tag{7.94}$$

Here, the dimensions of the matrices C_p and C_v are as indicated in Equation (4.25). Note that in Equation (7.93) the relative magnitudes of the elements $(S_m^T B_f \mathbf{u})_i$ are an indication of the degree of controllability for the ith mode. If $(S_m^T B_f \mathbf{u})_i$ is very small, the ith mode is hard to control. If it happens to be zero, then the ith mode is not controllable. If, on the other hand, $(S_m^T B_f \mathbf{u})_i$ is relatively large, the ith mode is very controllable.

If output feedback of the form suggested in Equation (4.26) is used, then $\mathbf{u} = -G_f \mathbf{y}$. Combining Equation (7.93) with Equation (7.94) and the control law, the system equation becomes

$$I\ddot{\mathbf{z}} + \Lambda_D \dot{\mathbf{z}} + \Lambda_K \mathbf{z} = -S_m^T B_f K_p S_m \mathbf{z} - S_m^T B_f K_v S_m \dot{\mathbf{z}} \tag{7.95}$$

where $K_p = G_f C_p$ and $K_v = G_f C_v$ represent measurement positions and B_f is taken to represent the control actuator locations. Note that, by this choice of $\mathbf{u}$, the discussion is now restricted to state variable feedback, i.e., position and velocity feedback. Equation (7.95) can be rewritten as

$$I\ddot{\mathbf{z}} + (\Lambda_D + S_m^T B_f K_v S_m)\dot{\mathbf{z}} + (\Lambda_K + S_m^T B_f K_p S_m)\mathbf{z} = \mathbf{0} \tag{7.96}$$

Now, Equation (7.96) is posed for modal pole placement control.

To cause the closed-loop system of Equation (7.96) to have the desired eigenvalues and, hence, a desired response, the control gain matrix B_f and the measurement matrices K_v and K_p must be chosen (a design process) appropriately. If, in addition, it is desired to control each mode independently, i.e., each z_i in Equation (7.96), then further restrictions must be satisfied. Namely, it can be seen from Equation (7.96) that an independent control of each mode is possible if and only if the matrices $S_m^T B_f K_v S_m$ and $S_m^T B_f K_p S_m$ are both diagonal. In the event where the matrix $B_f K_v$ and the matrix $B_f K_p$ are both symmetric, this will be true if and only if

$$B_f K_v M^{-1} D = D M^{-1} B_f K_v \tag{7.97}$$

and

$$B_f K_p M^{-1} K = K M^{-1} B_f K_p \tag{7.98}$$

Unfortunately, this puts *very* stringent requirements on the location and number of sensors and actuators. If, however, the conditions given by Equations (7.97) and (7.98) are satisfied, Equation (7.96) reduces to n single-degree-of-freedom control problems of the form

$$\ddot{z}_i + (2\zeta_i \omega_i + \alpha_i)\dot{z}_i + (\omega_i^2 + \beta_i) z_i = 0 \tag{7.99}$$

where α_i and β_i are the diagonal elements of the matrices $S_m^T B_f K_v S_m$ and $S_m^T B_f K_p S_m$ respectively. This last expression represents an independent set of equations that can be solved for α_i and β_i given desired modal response information. That is, if it is desired that the closed-loop system have a first-mode natural frequency of 10, for instance, then Equation (7.99) requires that $\omega_1^2 + \beta_1 = 10$, or $\beta_1 = 10 - \omega_1^2$. If the known open-loop system has $\omega_1^2 = 6$, then $\beta_1 = 4$ is the desired control gain.

While the modal control equations [Equations (7.99)] appear quite simple, the problem of independent control remains complicated and requires a larger number of sensor and actuator connections. This happens because the α_i and β_i of Equation (7.99), while independent, are not always capable of being independently implemented. The design problem is to choose actuator and sensor locations as well as gains such that B, K_v, and K_p satisfy Equations (7.97) and (7.98). The choice of these gains is not independent but rather coupled through the equations

$$S_m^T B_f K_v S_m = \mathrm{diag}[\alpha_i] \tag{7.100}$$

and

$$S_m^T B_f K_p S_m = \mathrm{diag}[\beta_i] \tag{7.101}$$

The modal control of a system in physical coordinates as well as the problem of performing independent modal control are illustrated in the following example.

Example 7.9.1

Consider the system of Figure 2.4 with coefficient values (dimensionless) of $m_1 = 9$, $m_2 = 1$, $c_1 = 8$, $c_2 = 1$, $k_1 = 24$, and $k_2 = 3$. This produces a system equivalent to the one of example 3.5.1, where it is shown that $DM^{-1}K = KM^{-1}D$, so that the system decouples. The matrix S_m is given by

$$S_m = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

The decoupled coordinates, listed in example 3.5.1, yield the modal frequencies and damping ratios as

$$\zeta_1 = 0.2357, \qquad \omega_1 = \sqrt{2} = 1.414 \text{ rad/s}$$
$$\zeta_2 = 0.3333, \qquad \omega_2 = 2.0 \text{ rad/s}$$

Consider the control problem of calculating a feedback law that will cause the closed-loop system (7.95) to have a response with a modal damping ratio of 0.4 in the first mode. Three cases of different sensor and actuator placements are considered.

In the first case, consider an SISO system with non-collocated control. Suppose one actuator is used to achieve the desired control and it is connected to only one mass, m_2, and one sensor is used to measure the velocity of m_1. Then

$$K_v = [g_1 \quad 0], \qquad K_p = \mathbf{0}$$
$$B_f = \begin{bmatrix} 0 \\ g_2 \end{bmatrix}$$

since u is a scalar in this case. Calculating the control and measurement quantities from Equation (7.96) yields

$$S_m^T B_f K_p S_m = 0$$
$$S_m^T B_f K_v S_m = \frac{g_1 g_2}{6} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

It should be clear from this last expression that no choice of g_1 and g_2 will allow just the first mode damping to be changed, i.e., $S_m^T B_f K_v S_n$ cannot be diagonal.

For the second case, consider using two actuators, one at each mass, and two sensors measuring the two velocities. This is a MIMO system with velocity feedback. Then

$$K_v = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

and

$$B_f = \begin{bmatrix} g_3 & 0 \\ 0 & g_4 \end{bmatrix}$$

Equation (7.96) then yields

$$S_m^T B_f K_v S_m = \frac{1}{18} \begin{bmatrix} g_1 g_3 + 9 g_2 g_4 & 9 g_2 g_4 - g_1 g_3 \\ 9 g_2 g_4 - g_1 g_3 & g_1 g_3 + 9 g_2 g_4 \end{bmatrix}$$

An obvious choice for decoupled control is $9 g_2 g_4 = g_1 g_3$, as this makes the above matrix diagonal. Then

$$S_m^T B_f K_v S_m = \begin{bmatrix} g_2 g_4 & 0 \\ 0 & g_2 g_4 \end{bmatrix}$$

Comparing the desired first mode damping ratio yields

$$2\zeta_1 \omega_1 + g_2 g_4 = 2(0.4)\sqrt{2}$$

Solving this equation for the two unknown $a = g_2 g_4$ yields $g_2 g_4 = 0.164$. This choice of the gain keeps the equations of motion decoupled and assigns the first damping ratio the desired value of 0.4. Unfortunately, because of the way the gains appear in closed-loop system, the effect on the second mode velocity coefficient is

$$2\zeta_2\omega_2 + g_2 g_4 = 2\hat{\zeta}_2\omega_2$$

This results in a new damping ratio for mode 2 of $\hat{\zeta}_2 = 0.449$. Hence, although a decoupled control has been found, it is still not possible independently to change the damping ratio of mode 1 without affecting mode 2. This would require $g_1 g_3 + 9 g_2 g_4$ to be zero, which of course would also not allow ζ_1 to be controlled at all.

For the third case, then consider a MIMO controller with the velocity signal at m_1 to be fed back to mass m_2, and vice versa. This means that K_v now has the form

$$K_v = \begin{bmatrix} g_1 & g_5 \\ g_6 & g_2 \end{bmatrix}$$

In this case, calculating the velocity feedback coefficient yields

$$S_m^T B_f K_v S_m = \frac{1}{18}\begin{bmatrix} g_3 g_1 + 3 g_3 g_5 + 9 g_2 g_4 + 3 g_4 g_6 & 9 g_2 g_4 + 3 g_3 g_5 - 3 g_4 g_6 - g_3 g_1 \\ 9 g_2 g_4 - 3 g_3 g_5 + 3 g_4 g_6 - g_3 g_1 & 9 g_2 g_4 + g_3 g_1 - 3 g_4 g_6 - 3 g_3 g_5 \end{bmatrix}$$

Examining this new feedback matrix shows that *independent* modal control will be possible if and only if the off-diagonals are set to zero, decoupling the system, and the element in the (2, 2) position is zero, leaving the second mode unchanged. To change the first mode-damping ratio to 0.4, the first entry in the matrix above requires that

$$2\zeta_1\omega_1 + \frac{g_1 g_3 + 3 g_3 g_5 + 9 g_2 g_4 + 3 g_4 g_6}{18} = 2(0.4)\omega_1$$

To ensure that the controller does not recouple the equations of motion, the off-diagonal elements are set to zero, resulting in the two equations

$$9 g_2 g_4 + 3 g_3 g_5 - 3 g_4 g_6 - g_3 g_1 = 0 \qquad \text{and} \qquad 9 g_2 g_4 - 3 g_3 g_5 + 3 g_4 g_6 - g_3 g_1 = 0$$

To ensure that the second mode damping is not changed, the element in the (2, 2) position is also set to zero, resulting in the additional condition

$$9 g_2 g_4 + g_3 g_1 - 3 g_4 g_6 - 3 g_3 g_5 = 0$$

This last set of equations can be recognized as four linear equations in the four unknowns

$$a = g_1 g_3, \qquad b = g_2 g_4, \qquad c = g_3 g_5, \qquad d = g_4 g_6$$

These equations in matrix form are

$$\begin{aligned} -a + 9b + 3c + 3d &= 0 \\ -a + 9b - 3c + 3d &= 0 \\ a + 9b - 3c - 3d &= 0 \\ \frac{a}{18} + \frac{1}{2}b + \frac{1}{6}c + \frac{1}{6}d &= 2(0.4)\omega_1 - 2\zeta_1\omega_1 \end{aligned}$$

Solving this set of equations with the open-loop values of ω_1 and ζ_1 given above yields

$$a = 2.09, \qquad b = 0.232, \qquad c = 0.697, \qquad d = 0.697$$

Substitution of these values along with the free choice of $g_3 = g_4 = 1$ into the feedback matrix yields

$$S_m^T B_f K_v S_m = \begin{bmatrix} 0.464 & 0 \\ 0 & 0 \end{bmatrix}$$

The resulting closed-loop system yields a new system with the desired first mode damping ratio of 0.4 and the second mode damping ratio unchanged. Furthermore, the closed-loop system remains decoupled. Hence, independent mode control is achieved.

Note that, of the six gain values, only four can be determined. In fact, B_f could have been chosen as the identity matrix from the start with the same result. However, in practice, each sensor and actuator will have a coefficient that needs to be accounted for. In many cases each sensor and actuator will have significant dynamics, ignored here, that could change the plant by adding additional poles and zeros.

Example 7.9.1 illustrates how many control actuators and sensors must be used in order to accomplish an independent control of one mode of a simple two-degree-of-freedom system. In the case of the last example, as many sensors and actuators were required as degrees of freedom in the system. Hence, it is important in practical control design to consider the placement of actuators and sensors and not just the relationships among the various coefficient matrices for the system.

7.10 ROBUSTNESS

The concept of robust control systems, or robust systems, has been defined in many ways, not all of which are consistent. However, the basic idea behind the concept of robustness is an attempt to measure just how stable a given system is in the presence of some uncertainty or perturbation in the system. That is, if a system is stable, is it still stable after some changes have been made in the physical or control parameters of the system? This is called *stability robustness*. This same question can be asked with respect to the performance of the system. In this latter case, the question is asked in terms of a given level of acceptability of a specific performance criterion such as overshoot or settling time. For example, if it is required that a given control system have an overshoot of less than 10% and there is an uncertainty in the control gain, does the overshoot still remain less than 10% in the presence of that uncertainty? This is called *performance robustness*.

An example of performance robustness is given by Davison (1976). The steady state error of a control system is defined to be the difference between the response of the system and the desired response of the system as t becomes large in the regulator problem of Section 7.4. A logical measure of control system performance is then whether or not the steady state error is zero. A given system is then said to be *robust* if there exists a control that regulates the system with zero steady state error when subjected to perturbations in any of the matrices A, B, or C in Equations (7.1) and (7.2).

In the remainder of this section, only stability robustness is discussed. The approach presented here follows that of Patel and Toda (1980). As an example of stability robustness, consider a closed-loop system under state variable feedback of the form

$$\dot{\mathbf{x}} = (A + BK_f)\mathbf{x} = A'\mathbf{x} \tag{7.102}$$

The matrix A' contains the measurement and control matrices and is such that the closed-loop system is asymptotically stable. Let the matrix E_e denote the uncertainty in the system parameters (A) and gains (BK_f). Then, rather than having Equation (7.102), the system may be of the form

$$\dot{\mathbf{x}} = (A' + E_e)\mathbf{x} \tag{7.103}$$

In this equation the matrix E_e is not known [recall the discussion following Equation (3.93)], but rather only bounds on its elements are known. In particular, it is assumed here that each element of the uncertainty is bounded in absolute value by the same number, ε, so that

$$\left|(E_e)_{\mathrm{ij}}\right| < \varepsilon \tag{7.104}$$

for each value of the indices i and j. It was shown by Patel and Toda (1980) that the system with the uncertainty given by Equation (7.104) is asymptotically stable if

$$\varepsilon < \mu_\varepsilon$$

where

$$\mu_\varepsilon = \frac{1}{2n}\left(\frac{1}{\sigma_{\max}[F]}\right) \tag{7.105}$$

Here, F is a solution of the Lyapunov matrix equation [Equation (4.29)] in the form

$$A'^T F + FA' = -2I \tag{7.106}$$

where I is the $2n \times 2n$ identity matrix. The notation $\sigma_{\max}[F]$ refers to the largest singular value of the matrix F.

Example 7.10.1

Consider a closed-loop system defined by the augmented state matrix given by

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & -9 & 4 \\ 1 & -1 & 4 & -4 \end{bmatrix}$$

the solution of the Lyapunov equation [Equation (7.106)] yields

$$F = \begin{bmatrix} 3.5495 & -0.6593 & 0.3626 & 0.0879 \\ 0.6593 & 4.6593 & 0.6374 & 1.6374 \\ 0.3626 & 0.6374 & 0.3956 & 0.5495 \\ 0.0879 & 1.6374 & 0.5495 & 1.2088 \end{bmatrix}$$

The singular values of F are calculated to be

$$\sigma_1 = 5.5728, \qquad \sigma_2 = 3.5202, \qquad \sigma_3 = 0.6288, \qquad \sigma_4 = 0.0914$$

From Equation (7.105) the value of μ_ε becomes $(n = 2)$

$$\mu_\varepsilon = \left(\frac{1}{4}\right)\left(\frac{1}{5.5728}\right)$$

Hence, as long as the parameters are not changed more than 1.3676, i.e., as long as

$$\left|(E_e)_{\mathrm{ij}}\right| < 1.3676$$

the system defined by $(A' + E_e)$ will be asymptotically stable.

Many other formulations and indices can be used to discuss stability robustness. For instance, Rew and Junkins (1986) use the sensitivity (see Section 6.5) of the closed-loop matrix A' in Equation (7.102) to discuss robustness in terms of the condition number of A'. Kissel and Hegg (1986) have discussed stability robustness with respect to neglected dynamics in control system design. They examine how stability of a closed-loop system is affected if a reduced-order model is used in designing the feedback law. Zhou and Doyle (1997) provide a complete account of the use of robustness principles in control design for both stability and performance.

7.11 POSITIVE POSITION FEEDBACK

A popular modal control method with experimentalists and structural control engineers is called *positive position feedback* (Goh and Caughey, 1985), which adds additional dynamics to the system through the control law. A unique feature of the positive position feedback (PPF) approach is that it can be designed around an experimental transfer function of the structure and does not require an analytical model of the system or plant to be controlled. Goh and Caughey proposed using a special dynamic feedback law designed specifically for use in the second-order form, compatible with Newton's formulation of the equations of motion of a structure. These PPF control circuits are designed to roll off at higher frequency and hence are able to avoid exciting residual modes and introducing spillover as discussed in Section 7.8.

To illustrate the PPF formulation, consider the single-degree-of-freedom system (or alternatively, a single mode of the system)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = bu \tag{7.107}$$

where ζ and ω_n are the damping ratio and natural frequency of the structure, and b is the input coefficient that determines the level of force applied to the mode of interest. The PPF control is implemented using an auxiliary dynamic system (compensator) defined by

$$\ddot{\eta} + 2\,\zeta_f\,\omega_f\,\dot{\eta} + \omega_f^2\,\eta = g\omega_f^2\,x$$
$$u = \frac{g}{b}\,\omega_f^2\,\eta \tag{7.108}$$

Here, ζ_f and ω_f are the damping ratio and natural frequency of the controller, and g is a constant. The particular form of Equation (7.108) is that of a second-order system much like a damped vibration absorber. The idea is to choose the PPF frequency and damping ratio so that the response of the structural mode has the desired damping. Combining Equations (7.107) and (7.108) gives the equations of motion in their usual second-order form, which, assuming no external force, is as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} 2\zeta\omega_n & 0 \\ 0 & 2\zeta_f\omega_f \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} \omega_n^2 & -g\omega_f^2 \\ -g\omega_f^2 & \omega_f^2 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.109}$$

Since the stiffness matrix couples the two coordinates, increasing the filter damping, ζ_f, will effectively add damping to the structural mode. Note also that this is a stable closed-loop system if the symmetric 'stiffness' matrix is positive definite for appropriate choices of g and ω_f, that is, if the determinant of displacement coefficient matrix is positive, which happens if

$$g^2\omega_f^2 < \omega_n^2 \tag{7.110}$$

Notice that the stability condition only depends on the natural frequency of the structure, and not on the damping or mode shapes. This is significant in practice because, when building an experiment, the frequencies of the structure are usually available with a reasonable accuracy, while mode shapes and damping ratios are much less reliable. The design of the controller then consists of choosing g and ω_f that satisfy inequality (7.110) and choosing ζ_f large enough to add significant damping to the structural mode. Note that the gains of the controller ζ_f, g, and ω_f are chosen electronically.

The stability property of PPF is also important because it can be applied to an entire structure, eliminating spillover by rolling off at higher frequencies. That is, the frequency response of the PPF controller has the characteristics of a low-pass filter. The transfer function of the controller is

$$\frac{\eta(s)}{X(s)} = \frac{g\omega_f^2}{s^2 + 2\zeta_f\omega_f s + \omega_f^2} \tag{7.111}$$

illustrating that it rolls off quickly at high frequencies. Thus, the approach is well suited to controlling a mode of a structure with frequencies that are well separated, as the controller is insensitive to the unmodeled high-frequency dynamics. If the problem is cast in the state space, the term $\mathbf{b}_{2n-k}$ in Equation (7.91) is zero, and no spillover results.

The positive position terminology in the name PPF comes from the fact that the position coordinate of the structure equation is positively fed to the filter, and the position coordinate of the compensator equation is positively fed back to the structure.

Next, suppose a multiple-degree-of-freedom analytical model of the structure is available. Following the formulation of output feedback discussed in Section 4.10 for an SISO system with no applied force yields

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u} \tag{7.112}$$

Coupling this with the PPF controller in the form given in Equation (7.108) written as

$$\begin{aligned} \ddot{\eta} + 2\,\zeta_f\,\omega_f\,\dot{\eta} + \omega_f^2\,\eta &= g\omega_f^2\,B_f^T\mathbf{q} \\ u &= g\,\omega_f^2\,\eta \end{aligned} \tag{7.113}$$

yields

$$\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 2\zeta_f\omega_f \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.114}$$

The system is SISO, so that B_f is a vector since u is a scalar. The augmented mass matrix in Equation (7.114) is symmetric and positive definite, the augmented damping matrix is symmetric and positive semidefinite, so the closed-loop stability will depend on the definiteness of the augmented stiffness matrix.

Consider then the definiteness of the augmented stiffness matrix defined by

$$\hat{K} = \begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix} \tag{7.115}$$

Let $\mathbf{x}$ be an arbitrary vector partitioned according to $\hat{K}$ and compute

$$\mathbf{x}^T\hat{K}\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}\begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}_1^T K\mathbf{x}_1 - g\omega_f\mathbf{x}_1^T B_f\mathbf{x}_2 - g\omega_f\mathbf{x}_2^T B_f^T\mathbf{x}_1 + \omega_f^2\mathbf{x}_2^T\mathbf{x}_2$$

Completing the square and factoring yields

$$\mathbf{x}^T\hat{K}\mathbf{x} = \mathbf{x}_1^T(K - g^2B_f^TB_f)\mathbf{x}_1 + \left(gB_f^T\mathbf{x}_1 - \omega_f\mathbf{x}_2\right)^T\left(gB_f^T\mathbf{x}_1 - \omega_f\mathbf{x}_2\right)$$

This is of the form

$$\mathbf{x}^T\hat{K}\mathbf{x} = \mathbf{x}_1^T(K - g^2B_f^TB_f)\mathbf{x}_1 + \mathbf{y}^T\mathbf{y}$$

for arbitrary $\mathbf{x}_1$ and $\mathbf{y}$. Since $\mathbf{y}^T\mathbf{y}$ is always nonnegative, $\hat{K}$ will be positive definite if the matrix $K - g^2B_f^TB_f$ is positive definite. If g is chosen such that $K - g^2B_f^TB_f$ is positive definite, then the closed-loop system will be stable (in fact asymptotically stable via the discussion in Section 4.5 since the coefficient matrices do not commute).

Example 7.11.1

Consider the two-degree-of-freedom system of example 7.9.1 and design a PPF controller to add damping to the first mode of the system without affecting the second mode.

For a single actuator at the location of the first mass, the input matrix becomes

$$B_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The augmented mass and stiffness matrices of Equation (7.114) become

$$\hat{M} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \hat{D} = \begin{bmatrix} 9 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2\zeta_f\omega_f \end{bmatrix}, \qquad \hat{K} = \begin{bmatrix} 27 & -3 & -g\omega_f \\ -3 & 3 & 0 \\ -g\omega_f & 0 & \omega_f^2 \end{bmatrix}$$

Following the constraint given by inequality (7.110) for controlling the first mode $g^2\omega_f^2 < \omega_1^2 = 2$, one free choice is $g = \omega_f = 1$. Choosing the PPF damping of $\zeta_f = 0.5$ results in the following closed-loop damping ratios and frequencies:

$$\begin{aligned} \omega_1 &= 1.43, & \zeta_1 &= 0.237 \\ \omega_2 &= 2.00, & \zeta_2 &= 0.332 \\ \omega_f &= 0.966, & \zeta_f &= 0.531 \end{aligned}$$

as computed from the corresponding state matrix. Note that damping is added to the first mode (from 0.235 to 0.237) while mode two damping is only slightly changed and the frequency is not changed at all. All the modes change slightly because the system is coupled by the filter and, as was shown in example 7.9.1, multiple sensors and actuators are required to affect only one mode.

7.12 MATLAB COMMANDS FOR CONTROL CALCULATIONS

Most of the calculations in this chapter are easily made in MATLAB. MATLAB contains a 'toolbox' just for controls, called the Control System Toolbox (Grace *et al.*, 1992). This is a series of algorithms expressed in m-files that implements common control design, analysis, and modeling methods. Many websites are devoted to using and understanding the Control System Toolbox. Table 7.1 lists some common commands useful in implementing calculations for active control. The control commands in MATLAB assume the control problem is a linear time-invariant system of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u} \tag{7.116}$$

The developments in this chapter assume that $D = 0$.

The following examples illustrate the use of some of the commands listed in Table 7.1 to perform some of the computations developed in the previous sections.

Table 7.1 MATLAB commands for control.

`ss2tf`	converts the state-space model to the system transfer function
`tf2ss`	converts a system transfer function to a state-space model
`step`	computes the step response of a system
`initial`	computes the response of a system to initial conditions
`impulse`	computes the response of a system to a unit impulse
`ctrb`	computes the controllability matrix
`obsv`	computes the observability matrix
`gram`	computes the controllability and observability grammians
`balreal`	computes the balanced realization
`place`	computes the pole placement gain matrix
`lqr`	computes the linear quadratic regulator solution
`modred`	computes the reduced-order model

Example 7.12.1

The function `initial` can be used to plot the response of the system of Equation (7.116) to a given initial condition. Consider the system in state-space form of a single-degree-of-freedom system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and plot the response using `initial` . Type the following in the command window:

```
>>clear all
>>A=[0 1;-4 -2];
>>b=[1;0];
>>c=[1 0];
>>d=[0];
>>x0=[1 0];
>>t=0:0.1:6;
>>initial(A,b,c,d,x0,t);
```

This results in the plot given in Figure 7.14.

Next, consider using MATLAB to solve the pole placement problem. MATLAB again works with the state-space model of Equation (7.116) and uses full-state feedback ($\mathbf{u} = -K\mathbf{x}$) to find the gain matrix K that causes the closed-loop system ($A - \mathbf{b}K$) to have the poles specified in the vector $\mathbf{p}$. The following example illustrates the use of the `place` command.

Example 7.12.2

Use the system of example 7.12.1 and compute the gain that causes the closed-loop system to have two real poles: -2 and -4. Using the code in Figure 7.12.1 to enter the state-space model (`A, b, c, d`), then type the following in the command window:

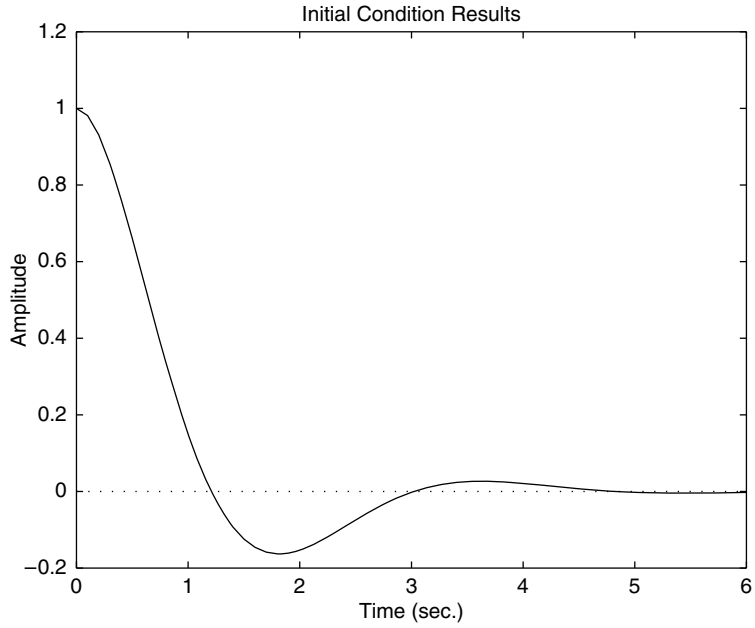

Figure 7.14 Response computed and plotted using the `initial` command.

```
>>p=[ -2 -4];
>>K=place(A,b,p)
place: ndigits= 15
K =
   4   1
```

Thus, the proper gain matrix is a vector in this case. To check that the result works, type

```
>>eig(A-b*K)
ans =
   -2
   -4
>>eig(A)
ans =
  -1.0000+ 1.7321i
  -1.0000- 1.7321i
```

This shows that the computed gain matrix causes the system to move its poles from the two complex values $-1 \pm 1.7321i$ to the two real values -2 and -4.

Next, consider solving the optimal control problem using the linear quadratic regulator problem defined by Equation (7.28). The MATLAB command `lqr` computes the solution of the

matrix Ricatta equation [Equation (7.32)], calculates the gain matrix K, and then calculates the eigenvalues of the resulting closed-loop system $(A - BK)$ for a given choice of the weighting matrices Q and R defined in Section 7.4. In this case, the output matrix B must have the same number of columns as the matrix R. The following example illustrates the procedure.

Example 7.12.3

Compute an optimal control for the system of example 7.12.1. Here, we chose **Q** to be the identity matrix and **R** to be 2**Q**. The following is typed into the command window:

```
>>A=[0 1;-4 -2];
>>B=eye(2);% the 2x2 identity matrix
>>c=[1 0];
>>d=[0];
>>Q=eye(2);R= 2*Q;% weighting matrices in the
cost function
>>[K,S,E]=lqr(A,B,Q,R)
K =
   0.5791    0.0205
   0.0205    0.1309
S =
   1.1582    0.0410
   0.0410    0.2617
E =
  -1.3550+ 1.8265i
  -1.3550- 1.8265i
```

The vector `E` contains new (closed-loop) eigenvalues of the system with the gain computed (K) from the solution of the Ricatta equation (S). Note that the closed-loop eigenvalues are more heavily damped then the open-loop eigenvalues $(-1 \pm 1.7321j)$.

Next, consider the model reduction problem of Section 7.7. The balanced realization can be obtained from the `balreal` command, and the corresponding reduced-order model can be obtained from the `modred` command. The procedure is illustrated in the next example.

Example 7.12.4

Compute the balanced realization of the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.3 & 0.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Use the balance realization to reduce the order of the model on the basis of its singular values. First the state-space model is typed into the command window. Then a balanced realization is formed and the singular values computed. The singular values (denoted by `g` in the code that follows) are then used with the model reduction routine to compute a reduced-order model. Type the following into the command window:

```
>>A=[0 0 1 0;0 0 0 1;-2 1 -0.3 0.1;1 -1 0.1 -0.1];
>>b=[0;0;0;1]; c=[0 1 0 0]; d=[0];
>>[ab,bb,cb,g]=balreal(A,b,c) % computes the balanced
version and singular values
ab =
   -0.0326    0.6166    0.0192   -0.0275
   -0.6166   -0.0334   -0.0218    0.0280
    0.0192    0.0218   -0.1030    1.6102
    0.0275    0.0280   -1.6102   -0.2309
bb =
    0.7821
    0.7510
   -0.2369
   -0.3223
cb =
  0.7821  -0.7510  -0.2369  0.3223
g =
   9.3836
   8.4310
   0.2724
   0.2250
>>elim=find(g<1); %eliminate states with singular values less
than 1
>>[ar,br,cr]=modred(ab,bb,cb,d,elim) % compute the reduced
order model based on eliminating the states with g<1
ar =
  -0.0319    0.6173
  -0.6173   -0.0342
br =
   0.7743
   0.7595
cr =
   0.7743  -0.7595
```

The matrix `ar` and the vectors `br` and `cr` are the reduced-order state matrix and the input and output vectors and form the reduced-order model as defined in Equations (7.76) and (7.77).

There are other commands in the Control System Toolbox that can be used to simplify the calculation of control laws and models and to perform analysis of control systems. Texts in control and the web should be consulted for additional samples and commands.

CHAPTER NOTES

This chapter attempts to introduce some control concepts useful in vibration analysis and design. Linear control engineering is, of course, a topic in its own right, and hence many texts have been written on the subject. The reader is referred to the excellent text by Kuo and Gholoaraghi (2003) for an introduction to control topics and to Kailath (1980) for a more advanced treatment of linear systems and control topics. Section 7.2 on controllability and observability stresses the use of these concepts in physical coordinate systems. However, literally hundreds of papers have been written on the topic in the state-space formulation. In addition to those references listed in the text, useful conditions for controllability and observability in second-order models are given by Laub and Arnold (1984) and Bender and Laub (1985). The early important papers in linear control and computation are presented in Patel, Laub, and Van Dooren (1994).

Section 7.3 on eigenstructure assignment extends the pole placement topics of Section 6.6 to include placing eigenvectors as well as eigenvalues. Again, hundreds of papers appear in the literature on pole placement methods in the state space. The most comprehensive treatment of eigenstructure assignment with structures in mind is that by Andry, Shapiro, and Chung (1983), which motivated the development of Section 7.3. An excellent text on eigenstructure assignment including MATLAB codes is the one by Liu and Patton (1998).

The section on optimal control, Section 7.4, should be viewed by the reader as a very brief introduction. Again, optimal control is a topic that should be studied in detail by using one of the many texts written on the subject (see, for instance, Kirk, 1970). In addition, optimal control is an intense area of current research, with several journals devoted to the topic. Optimal control is an extremely useful tool in controlling unwanted vibration. As pointed out by example 7.4.1, however, an element of art remains in optimal control by virtue of the freedom to choose a cost function. Section 7.5 on observers and estimators follows the development given by Chen (1970, 1998) and Kailath (1980). Section 7.6 on realization also closely follows the development in Chen (1998). The purpose of introducing realization is to set the stage for its use in vibration testing in Section 8.6. This illustrates a recent application of a standard control topic (realization) to a vibration problem (testing) and hence should motivate serious vibration engineers to pay more attention to the topic of linear systems and control theory as a source of alternative solutions.

Model reduction first introduced in Section 6.8 is revisited in Section 7.7 in the state space. The balancing method presented is improved numerically by Laub (1980). This is again an important area of vibration research and design that benefits from developments that have taken place in linear system theory. There are many other approaches to model reduction, notably the work of Hyland and Bernstein (1985) and Skelton, Hughes, and Hablani (1982).

Section 7.8 on modal control in the state space follows the development given by Takahashi, Rabins, and Auslander (1970). Modal control is popular and appears as a tool in a large number of papers. The section on modal control in physical coordinates is an attempt to extend the idea of modal coordinates to second-order systems. This is again a popular method in vibration control problems (see, for instance, Meirovitch and Baruh, 1982). The section on robustness is intended to provide a brief introduction to a very popular topic in control. The development follows that used by Ridgely and

Banda (1986). Note the similarity between robustness and design sensitivity discussed in Section 6.5. Much activity surrounds the topic of robust control, and space has prevented the inclusion of much material in this area. Instead the reader is referred to Zhou and Doyle (1997). The material on PPF control of Section 7.10 comes from the article by Friswell and Inman (1999), which contains additional references to PPF formulations in first-order form and connects PPF to standard output feedback and optimal control laws.

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PROBLEMS

7.1 Consider the system of example 7.3.1 and determine if the system is controllable with one actuator placed at m_1 and observable with one sensor placed at m_2.

7.2 Recalculate the gain matrix G_f for example 7.3.1 with one actuator placed at m_2 and one sensor placed at m_1 to measure $q_1(t)$ only. Does the resulting system have the desired eigenstructure?

7.3 Consider the system given by $M = I$

$$D = \text{diag}[0.1 \quad 0.2] \qquad \text{and} \qquad K = \text{diag}[1 \quad 2]$$

Is the system controllable for $B = [0 \quad 1]^T$? For $B = [1 \quad 0]^T$?

7.4 Repeat example 7.3.1 with desired eigenvectors $[1 \quad -1]$ and $[1 \quad 1]$, desired eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, and desired mass matrix

$$M_O = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Assume the same structures as in example 7.3.1, and place the sensors and actuators as you see fit.

7.5 Calculate G_f in example 7.3.1 if the desired eigenvalues are $\lambda_1 = 10$ and $\lambda_2 = 20$ with no specification of the system eigenvectors.

7.6 Show that for $\mathbf{b} = [1 \; 0 \; 0 \; \cdots]^T$ the system of Equation (7.12) can be assigned any eigenvalues by proper choice of the gain g_i.

7.7 Show that the mode participation factors are unique for a given set of initial conditions.

7.8 Show that the control given in example 7.4.1 for the case $Q = R = I$ satisfies inequality (7.27) by calculating some other values of $\mathbf{u}$ (i.e., H_f).

7.9 Consider example 7.5.1. Let the observer poles be placed at -2 and -2 so that the speed of the observer is comparable with that of the system. Calculate $\mathbf{e}(t)$ versus time and $\mathbf{x}(t)$ versus time for $\mathbf{e}(0) = [1 \;\; 0]^T$ and plot the results. Note that in this case, the observer does not do as well.

7.10 Calculate a realization for the vibration absorber problem of Section 6.2.

7.11 Calculate W_C and W_O for the vibration absorber problem of Equation (6.14). Use $m_0 = 1$, $m = 10$, $k = 5$, and $k_0 = \omega^2$.

7.12 Calculate the singular value decomposition of the following matrices

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 6 \\ 5 & -1 \\ 1 & 0 \end{bmatrix}$$

7.13 Consider a two-degree-of-freedom system similar to that of Figure 2.4. Discuss in physical terms how such a system would have to behave if it could be reduced to a single-degree-of-freedom model very accurately.

7.14 Using available software subroutines (for things like SVD), write a program to perform model reduction using balancing. Then reproduce the reduction illustrated in example 7.7.1.

7.15 Consider example 2.4.4. Write the system of Equation (2.26) in state-space form. Let $m_1 = 1$, $m_2 = 2$, $c_1 = c_2 = 0.1$, $k_1 = 1$, and $k_2 = 4$. Use the modal control approach of Section 7.8 to design a control (i.e., choose g_1 and g_2) to raise the natural frequencies of the structure above 5 Hz. Add additional controls or measurements as necessary.

7.16 Referring to the statement of problem 7.15, calculate the minimum number of sensors and actuators required to control independently the lowest frequency; that is, for $k = 2$ in Equation (7.91), what control configuration is required for

$$\mathbf{b}_{2n-k} = \mathbf{c}_{2n-k}^T = \mathbf{0}$$

7.17 Again consider the feedback control problem for the example of Figure 2.4. Calculate B_f, K_v, and K_p for this system such that Equation (7.96) is diagonal (decoupled).

7.18 Make a physical interpretation of the calculation in problem 7.16 (i.e., where are the sensors and actuators located and does it make sense?).

7.19 Prove, or derive, that Equations (7.97) and (7.98) are in fact sufficient conditions for the closed-loop system to decouple.

7.20 Suppose that the commutivity condition assumed in Section 7.9 is not satisfied. Then the coefficient of $\dot{\mathbf{z}}$ in Equation (7.93) is *not* diagonal. Calculate conditions on the matrices B_f and K_v so that the coefficient $\dot{\mathbf{z}}$ in Equation (7.96) becomes diagonal. Such a control decouples a coupled system.

7.21 Based on your answer to problem 7.20, discuss the physical implications of the control in terms of, say, Figure 2.4.

7.22 Is the condition of problem 7.20 robust? Please discuss your answer.

7.23 Repeat example 7.10.1 for the system of problem 7.15, assuming that the coefficients c_1 and c_2 are only accurate to within 25%.

7.24 Compare the controllability norm of Equation (7.8) with the controllability grammian of Equation (7.58). Is there a mathematical relationship between them?

7.25 Show that the realization of example 7.6.1 is both controllable and observable. In general, show that Equations (7.56) and (7.57) are controllable and observable.

7.26 Using the criterion given in Equation (7.3), show that the system of Figure 7.1(a) is uncontrollable and that the system of Figure 7.1(b) is controllable.

7.27 Show that Equations (7.56) and (7.53) reduce to Equation (7.53) by using the definition of a transfer function given in Equation (7.51).

7.28 One method of performing a model reduction (Section 7.7) is to represent the open-loop system by the first few modes of the structure. Try this by using the first two (state-space) modes of example 7.7.1 as the ROM of the system. Then compare the response of this ROM to the ROM obtained in example 7.7.1 by the balancing method.

7.29 Design a PPF controller for the mode $\ddot{r}_i + 0.6\dot{r}_i + 9r_i = 0$ that increases the modal damping ratio to 0.3.

7.30 Compute the mode shapes, natural frequencies, and modal damping ratios for the closed-loop system of example 7.9.1, case 3, to see if the controller does just change the first mode damping ratio.

7.31 Consider example 7.9.1 and use pole placement to attempt to change just the first mode damping ratio (to 0.4).

7.32 Consider example 7.9.1 and use eigenstructure assignment to attempt to change just the first mode damping ratio (to 0.4).

7.33 Consider the system of example 7.9.1. Using pole placement, derive a control law that will cause the first natural frequency of the closed-loop system to be 1.6 rad/s and leave the second natural frequency fixed at 2 rad/s.

7.34 Use the eigenstructure assignment method to place the mode shapes of the system of example 7.3.1 to shift the second eigenvector to $[0.316 \quad 0.949]$.

7.35 Compute the time response of the system of example 7.9.1 using the gains from cases 1 and 3 and compare your results. Use an initial condition of zero velocity and the first coordinate displaced by 0.1 units.

8
Modal Testing

8.1 INTRODUCTION

This chapter introduces the basic topics associated with dynamic measurement and testing of structures. In the previous chapters, the differential equations, or models, of the system were assumed to be known, and the theory developed consists of calculating and characterizing the response of the system to known inputs. This is called the *forward problem*. In this chapter, however, the interest lies in measuring the response of a structure and in some way determining the equations of motion from test data. The problem of determining a system of equations from information about inputs and responses belongs to a class of problems called *inverse problems*. Although measurements of vibrating systems are made for a variety of reasons, this chapter focuses on the use of vibration measurement to identify, or verify, a mathematical model of a test structure. Other uses of vibration measurements include environmental testing, balancing of rotating machinery, prediction of failures, structural health monitoring, and machinery diagnostics for maintenance (Inman *et al.*, 2005).

The previous chapters have considered only structures modeled by lumped-mass elements. As mentioned in Section 2.5, however, many structures are not configured in clear, simple lumped-mass arrangements but rather have mass distributed throughout the structure. Such distributed-mass models form the topics of Chapters 9 through 13. However, measurements of distributed-mass structures often result in data that resemble the response of a lumped-mass structure. Hence, many test procedures, including those described here, assume that the structure being tested can be adequately described by a lumped-parameter model. The significance of such an assumption is discussed in Chapter 13 on finite-dimensional modeling of distributed-parameter systems.

There are several other assumptions commonly made but not stated (or understated) in vibration testing. The most obvious of these is that the system under test is linear and is driven by the test input only in its linear range. This assumption is essential and should not be neglected. Measurement of nonlinear systems is presented in Virgin (2000).

Vibration testing and measurement for modeling purposes forms a large industry. This field is referred to as *modal testing*, *modal analysis*, or *experimental modal analysis* (abbreviated EMA). Understanding modal testing requires knowledge of several areas. These include instrumentation, signal processing, parameter estimation, and vibration analysis. These topics are introduced in the following sections, but Ewins (2000) should be consulted for a complete description.

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

It is worth noting that modal testing activity is centered mainly around determining the modal parameters of a lumped-parameter model of the test structure. The process of determining a mathematical model of a system from measured inputs and outputs is called *system identification theory*. This theory has grown rapidly in the discipline of control theory rather than vibrations. System identification theory, in turn, is a subclass of inverse problems, as opposed to the direct problem discussed in earlier chapters. Much of the early work in modal analysis did not take advantage of the disciplines of system identification and parameter estimation, a situation rectified by efforts such as the book by Juang (1994). In the material that follows, a few of the basic approaches are presented.

As mentioned, the first seven chapters consider only the *forward problem*, i.e., given the matrices M, D, and K along with the appropriate initial conditions and forcing functions, determine the solution of Equation (2.13) describing the dynamics of the structure. The *inverse problem*, on the other hand, is to determine the matrices M, D, and K from knowledge of the measurements of the responses (position, velocity, or acceleration). The modal testing problem, a subclass of inverse problems, is to recover the mode shapes, natural frequencies, and damping ratios from response measurements. In general, modal testing methods cannot fully determine the matrices of physical parameters M, D, and K. While forward problems have a unique solution for linear systems, inverse problems do not. Model updating, introduced in Section 8.8, is an attempt to justify the analytically derived values of M, D, and K with measured modal data.

The fundamental idea behind modal testing is that of resonance first introduced in Section 1.4. If a structure is excited at resonance, its response exhibits two distinct phenomena, as indicated by Equations (1.19) and (1.20). As the driving frequency approaches the natural frequency of the structure, the magnitude at resonance rapidly approaches a sharp maximum value, provided the damping ratio is less than about 0.5. The second, often neglected, phenomenon of resonance is that the phase of the response shifts by 180° as the frequency sweeps through resonance, with the value of the phase at resonance being 90°.

The first few sections of this chapter deal with the hardware considerations and digital signal analysis necessary for making a vibration measurement for any purpose. The data acquisition and signal processing hardware has changed considerably over the past decade and continues to change rapidly as a result of advances in solid-state and computer technology. The remaining sections discuss the modal analysis version of model identification and parameter estimation. The last section introduces model updating.

8.2 MEASUREMENT HARDWARE

A vibration measurement generally requires several hardware components. The basic hardware elements required consist of a source of excitation for providing a known or controlled input to the structure, a *transducer* to convert the mechanical motion of the structure into an electrical signal, a signal conditioning amplifier to match the characteristics of the transducer to the input electronics of the digital data acquisition system, and an analysis system (or analyzer), in which signal processing and modal analysis programs reside. The arrangement is illustrated in Figure 8.1; it includes a power amplifier and a signal generator for the exciter, as well as a transducer to measure, and possibly control, the driving force or other input. Each of these devices and their functions are discussed briefly in this section.

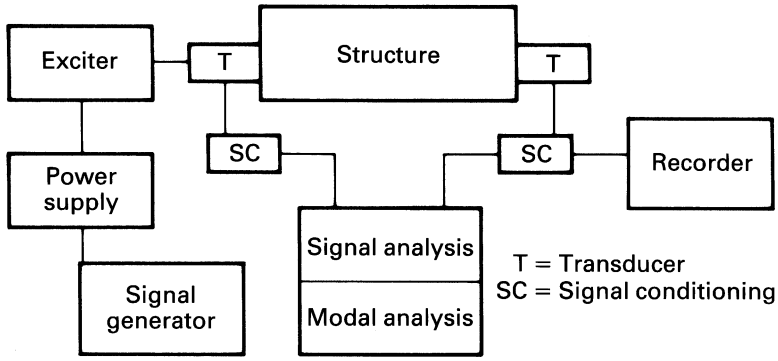

Figure 8.1 Components of a vibration measurement system for modal analysis.

First consider the excitation system, denoted 'exciter' in Figure 8.1. This system provides an input motion or, more commonly, a driving force $f_i(t)$, as in Equation (2.13). The physical device may take several forms, depending on the desired input and the physical properties of the test structure. The two most commonly used exciters in modal testing are the shaker (electromagnetic or electrohydraulic) and the impulse hammer. The preferred device is often the electromagnetic exciter, which has the ability, when properly sized, to provide inputs large enough to result in easily measured responses. Also, the output is easily controlled electronically, sometimes using force feedback. The excitation signal, which can be tailored to match the requirements of the structure being tested, can be a swept sine, random, or other appropriate signal. The *electromagnetic shaker* is basically a linear electric motor consisting of coils of wire surrounding a shaft in a magnetic field. An alternating current applied to the coil causes a force to be applied to the shaft, which, in turn, transfers force to the structure. The input electrical signal to the shaker is usually a voltage that causes a proportional force to be applied to the test structure.

Since shakers are attached to the test structure and since they have significant mass, care should be taken by the experimenter in choosing the size of shaker and method of attachment to minimize the effect of the shaker on the structure. The shaker and its attachment can add mass to the structure under test (called *mass loading*) as well as otherwise constraining the structure.

Mass loading and other constraints can be minimized by attaching the shaker to the structure through a *stinger*. A stinger consists of a short thin rod (usually made of steel or nylon) running from the driving point of the shaker to a force transducer mounted directly on the structure. The stinger serves to isolate the shaker from the structure, reduces the added mass, and causes the force to be transmitted axially along the stinger, controlling the direction of the applied force.

Excitation can also be applied by an *impulse* by using an *impact hammer*. An impact hammer consists of a hammer with a force transducer built into its head. The hammer is then used to hit (impact) the test structure and thus excite a broad range of frequencies. The peak impact force is nearly proportional to the hammer head mass and the impact velocity. The upper frequency limit excited by the hammer is decreased by increasing the hammer head mass, and increased with increasing stiffness of the tip of the hammer.

As a simple example of how the choice of excitation device is critical, consider Figure 8.2. This plot illustrates the frequency spectrum of a typical force resulting from a hammer

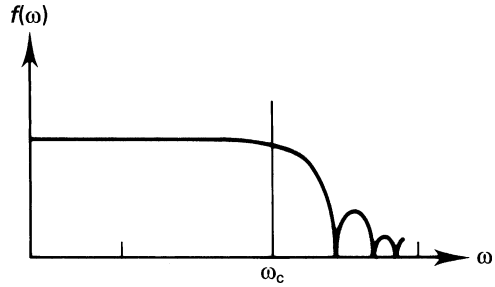

Figure 8.2 Frequency spectrum of a typical hammer impact.

impact or hit. It shows that the impact force is approximately constant up to a certain value of frequency, denoted by ω_c. This hammer hit is less effective in exciting the modes of the structure with frequencies larger than ω_c than it is in exciting the modes with frequencies less than ω_c. For a given hammer mass, ω_c can be lowered by using a softer tip.

The built-in force transducer in impact hammers should be dynamically calibrated for each tip used, as this will affect the sensitivity. Although the impact hammer is simple and does not add mass loading to the structure, it is often incapable of transforming sufficient energy to the structure to obtain adequate response signals in the frequency range of interest. Also, peak impact loads are potentially damaging, and the direction of the applied load is difficult to control. Nonetheless, impact hammers remain a popular and useful excitation device, as they generally are much faster to use than shakers.

Next, consider the transducers required to measure the response of the structure as well as the impact force. The most popular and widely used transducers are made from piezoelectric materials. Piezoelectric materials generate electrical charge when strained. By various designs, transducers incorporating these materials can be built to produce signals proportional to either force or local acceleration. *Accelerometers*, as they are called, actually consist of two masses, one of which is attached to the structure, separated by a piezoelectric material, which acts like a very stiff spring. This causes the transducer to have a resonant frequency. The maximum measurable frequency is usually a fraction of the resonance frequency of the accelerometer. In fact, the upper frequency limit is usually determined by the so-called mounted resonance, since the connection of the transducer to the structure is always somewhat compliant.

Piezoelectric materials also produce a strain when excited with a voltage. Hence, piezoelectric materials are emerging as a reasonable means of vibration excitation for special-purpose situations, as described, for instance, by Cole, Saunders, and Robertshaw (1995) and Inman *et al.* (2005).

The output impedance of most transducers is not well suited for direct input into signal analysis equipment. Hence, *signal conditioners*, which may be charge amplifiers or voltage amplifiers, match and often amplify signals prior to analyzing the signal. It is very important that each set of transducers along with signal conditioning is properly calibrated in terms of both magnitude and phase over the frequency range of interest. While accelerometers are convenient for many applications, they provide weak signals if one is interested in lower-frequency vibrations incurred in terms of velocity of displacement. Even substantial low-frequency vibration displacements may result in only small accelerations, recalling that, for a harmonic displacement of amplitude X, the acceleration amplitude is $-\omega^2 X$. Strain gauges

and potentiometers as well as various optical (fiber optics, laser vibrometers), capacitive, and inductive transducers are often more suitable than accelerometers for low-frequency motion measurement.

Once the response signal has been properly conditioned, it is routed to an analyzer for signal processing. The standard is called a digital or discrete Fourier analyzer (often abbreviated DFA), also called the fast Fourier transform (often abbreviated FFT) analyzer; and is introduced briefly here. Basically, the DFA accepts analog voltage signals which represent the acceleration (force, velocity, displacement, or strain) from a signal conditioning amplifier. This signal is filtered and digitized for computation. Discrete frequency spectra of individual signals and cross-spectra between the input and various outputs are computed. The analyzed signals can then be manipulated in a variety of ways to produce such information as natural frequencies, damping ratios, and mode shapes in numerical or graphic displays.

While almost all the commercially available analyzers are marketed as turnkey devices, it is important to understand a few details of the signal processing performed by these analysis units in order to carry out valid experiments. This forms the topic of the next two sections.

8.3 DIGITAL SIGNAL PROCESSING

Much of the analysis done in modal testing is performed in the frequency domain, as discussed in Section 5.7. The analyzer's task is to convert analog time domain signals into digital frequency domain information compatible with digital computing and then to perform the required computations with these signals. The method used to change an analog signal, $x(t)$, into frequency domain information is the Fourier transform (similar to the Laplace transform used in Chapter 5), or a *Fourier series*. The Fourier series is used here to introduce the digital Fourier transform (DFT). See Newland (1985, p. 38) for the exact relationship between a Fourier transform and a Fourier series.

Any periodic in time signal, $x(t)$, of period T can be represented by a Fourier series of the form

$$x(t) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left[a_i \cos \frac{2\pi i t}{T} + b_i \sin \frac{2\pi i t}{T} \right] \tag{8.1}$$

where the constants a_i and b_i, called the *Fourier coefficients*, or *spectral coefficients*, are defined by

$$\begin{aligned} a_i &= \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi i t}{T} dt \\ b_i &= \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi i t}{T} dt \end{aligned} \tag{8.2}$$

The expression in Equation (8.1) is referred to as the Fourier series for the periodic function $x(t)$. The spectral coefficients represent frequency domain information about the time signal $x(t)$.

The coefficients a_i and b_i also represent the connection to vibration experiments. The analog output signals of accelerometers and force transducers, represented by $x(t)$, are inputs to the analyzer. The analyzer in turn calculates the spectral coefficients of these signals,

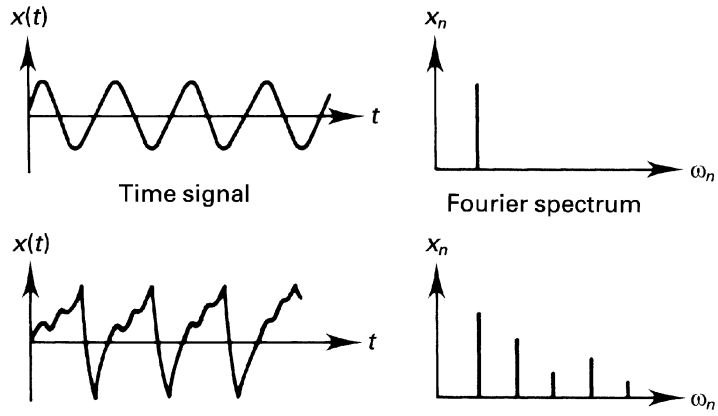

Figure 8.3 Some signals and their Fourier spectrum.

thus setting the stage for a frequency domain analysis of the signals. Some signals and their Fourier spectrum are illustrated in Figure 8.3. The analyzer first converts the analog signals into digital records. It samples the signals $x(t)$ at many different equally spaced values and produces a digital record of the signal in the form of a set of numbers $\{x(t_k)\}$. Here, $k = 1, 2, \ldots, N$, the digit N denotes the number of samples, and t_k indicates a discrete value of the time.

This process is performed by an analog-to-digital converter (often denoted by A/D). This conversion from an analog to digital signal can be thought of in two ways. First, one can imagine a grate that samples the signal every t_k seconds and passes through the signal $x(t_k)$. One can also consider the analog-to-digital converters as multiplying the signal $x(t)$ by a square-wave function, which is zero over alternate values of t_k and has the value of 1 at each t_k for a short time. Some signals and their digital representation are illustrated in Figure 8.4.

In calculating digital Fourier transforms, one must be careful in choosing the sampling time, i.e., the time elapsed between successive t_k. A common error introduced in digital signal analysis caused by improper sampling time is called *aliasing*. Aliasing results from analog-to-digital conversion and refers to the misrepresentation of the analog signal by the digital record. Basically, if the sampling rate is too slow to catch the details of the analog

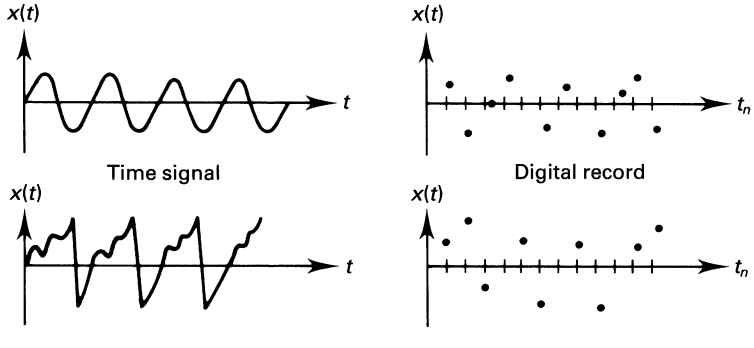

Figure 8.4 Sample time histories and their corresponding digital record.

signal, the digital representation will cause high frequencies to appear as low frequencies. The following example illustrates two analog signals of different frequency that produce the same digital record.

Example 8.3.1

Consider the signals $x_1(t) = \sin[(\pi/4)t]$ and $x_2(t) = -\sin[(7\pi/4)t]$, and suppose these signals are both sampled at 1 intervals. The digital record of each signal is given in the following table.

t_k	0	1	2	3	4	5	6	7	ω_i
x_1	0	0.707	1	0.707	0	−0.707	−1	−0.707	$\frac{1}{8}$
x_2	0	0.707	1	0.707	0	−0.707	−1	−0.707	$\frac{7}{8}$

As is easily seen from the table, the digital sample records of x_1 and x_2 are the same, i.e., $x_1(t_k) = x_2(t_k)$ for each value of k. Hence, no matter what analysis is performed on the digital record, x_1 and x_2 will appear the same. Here, the sampling frequency, $\Delta\omega$, is 1. Note that $\omega_1 - \Delta\omega = 1/8–1 = -7/8$, which is the frequency of $x_2(t)$.

To avoid aliasing, the sampling interval, denoted by Δt, must be chosen small enough to provide at least two samples per cycle of the highest frequency to be calculated. That is, in order to recover a signal from its digital samples, the signal must be sampled at a rate at least twice the highest frequency in the signal. In fact, experience (see Otnes and Enochson, 1972) indicates that 2.5 samples per cycle is a better choice. This is called the *sampling theorem*, or *Shannon's sampling theorem*.

Aliasing can be avoided in signals containing many frequencies by subjecting the analog signal $x(t)$ to an *antialiasing* filter. An antialiasing filter is a low-pass (i.e., only allowing low frequencies through) sharp cut-off filter. The filter effectively cuts off frequencies higher than about half the maximum frequency of interest, denoted by $\omega_{\max}$, and also called the *Nyquist frequency*. Most digital analyzers provide built-in antialiasing filters.

Once the digital record of the signal is available, the discrete version of the Fourier transform is performed. This is accomplished by a digital Fourier transform or series defined by

$$x_k = x(t_k) = \frac{a_0}{2} + \sum_{i=1}^{N/2}\left[a_i \cos\frac{2\pi i t_k}{T} + b_i \sin\frac{2\pi i t_k}{T}\right], \qquad k = 1, 2, \ldots, N \tag{8.3}$$

where the *digital spectral coefficients* are given by

$$\begin{aligned} a_0 &= \frac{1}{N}\sum_{k=1}^{N} x_k \\ a_i &= \frac{1}{N}\sum_{k=1}^{N} x_k \cos\frac{2\pi i k}{N} \\ b_i &= \frac{1}{N}\sum_{k=1}^{N} x_k \sin\frac{2\pi i k}{N} \end{aligned} \tag{8.4}$$

The task of the analyzer is to solve Equations (8.3) given the digital record $x(t_k)$, also denoted by x_k, for the measured signals. The transform size or number of samples, N, is usually fixed for a given analyzer and is a power of 2.

Writing out Equations (8.3) for each of the N samples yields N linear equations in the N spectral coefficients $(a_1, \ldots, a_{N/2}, b_1, \ldots, b_{N/2})$. These equations can also be written in the form of matrix equations. Equations (8.3) in matrix form become

$$\mathbf{x} = C\mathbf{a} \tag{8.5}$$

where $\mathbf{x}$ is the vector of samples with elements x_k and $\mathbf{a}$ is the vector of spectral coefficients. The solution of Equations (8.5) for the spectral coefficients is then given simply by

$$\mathbf{a} = C^{-1}\mathbf{x} \tag{8.6}$$

The task of the analyzer is to compute C^{-1} and hence the coefficients $\mathbf{a}$. The most widely used method of computing the inverse of this matrix C is called the fast Fourier transform (FFT) developed by Cooley and Tukey (1965).

In order to make digital analysis feasible, the periodic signal must be sampled over a finite time (N must obviously be finite). This can give rise to another problem referred to as *leakage*. To make the signal finite, one could simply cut the signal at any integral multiple of its period. Unfortunately, there is no convenient way to do this for complicated signals containing many different frequencies. Hence, if no further steps are taken, the signal may be cut off midperiod. This causes erroneous frequencies to appear in the digital representation because the digital Fourier transform of the finite-length signal assumes that the signal is periodic within the sample record length. Thus, the actual frequency will 'leak' into a number of fictitious frequencies. This is illustrated in Figure 8.5.

Leakage can be corrected to some degree by the use of a *window function*. Windowing, as it is called, involves multiplying the original analog signal by a weighting function, or window function, $w(t)$, which forces the signal to be zero outside the sampling period.

A common window function, called the *Hanning window*, is illustrated in Figure 8.6, along with the effect it has on a periodic signal. A properly windowed signal will yield a spectral plot with much less leakage. This is also illustrated in the figures.

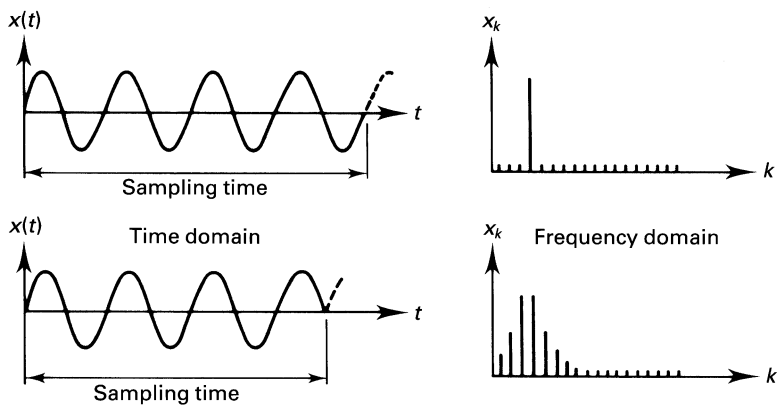

Figure 8.5 Illustration of leakage.

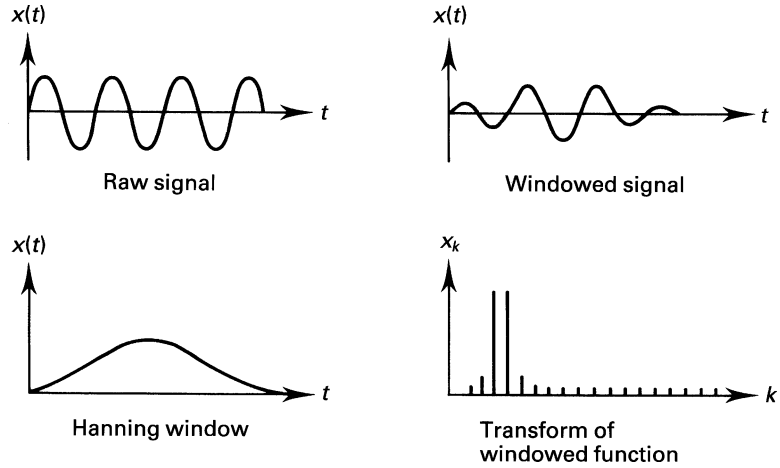

Figure 8.6 Hanning window and its effect on transform signals.

8.4 RANDOM SIGNAL ANALYSIS

This section introduces the topic of transfer function identification from signals, which is the most commonly used approach to system identification concerned with trying to determine the modal parameters of a linear time-invariant system. In most test situations, it becomes desirable to average measurements to increase confidence in the measured parameter. This, along with the fact that a random excitation is often used in testing, requires some background in random processes.

First, some concepts and terminology required to describe random signals need to be established. Consider a random signal $x(t)$ such as pictured in Figure 8.7. The first distinction to be made about a random time history is whether or not the signal is stationary. A random signal is *stationary* if its statistical properties (such as its root mean square value, defined later) are time-independent (i.e., do not change with time). For random signals, it is not possible to focus on the details of the signal, as it is with a pure deterministic signal. Hence, random signals are classified and manipulated in terms of their statistical properties.

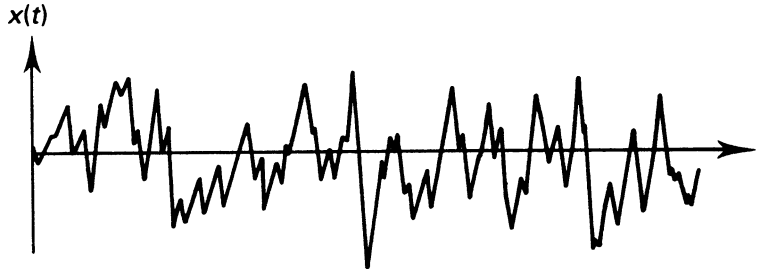

Figure 8.7 Sample random signal.

The *average* of the random signal $x(t)$ is defined and denoted by

$$\overline{x}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,dt \tag{8.7}$$

For a digital signal, the definition of average becomes

$$\overline{x} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x(t_k) \tag{8.8}$$

Often, it is convenient to consider signals with zero average, or *mean*, i.e., $\overline{x}(t) = 0$. This is not too restrictive an assumption, since, if $\overline{x}(t) \neq 0$, a new variable $x' = x(t) - \overline{x}(t)$ can be defined. The new variable $x'(t)$ now has zero mean. In what follows, the signals are assumed to have zero mean.

The *mean square value* of the random variable $x(t)$ is denoted by $\overline{x}^2(t)$ and is defined by

$$\overline{x}^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)\,dt \tag{8.9}$$

or, in digital form,

$$\overline{x}^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} x^2(t_k) \tag{8.10}$$

The mean square value, or *variance* as it is often called, provides a measure of the magnitude of the fluctuations in the signal. A related quantity, called the *root mean square* value, or simply the RMS value, of $x(t)$ is just the square root of the variance, i.e.,

$$x_{\mathrm{RMS}} = \sqrt{\overline{x}^2} \tag{8.11}$$

Another question of interest for a random variable is that of calculating the probability that the variable $x(t)$ will lie in a given interval. However, neither the root mean square value of $x(t)$ nor its probability yields information about how 'fast' the values of $x(t)$ change and hence how long it takes to measure enough $x(t_k)$ to compute statistically meaningful root mean square values and probabilities. A measure of the speed with which the random variable $x(t)$ is changing is provided in the time domain by the *autocorrelation* function. For analog signals, the autocorrelation function, denoted by $R(t)$, defined by

$$R(t) = x^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(\tau)x(\tau+t)\,d\tau \tag{8.12}$$

is used as a measure of how fast the signal $x(t)$ is changing. Note that $R(0)$ is the mean square value of $x(t)$. The autocorrelation function is also useful for detecting the presence of periodic signals contained in, or buried in, random signals. If $x(t)$ is periodic or has a periodic component, then $R(t)$ will be periodic instead of approaching zero as $T \to \infty$, as required for a purely random $x(t)$.

The digital form of the autocorrelation function is

$$R(r, \Delta t) = \frac{1}{N-r}\sum_{k=0}^{N-r} x_k x_{k+r} \tag{8.13}$$

Here, N is the number of samples, Δt is the sampling interval, and r is an adjustable parameter, called a *lag value*, that controls the number of points used to calculate the autocorrelation function.

In the frequency domain, the *power spectral density* (often denoted PSD) is used to measure the speed with which the random variable $x(t)$ changes. The power spectral density, denoted by $S(\omega)$, is defined to be the Fourier transform of $R(t)$, i.e.,

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau \tag{8.14}$$

The digital version of the power spectral density is given by

$$S(\Delta\omega) = \frac{|x(\omega)|^2}{N\,\Delta t} \tag{8.15}$$

where $|x(\omega)|^2$ is the magnitude of the Fourier transform of the sampled data corresponding to $x(t)$.

This definition of the autocorrelation function can also be applied to two different signals to provide a measure of the transfer function between the two signals. The *cross-correlation* function, denoted by $R_{xf}(t)$, for the two signals $x(t)$ and $f(t)$ is defined to be

$$R_{xf}(t) = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} x(\tau) f(\tau + t)\, d\tau \tag{8.16}$$

Likewise, the *cross-spectral density* is defined as the Fourier transform of the cross-correlation, i.e.,

$$S_{xf}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xf}(\tau)\, e^{-j\omega\tau}\, d\tau \tag{8.17}$$

If the function $f(\tau + t)$ is replaced with $x(\tau + t)$ in Equation (8.16), the power spectral density [Equation (8.14)], also denoted by S_{xx}, results. These correlation and density functions allow calculation of the transfer functions of test structures. The frequency response function (recall Section 1.5), $G(j\omega)$, can be shown (see, for instance, Ewins, 2000) to be related to the spectral density functions by the two equations

$$S_{fx}(\omega) = G(j\omega) S_{ff}(\omega) \tag{8.18}$$

and

$$S_{xx}(\omega) = G(j\omega) S_{xf}(\omega) \tag{8.19}$$

These hold if the structure is excited by a random input f resulting in the response x. Here, S_{ff} denotes the power spectral density for the function $f(t)$, which is taken to be the excitation force in a vibration test.

The spectrum analyzer calculates (or estimates) the various spectral density functions. Then, using Equation (8.18) or (8.19), the analyzer can calculate the desired frequency response function $G(j\omega)$. Note that Equations (8.18) and (8.19) use different power spectral densities to calculate the same quantity. This can be used to check the value of $G(j\omega)$.

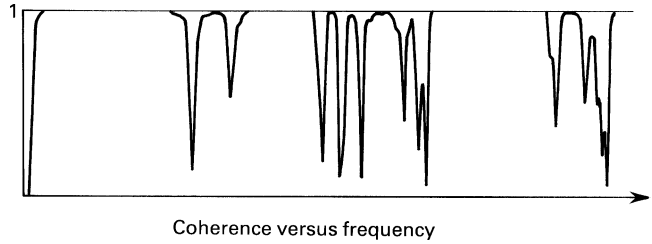

Figure 8.8 Plot of the coherence of a sample signal versus frequency.

The *coherence function*, denoted by γ^2, is defined to be the ratio of the two values of $G(j\omega)$ calculated from Equations (8.18) and (8.19). Let $G'(j\omega) = S_{fx}(\omega)/S_{ff}(\omega)$ denote the value of G obtained from Equation (8.18) from measurements of $S_{fx}(\omega)$ and $S_{ff}(\omega)$. Likewise, let $G''(j\omega) = S_{xx}(\omega)/S_{fx}(\omega)$ denote the frequency response as determined by measurements made for Equation (8.19). Then the coherence function becomes

$$\gamma^2 = \frac{G'(j\omega)}{G''(j\omega)} \tag{8.20}$$

which is always less than or equal to 1. In fact, if the measurements are consistent, $G(j\omega)$ should be the same value, independent of how it is calculated, and the coherence should be 1 ($\gamma^2 = 1$). In practice, coherence versus frequency is plotted (see Figure 8.8) and is taken as an indication of how accurate the measurement process is over a given range of frequencies. Generally, the values of $\gamma^2 = 1$ should occur at values of ω near the resonant frequencies of the structure.

8.5 MODAL DATA EXTRACTION (FREQUENCY DOMAIN)

Once the frequency response of a test structure is calculated from Equation (8.18) or (8.19), the analyzer is used to construct various vibrational information from the processed measurements. This is what is referred to as *experimental modal analysis*. In what follows, it is assumed that the frequency response function $G(j\omega)$ has been measured via Equation (8.18) or (8.19) or their equivalents.

The task of interest is to measure the natural frequencies, damping ratios, and modal amplitudes associated with each resonant peak of the frequency response function. There are several ways to examine the measured frequency response function to extract the desired modal data. To examine all of them is beyond the scope of this text. To illustrate the basic method, consider the somewhat idealized frequency response function record of Figure 8.9, resulting from measurements taken between two points on a simple structure.

One of the gray areas in modal testing is deciding on the number of degrees of freedom to assign to a test structure. In many cases, simply counting the number of clearly defined peaks or resonances, three in Figure 8.9, determines the order, and the procedure continues with a three-mode model. However, as is illustrated later on, this procedure may not be accurate if the structure has closely spaced natural frequencies or high damping values.

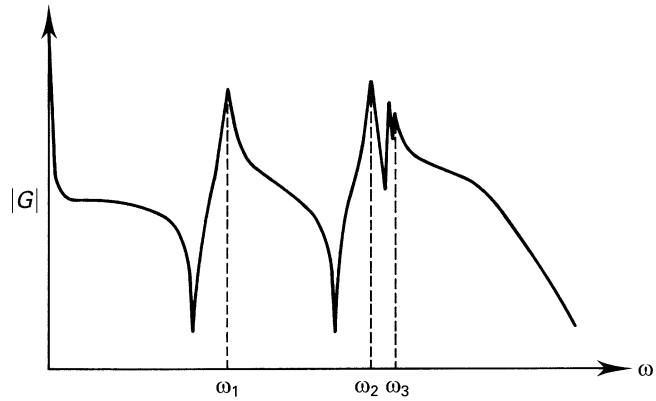

Figure 8.9 Idealized frequency response function.

The easiest method to use on these data is the so-called single-degree-of-freedom curve fit (often called the SDOF method), an approach alluded to in Section 2.5. In this approach the frequency response function is sectioned off into frequency ranges bracketing each successive peak. Each peak is then analyzed by assuming it is the frequency response of a single-degree-of-freedom system. This assumes that, in the vicinity of the resonance, the frequency response function is dominated by that single mode.

In other words, in the frequency range around the first resonant peak, it is assumed that the curve is due to the response of a damped single-degree-of-freedom system to a harmonic input at and near the first natural frequency. Recall from Section 1.4 that the point of resonance corresponds to that value of frequency for which the magnification curve has its maximum or peak value and the phase changes 180°.

The method is basically the peak picking method referred to in Section 1.6 and illustrated in Figure 1.8. The approach is to assume that the peak occurs at the frequency corresponding to the damped natural frequency of that particular mode, denoted by ω_{di} (for the ith mode). The two frequencies on each side of this peak correspond to the points on the curve that are 0.707 of $|G(j\omega_{di})|$ (also called the half-power points). Denoting these two frequencies as ω_{1i} and ω_{2i}, the formula for calculating the damping ratio for the ith peak is (see Blevins, 1994, p. 318)

$$\zeta_i = \frac{\omega_{2i} - \omega_{1i}}{2\omega_{di}} \tag{8.21}$$

Next, even though these formulae result from examining a single-degree-of-freedom model, it is recognized that the system is a distributed mass system that is being modeled as a multiple-degree-of-freedom system with three modes. The frequency response function for a multiple-degree-of-freedom system is discussed in Section 5.7 and leads to the concept of a receptance matrix, $\alpha(\omega)$, which relates the input vector (driving force) to the response (position in this case) via Equation (5.67). For a three-degree-of-freedom system, with the assumption of proportional damping and that the frequency response near resonance ω_{d1} is not influenced by contributions from ω_{d2} and ω_{d3}, the magnitude of the measured frequency response at ω_{d1}, denoted by $G(j\omega_{d1})$, allows the calculation of one of the elements of the receptance matrix.

If the structure is underdamped, the ikth element of the receptance matrix is given by

$$\alpha_{ik} = \sum_{r=1}^{n} \frac{[\mathbf{s}_r \mathbf{s}_r^T]_{ik}}{\omega_r^2 + 2j\zeta_r \omega_r \omega - \omega^2} \tag{8.22}$$

where $\mathbf{s}_r$ is the rth eigenvector of the system. The magnitude of $\alpha_{ij}(\omega_r)$ is the measured amplitude of the peak at ω_r. In this case, Equation (8.22) becomes

$$\alpha_{ik}(\omega_r) = \frac{[\mathbf{s}_r \mathbf{s}_r^T]_{ik}}{\omega_r^2 + 2\zeta_r \omega_r^2 j - \omega_r^2} \tag{8.23}$$

Hence

$$\left|[\mathbf{s}_r \mathbf{s}_r^T]_{ik}\right| = |G_{ik}(j\omega_r)| \left|2\zeta_r \omega_r^2 j\right| = 2\zeta_r \omega_r^2 \, |G_{ik}(j\omega_r)| \tag{8.24}$$

Here, the measured value of the maximum of the frequency response function at $\omega_r = \omega$ with input at point i and response measured at point k is denoted by $G_{ik}(\omega)$ and is approximated by $|\alpha_{ik}(\omega)|$. With the assumption, stated earlier, that the response is due to only one frequency near resonance, Equation (8.24) yields one value for the *modal constant*, which is defined as the magnitude of the ikth element of the matrix $\mathbf{s}_1 \mathbf{s}_1^T$. The phase plot yields the relative sign of the ikth element of $\mathbf{s}_1 \mathbf{s}_1^T$. Equation (8.24) is the mathematical equivalent of assuming that the first peak in the curve of Figure 8.9 results from only a single-degree-of-freedom system.

The subscripts ik denote the output coordinate and the relative position of the input force. In order words, the quantity $|\alpha_{ik}(\omega_r)|$ represents the magnitude of the transfer function between an input at point i and a measured output at point k.

Here, an estimate of the eigenvectors, or mode shapes, can be calculated by making a series of measurements at different points, applying the modal constant formula, [Equation (8.24)], and examining the relative phase shift at ω_{di}. To see this, suppose that the transfer function of Figure 8.9 is between the input f_1 (at point 1) and the measured response x_3 (at point 3 on the structure as labeled in Figure 8.10). Then, from Equation (8.24) with $i = 1, k = 3$, and $r = 1, 2, 3$, the three values of the matrix elements $\left[\mathbf{s}_1 \mathbf{s}_1^T\right]_{13}$, $\left[\mathbf{s}_2 \mathbf{s}_2^T\right]_{13}$, and $\left[\mathbf{s}_3 \mathbf{s}_3^T\right]_{13}$ can be measured. Furthermore, if the preceding experiment is completed two more times with the response measurement (or the force) being moved to the two remaining positions ($k = 2, 1$) shown in Figure 8.10, the following matrix elements can also be measured: $\left[\mathbf{s}_1 \mathbf{s}_1^T\right]_{12}$, $\left[\mathbf{s}_2 \mathbf{s}_2^T\right]_{12}$, and $\left[\mathbf{s}_3 \mathbf{s}_3^T\right]_{12}$ as well as $\left[\mathbf{s}_1 \mathbf{s}_1^T\right]_{11}$, $\left[\mathbf{s}_2 \mathbf{s}_2^T\right]_{11}$, and $\left[\mathbf{s}_3 \mathbf{s}_3^T\right]_{11}$. Gathering up these nine elements of the three different outer product matrices $\mathbf{s}_1 \mathbf{s}_1^T$, $\mathbf{s}_2 \mathbf{s}_2^T$, and $\mathbf{s}_3 \mathbf{s}_3^T$ allows the three vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ to be calculated (problem 8.5). Note that, because of its special structure, knowing one row or one column of the matrix $\mathbf{s}_i \mathbf{s}_i^T$ is enough to determine the entire matrix and hence the value of $\mathbf{s}_i$ to within a multiplicative constant.

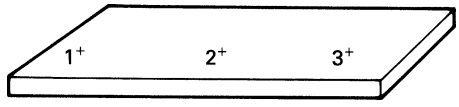

Figure 8.10 Measurement positions marked on a simple plate for performing a modal test.

Hence, through measurements taken from the plot in Figure 8.9, along with the above computations and two more measurements at the remaining two measurement points, the receptance matrix, given by Equation (5.68), is completely determined, as are the eigenvalues and eigenvectors of the system. However, in general this does *not* provide enough information to determine the values of the matrices M, D, and K, because the eigenvectors may not be scaled (normalized) properly. Also, the mathematical model constructed in this way is dependent upon the choice of measurement points and as such should not be considered to be unique.

Given the modal data, however, the receptance matrix can be used to predict the response of the system as the result of any input, to compare the modal data with theoretical models, or, along with other information, to construct physical models of the structure. The question of uniqueness remains, and the physical models developed should always be referenced to the measurement and disturbance position used in performing the test.

The single-degree-of-freedom method is really not based on a principle of parameter identification but rather is an unsophisticated first attempt to 'read' the experimental data and to fit the data into a specific underdamped vibration model. An attempt to make the single-degree-of-freedom approach more sophisticated is to plot the real part of the frequency response function versus the imaginary part near the resonant frequency. This yields the Nyquist plot of Figure 1.14. In this domain, the theory predicts a circle; hence, the experimental data around a resonant peak can be 'fit' using a least-squares circle fit to produce the 'best' circle for the given data. Then the frequency and damping values can be taken from the Nyquist plot. This method is referred to as the *circle fit method* and brings the problem a little closer to using the science of parameter estimation (see Ewins, 2000, or Zaveri, 1984, for a more complete account).

The single-degree-of-freedom method is in obvious error for systems with closely spaced modes and/or highly coupled modes. Hence, several other methods have been developed to take into consideration the effects of coupling. These methods are usually referred to as multiple-degree-of-freedom curve fits and consist largely of adding correction terms to Equation (8.23) to take into consideration mode coupling effects. Because of space limitations, these methods are not discussed here. The reader is referred to the note section at the end of the chapter for references to more advanced methods.

8.6 MODAL DATA EXTRACTION (TIME DOMAIN)

An alternative approach to extracting the modal data of a structure from the vibrational response of the structure is based in the time domain. Time domain methods have been more successful in identifying closely spaced or repeated natural frequencies. They also offer a more systematic way of determining the appropriate order (number of degrees of freedom) of a test structure and generally identify a larger number of natural frequencies than frequency domain methods do. Time domain methods are also referred to as damped complex exponential response methods (Allemang, 1984).

The first time domain method to make an impact on the vibration testing community was developed by Ibrahim (1973) and has since become known as the Ibrahim time domain (ITD) method. The original method is based on the state equations of a dynamic system [i.e., equation (2.20)]. A simplified version of the method is presented here.

The solution for the dynamic system in physical coordinates in the form of Equation (2.13) is given in Equation (3.68) and repeated here as

$$\mathbf{q}(t) = \sum_{r=1}^{2n} c_r \mathbf{r}_r e^{\lambda_r t} \tag{8.25}$$

where λ_r are the complex latent roots, or eigenvalues, of the system and the $\mathbf{u}_r$ are the system eigenvectors. Here, it is convenient to absorb the constant c_r into the vector $\mathbf{u}_r$ and write Equation (8.25) as

$$\mathbf{q}(t) = \sum_{r=1}^{2n} \mathbf{p}_r e^{\lambda_r t} \tag{8.26}$$

where the vector $\mathbf{p}_r$ has an arbitrary norm. If the response is measured at discrete times, t_i, then Equation (8.26) becomes simply

$$\mathbf{q}(t_i) = \sum_{r=1}^{2n} \mathbf{p}_r e^{\lambda_r t_i} \tag{8.27}$$

For simplicity of explanation, assume that the structure is measured in n places (this assumption can be relaxed) at $2n$ times, where n is the number of degrees of freedom exhibited by the test structure. Writing Equation (8.27) $2n$ times results in the matrix equation

$$X = PE(t_i) \tag{8.28}$$

where $\mathbf{X}$ is an $n \times 2n$ matrix with columns consisting of the $2n$ vectors $\mathbf{q}(t_1), \mathbf{q}(t_2), \ldots, \mathbf{q}(t_{2n})$, P is the $n \times 2n$ matrix with columns consisting of the $2n$ vectors $\mathbf{p}_r$, and $E(t_i)$ is the $2n \times 2n$ matrix given by

$$E(t_i) = \begin{bmatrix} e^{\lambda_1 t_1} & e^{\lambda_1 t_2} & \cdots & e^{\lambda_1 t_{2n}} \\ e^{\lambda_2 t_1} & e^{\lambda_2 t_2} & & e^{\lambda_2 t_{2n}} \\ \vdots & \vdots & & \vdots \\ e^{\lambda_{2n} t_1} & e^{\lambda_{2n} t_2} & \cdots & e^{\lambda_{2n} t_{2n}} \end{bmatrix} \tag{8.29}$$

Likewise, if these same responses are measured Δt seconds later, i.e., at time $t_i + \Delta t$, Equation (8.28) becomes

$$Y = PE(t_i + \Delta t) \tag{8.30}$$

where the columns of Y are defined by $\mathbf{y}(t_i) = \mathbf{q}(t_i + \Delta t)$, and $E(t_i + \Delta t)$ is the matrix with ijth element equal to $e^{\lambda_i (t_j + \Delta t)}$. Equation (8.30) can be factored into the form

$$Y = P' E(t_i) \tag{8.31}$$

where the matrix P' has as its ith column the vector $\mathbf{p}_i e^{\lambda_i \Delta t}$. This process can be repeated for another time increment Δt later to provide the equations

$$Z = P'' E(t_i) \tag{8.32}$$

where the columns of Z are the vectors $\mathbf{q}(t_i + 2\Delta t)$, and so on.

Collecting Equations (8.28) and (8.31) together yields

$$\Phi = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} P \\ P' \end{bmatrix} E(t_i) = \Psi E(t_i) \tag{8.33}$$

where the $2n \times 2n$ matrices Φ and Ψ have obvious definitions. Likewise, Equations (8.31) and (8.32) can be combined to form

$$\Phi' = \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} P' \\ P'' \end{bmatrix} E(t_i) = \Psi' E(t_i) \tag{8.34}$$

where $\Phi' = [Y^T \ Z^T]^T$ and $\Psi' = [P'^T \ P''^T]^T$. Note that the matrices Φ, Φ', Ψ, and Ψ' are all square $2n \times 2n$ matrices that are assumed to be nonsingular (Ibrahim and Mikulcik, 1976).

Equations (8.33) and (8.34) can be used to calculate a relationship between the vectors that make up the columns of the matrix Ψ and those of Ψ', namely

$$\Phi' \Phi^{-1} \Psi = \Psi' \tag{8.35}$$

so that

$$\Phi' \Phi^{-1} \Psi_i = \Psi_i', \qquad i = 1, 2, \ldots, 2n \tag{8.36}$$

where the subscript i denotes the ith column of the indexed matrix. However, the ith column of P' is $\mathbf{p}_i = e^{\lambda_i \Delta t}$, so that

$$\Psi_i' = e^{\lambda_i \Delta t} \Psi_i \tag{8.37}$$

Comparison of Equations (8.37) and (8.36) yields

$$\Phi' \Phi^{-1} \Psi_i = e^{\lambda_i \Delta t} \Psi_i \tag{8.38}$$

Note that this last expression states that the complex scalar $e^{\lambda_i \Delta t} = \beta + \gamma j$ is an eigenvalue of the matrix $\Phi' \Phi^{-1}$ with complex eigenvector Ψ_i. The $2n \times 1$ eigenvector Ψ_i has as its first n elements the eigenvector or mode shape of the structure. That is, $\Psi_i = [\mathbf{p}_i^T \quad \mathbf{p}_i^T e^{\lambda_i \Delta t}]^T$, where $\mathbf{p}_i$ are the system eigenvectors.

The eigenvalues of Equation (8.38) can be used to calculate the eigenvalues of the system and hence the damping ratios and natural frequencies. In particular, since $e^{\lambda_i \Delta t} = \beta + \gamma j$, it follows that

$$\mathrm{Re}\lambda_i = \frac{1}{\Delta t} \ln(\gamma^2 + \beta^2) \tag{8.39}$$

and

$$\mathrm{Im}\lambda_i = \frac{1}{\Delta t} \tan^{-1}\left(\frac{\gamma}{\beta}\right) \tag{8.40}$$

With proper consideration for sampling time and the interval for the arc tangent in Equation (8.40), Equation (8.38) yields the desired modal data.

To summarize, this method first constructs a matrix of measured time responses, forms the product $\Phi'\Phi^{-1}$, and then numerically calculates the eigenvalue–eigenvector problem for this matrix. The resulting calculation then yields the mode shapes, natural frequencies, and damping ratios of the structure. The natural frequencies and damping ratios follow from Equations (8.39) and (8.40) since (see Chapter 1)

$$\omega_i = [(\mathrm{Re}\lambda_i)^2 + (\mathrm{Im}\lambda_i)^2]^{1/2} \tag{8.41}$$

and

$$\zeta_i = \frac{\mathrm{Re}\lambda_i}{\omega_i} \tag{8.42}$$

Note that the key difference between the time domain approach to modal analysis and the frequency domain approach is that the time domain approach constructs a matrix from the time response and numerically computes the modal data by solving an eigenvalue problem. On the other hand, the frequency domain method extracts and curve-fits the modal data from the digital Fourier transform of the measured input and response data.

Two other major approaches in the time domain have been developed. One, called the polyreference method (Vold and Rocklin, 1982), again uses a state-space approach and has been developed commercially. The other time domain method is based on realization theory developed by control theorists, as introduced in Section 7.6. This method, advocated by Juang and Pappa (1985), also uses a state-space approach and is called the *eigensystem realization algorithm* (often abbreviated ERA). The eigensystem realization algorithm method introduces several important aspects to modal test analysis and hence is discussed next. The method is based on the realization theory introduced in Section 7.6 and puts modal testing on the firm theoretical foundation of linear systems theory (Juang, 1994).

Unlike the other time domain approaches, the eigensystem realization algorithm determines a complete state-space model of the structure under test. In particular, using the notation of Equations (7.1) and (7.2), the eigensystem realization algorithm identifies all three of the matrices A, B, and C. This is an important aspect because it addresses the nonuniqueness of modal analysis by specifying the measurement locations through the matrices B and C. In addition, the eigensystem realization algorithm approach firmly attaches the modal testing problem to the more mature discipline of realization used in control theory (Section 7.6). This approach is computationally more efficient and provides a systematic means of determining the order of the test model. The method is based on a discrete-time version of the state-space solution given by Equation (5.16) with $\mathbf{f}(\tau) = B\mathbf{u}(\tau)$. The solution of Equation (7.1), starting at time t_0, can be written as

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\mathbf{u}(\boldsymbol{\tau})\,d\boldsymbol{\tau} \tag{8.43}$$

This expression can be written at discrete time intervals equally spaced at times 0, $\Delta t, 2\Delta t, \ldots, k\Delta t, \ldots$, by making the substitutions $t = (k+1)\Delta t$ and $t_0 = k\Delta t$. This yields

$$\mathbf{x}((k+1)\Delta t) = e^{A\Delta t}\mathbf{x}(k\Delta t) + \int_{k\Delta t}^{(k+1)\Delta t} e^{A((k+1)\Delta t-\tau)}B\mathbf{u}(\tau)\,d\tau \tag{8.44}$$

Note that the integration period is now performed over a small increment of time, Δt. If $\mathbf{u}(t)$ is assumed to be constant over the interval Δt and to have the value $\mathbf{u}\ (k\Delta t)$, this last expression can be simplified to

$$\mathbf{x}(k+1) = e^{A\Delta t}\mathbf{x}(k) + \left[\int_0^{\Delta t} e^{A\sigma}\, d\sigma B\right]\mathbf{u}(k) \tag{8.45}$$

Here, $\mathbf{x}(k+1)$ is used to denote $\mathbf{x}((k+1)\Delta t)$, and so on, and the integration variable has been changed to $\sigma = (k+1)\Delta t - \tau$.

Equation (8.45) can be further reduced in form by defining two new matrices A' and B' by

$$A' = e^{A\,\Delta t}, \qquad B' = \int_0^{\Delta t} e^{A\sigma}\, d\sigma B \tag{8.46}$$

Thus, a discrete-time version of Equations (7.1) and (7.2) becomes

$$\begin{aligned} \mathbf{x}(k+1) &= A'\mathbf{x}(k) + B'\mathbf{u} \\ \mathbf{y}(k) &= C\mathbf{x}(k) \end{aligned} \tag{8.47}$$

The last expression allows the measurement taken at discrete times, $\mathbf{y}(k)$, to be related to the state matrix by the definition of the matrices A' and B'.

Consider a system subject to an impulse, $\mathbf{y}(1) = CB'\mathbf{u}(1)$, where $\mathbf{u}(1)$ is a vector of zeros and ones, depending upon where the impulse is applied. If the impulse is applied at m different points one at a time, all the response vectors at $k = 1, \mathbf{y}(k)$, corresponding to each of these m impulses, can be gathered together into a single $s \times m$ matrix Y, given by

$$Y(1) = CB' \tag{8.48}$$

In a similar fashioin, it can be shown using Equation (8.47) that for any measurement time k

$$Y(k) = CA^{k-1}B' \tag{8.49}$$

The matrices $Y(k)$ defined in this manner are called *Markov parameters*.

The Markov parameters for the test structure can next be used to define a larger matrix of measurements, denoted by $H_{ij}(k)$, for any arbitrary positive integers i and j by

$$H_{ij}(k) = \begin{bmatrix} Y(k) & Y(k+1) & \cdots & Y(k+j) \\ Y(k+1) & Y(k+2) & & \\ \vdots & \vdots & & \vdots \\ Y(k+i) & Y(k+u+1) & & Y(k+i+j) \end{bmatrix} \tag{8.50}$$

This $(i+1) \times (j+1)$ block matrix is called a *generalized Hankel matrix*. If the arbitrarily chosen integers $i+1$ and $j+1$ are both greater than $2n$, the rank of $H_{ij}(k)$ will be $2n$. This follows from realizing that the Hankel matrix can be factored and written as

$$H_{ij}(k) = O_i A^{k-1} R_j \tag{8.51}$$

where O_i is the observability matrix and R_j is the controllability matrix given by

$$O_i = \begin{bmatrix} C \\ CA' \\ \vdots \\ C(A')^i \end{bmatrix}, \qquad R_j = [B'A'B' \cdots (A')^j B'] \tag{8.52}$$

introduced in Section 7.2. Note that the rank of $H_{ij}(k)$ is the order of the matrix A', and hence the order of the system state matrix A, as long as i and j are large enough. Equation (8.52) also indicates that the smallest-order realization of A will have an order equal to the rank of O_i and R_j for a controllable and observable system. (Recall the balanced model reduction problem of Section 7.7.)

The eigensystem realization algorithm is based on a singular-value decomposition (see Section 7.7) of the Hankel matrix $H(0)$. Using Equation (7.60), the matrix $H(0)$ can be written as

$$H(0) = U\boldsymbol{\Sigma}V^T \tag{8.53}$$

where Σ is the diagonal matrix of the singular values, σ_i, of $H(0)$, and U and V are defined in Equations (7.60), (7.62), and (7.63). If enough measurements are taken, so that i and j are large enough, there will be an index r such that $\sigma_r >> \sigma_r + 1$. Just as in the model reduction problem of Section 7.7, this value of r yields a logical index for defining the order of the structure under test, i.e., $r = 2n$.

Partitioning U and V^T into the first r columns and rows respectively, and denoting the reduced matrices by U_r and V_r yields

$$H(0) = U_r \boldsymbol{\Sigma}_r V_r^T \tag{8.54}$$

Here, $\boldsymbol{\Sigma}_r$ denotes the $r \times r$ diagonal matrix of the first r singular values of $H(0)$. Juang and Pappa (1985) showed that

$$A' = \boldsymbol{\Sigma}_r^{-\min 1/2} U_r^T H(1) V_r \boldsymbol{\Sigma}_r^{-\min 1/2} \tag{8.55}$$

$$B' = \boldsymbol{\Sigma}_r^{1/2} V_r E_p \tag{8.56}$$

$$C = E_p^T U_r \boldsymbol{\Sigma}_r^{1/2} \tag{8.57}$$

where E_p is defined as the $p \times r$ matrix

$$E_p^T = [I_p \quad \mathbf{0}_p \quad \cdots \quad \mathbf{0}_p] \tag{8.58}$$

Here I_p is a $p \times p$ identity matrix and $\mathbf{0}_p$ is a $p \times p$ matrix of zeros.

The modal parameters of the test structure are calculated from the numerical eigensolution of the matrix A'. The mode shapes of the structure are taken from the eigenvectors of A', and the natural frequencies and damping ratios are taken from the eigenvalues of A.

The eigenvalues of A, the continuous-time state matrix, denoted by λ_i, are found from the eigenvalues of A' via the formula (see Juang and Pappa, 1985)

$$\lambda_i = \frac{\ln \lambda_i(A')}{\Delta t} \tag{8.59}$$

For underdamped structures, the modal parameters are determined from Equations (8.41) and (8.42).

8.7 MODEL IDENTIFICATION

A major reason for performing a modal test is for validation and verification of analytical models of the test structure. What is often desired is a mathematical model of the structure under consideration for the purpose of predicting how the structure will behave under a variety of different loadings, to provide the plant in a control design, or to aid in the design process in general. This section discusses three types of model of the structure that result from using modal test data. The three types of model considered are the *modal model*, discussed in the previous section, the *response model*, and the *physical model* (also called the *spatial model*).

The modal model is simply the natural frequencies, damping ratios, and mode shapes as given in the previous two sections. The modal model is useful in several ways. First, it can be used to generate both static and dynamic displays of the mode shapes, lending visual insight into the manner in which the structure vibrates. These displays are much more useful to examine, in many cases, than reading a list of numbers. An example of such a display is given in Figure 8.11.

A second use of modal models is for direct comparison with the predicted frequencies and modal data for an analytical model. Frequently, the analytical model will not predict the measured frequencies. As a result, the analytical model is changed, iteratively, until it produces the measured natural frequencies. The modified analytical model is then considered an improvement over the previous model. This procedure is referred to as model updating and is introduced in the following section.

The frequency response model of the system is given by the receptance matrix defined in Section 5.7 and used in Section 8.5 to extract the modal parameters from the measured frequency response functions. From Equations (5.68) and (5.65) the measured receptance

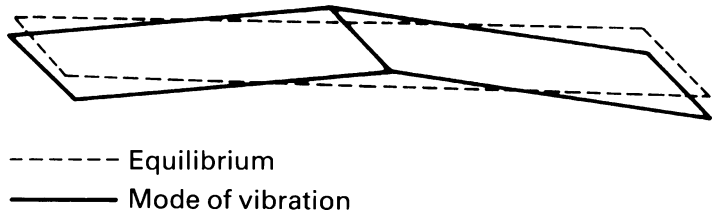

Figure 8.11 Display of the mode shape of a simple test structure.

matrix can be used to calculate the response of the test structure data to any input $\mathbf{f}(t)$. To see this, consider taking the Laplace transform, denoted £[·], of the dynamic equation (2.13), i.e.,

$$(s^2M + sD + K)\pounds[\mathbf{q}] = \pounds[\mathbf{f}] \tag{8.60}$$

so that

$$\pounds[\mathbf{q}(t)] = \frac{1}{s^2M + sD + K}\pounds[\mathbf{f}] \tag{8.61}$$

Letting $s = j\omega$ in Equation (8.61) yields

$$\Im(\mathbf{q}) = \alpha(\omega)\Im(\mathbf{f}) \tag{8.62}$$

where it is noted that the Laplace transform of a function evaluated at $s = j\omega$ yields the exponential Fourier transform, denoted by $\Im[.]$. Hence, Equation (8.62) predicts the response of the structure for any input $\mathbf{f}(t)$.

In some instances, such as control or design applications, it would be productive to have a physical model of the structure in spatial coordinates, i.e., a model of the form

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{8.63}$$

where $\mathbf{q}(t)$ denotes a vector of physical positions on the structure. The obvious way to construct this type of model from measured data is to use the orthogonal mass normalized matrix of eigenvectors, S_m, defined in Section 3.5, and to 'undo' the theoretical modal analysis by using Equations (3.69) and (3.70). That is, if these equations are pre- and postmultiplied by the inverses of S_m^T and S_m, they yield

$$\begin{aligned} M &= (S_m^T)^{-1}S_m^{-1} \\ D &= (S_m^T)^{-1}\mathrm{diag}[2\zeta_i\omega_i]S_m^{-1} \\ K &= (S_m^T)^{-1}\mathrm{diag}[\omega_i^2]S_m^{-1} \end{aligned} \tag{8.64}$$

where the ω_i, ζ_i, and the columns of S_m are the measured modal data. Unfortunately, this formulation requires the damping mechanism to be proportional to velocity and measured data for all n modes to be available. These two assumptions are very seldom met. Usually, the modal data are too incomplete to form all S_m. In addition, the measured eigenvectors are seldom scaled properly.

The problem of calculating the mass, damping, and stiffness matrices from experimental data is very difficult and does not result in a unique solution (it is, after all, an inverse problem – see Lancaster and Maroulas, 1987, and Starek and Inman, 1997). Research continues in this area by examining reduced-order models and approaching the problem as one of improving existing analytical models, the topic of the next section.

8.8 MODEL UPDATING

Analytical models are only useful if they are verified against an experimental model. Often, analysis gets fairly close to predicting experimental data, and an approach called *model updating* is used to change slightly or update the analytical model to predict or agree with experimental data. Friswell and Mottershead (1995) present both an introduction to model updating and a good summary of most techniques developed up to that date. One approach to model updating is to use the analytical model of a structure to compute natural frequencies, damping ratios, and mode shapes, and then to test, or perform an experimental modal analysis on the structure and examine how well the analytical modal data predict the measured modal data. If there is some difference, then the analytical model is adjusted or updated until the updated analytical model predicts the measured frequencies and mode shapes. The procedure is illustrated by the following simple example.

Example 8.8.1

Suppose the analytical model of a structure is derived as the following two-degree-of-freedom model (from example 3.3.2):

$$I\ddot{\mathbf{q}}(t)+\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{q}(t)=0$$

This has natural frequencies computed to be $\omega_1=\sqrt{2-\sqrt{2}}$ and $\omega_2=\sqrt{2+\sqrt{2}}$. Suppose a modal test is performed and that the measured frequencies are $\omega_1=\sqrt{2}$ and $\omega_2=\sqrt{3}$. Find adjustments to the analytical model such that the new updated model predicts the measured frequencies.

Let ΔK denote the desired correction in the analytical model. To this end, consider the correction matrix

$$\Delta K=\begin{bmatrix} 0 & 0 \\ \Delta k_1 & \Delta k_2 \end{bmatrix} \tag{8.65}$$

The characteristic equation of the updated model is then

$$\lambda^2-(4+\Delta k_1)\lambda+(2+3\Delta k_2+\Delta k_1)=0 \tag{8.66}$$

The characteristic equation of the system with the experimental frequencies would be

$$\lambda^2-5\lambda+6=0 \tag{8.67}$$

Comparing coefficients in Equations (8.66) and (8.67) yields the updated parameters

$$\Delta k_1=1 \qquad \text{and} \qquad \Delta k_2=1$$

The new updated model is

$$I\ddot{\mathbf{q}}(t)+\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}\mathbf{q}(t)=0$$

which has frequencies $\omega_1=\sqrt{2}$ and $\omega_2=\sqrt{3}$ as measured.

While the updated model in example 8.8.1 does in fact produce the correct frequencies, the stiffness matrix of the updated model is no longer symmetric, nor does it retain the original connectivity of, in this case, the two-degree-of-freedom spring–mass system. Other methods exist that address these issues (see, for instance, Halevi, Tarazaga, and Inman, 2004). Model updating, as stated here, is related to the pole placement and eigenstructure assignment methods used in Section 7.3. Instead of computing control gains, as done in Section 7.3, the same computation yields an updating procedure if the gain matrices are considered to be the correction matrices (e.g., ΔK). Basically, if measured mode shapes, damping ratios, and frequencies are available, then an eigenstructure assignment method may be used to compute updated damping and stiffness matrices. If only damping ratios and frequencies are available, then pole placement can be used to update the damping and stiffness matrices. Friswell and Mottershead (1995) should be consulted for other methods. The literature should be consulted for latest approaches.

CHAPTER NOTES

Modal testing dominates the field of vibration testing. There are, of course, other identification methods and other types of vibration experiment. The main reference for this chapter is the text by Ewins (1984, 2000), which discusses each topic of this chapter, with the exception of Section 8.6 on time domain methods, in much more detail. Ewins (1984) was the first book to describe modal testing. Another good general reference for modal testing is the book by Zaveri (1984). Allemang (1984) has provided a review article on the topic as well as an extended bibliography. Each year since 1981, an International Modal Analysis Conference (IMAC) has been held, indicating the continued activity in this area. McConnell (1995) presents a more general look at vibration testing. Maia and Silva (1997) present an edited volume on modal analysis and testing written by Ewins' former students.

The material in Section 8.3 is a brief introduction to signal processing. More complete treatments along with the required material from Section 8.4 on random signals can be found in Otnes and Enochson (1972) or Doeblin (1980, 1983). An excellent introduction to signal processing and the appropriate transforms is the book by Newland (1985). There are many methods of extracting modal data from frequency response data, and only one is covered in Section 8.5. Ewins (2000) and Zaveri (1984) discuss others. Most commercially available modal software packages include details of the various modal data extraction methods used in the package.

The time domain methods are mentioned briefly in Ewins (2000) and by Allemang (1984). Leuridan, Brown, and Allemang (1986) discuss multiple-input and multiple-output time domain methods and present some comparison of methodology. However, the eigensystem realization algorithm is not included in these works. An excellent treatment of identification methods driven by structural dynamics is given in the text by Juang (1994). A good introduction to the realization theory upon which the eigensystem realization algorithm depends can be found in Chen (1970). A least-squares regression technique for system identification of discrete-time dynamic systems is given by Graupe (1976). The topic of physical modeling using measurements discussed in Section 8.7 is really an inverse problem and/or an identification problem such as defined in Rajaram and Junkins (1985) and Lancaster and Maroulas (1987). Virgin (2000) provides an excellent introduction to experimental

analysis for nonlinear systems. The use of vibration testing in structural health monitoring is reviewed in Doebling, Farrar, and Prime (1998). Inman *et al.* (2005) provide reviews of health monitoring and machinery diagnostics.

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PROBLEMS

8.1 Plot the error in measuring the natural frequency of a single-degree-of-freedom system of mass 10 and stiffness 35 if the mass of the exciter is included in the calculation and ranges from 0.4 to 5.0.

8.2 Calculate the Fourier transform of $f(t) = 3\sin 2t + 2\sin t - \cos t$ and plot the spectral coefficients.

8.3 Consider a signal $x(t)$ with a maximum frequency of 500 Hz. Discuss the choice of record length and sampling interval.

8.4 The eigenvalues and mode shapes of a structure are given next. Develop a two-degree-of-freedom model of the structure that yields the same modal data if the mass matrix is known to be diag[4 1]:

$$\lambda_1 = -0.2134 \pm 1.2890j$$
$$\lambda_2 = -0.0366 \pm 0.5400j$$
$$\mathbf{u}_1 = [0.4142 \quad 1]^T$$
$$\mathbf{u}_2 = [1.000 \quad -0.4142]^T$$

8.5 Consider the vector $\mathbf{s}^T = [s_1 \ s_2 \ s_3]$. Write the outer product matrix $\mathbf{s}\mathbf{s}^T$ and show that the elements in one row (or one column) of the matrix completely determine the other six elements of the matrix. In particular, calculate the receptance matrix of Section 8.5 for the following measured data:

$\zeta_1 = 0.01,$	$\zeta_2 = 0.2,$	$\zeta_3 = 0.01$	
$\omega_{d1} = 2,$	$\omega_{d2} = 10,$	$\omega_{d3} = 12$	
$G(\omega_{d1}) = 1,$	$G(\omega_{d2}) = -1,$	$G(\omega_3) = 3$	(force at position 1 and transducer at position 3)
$G(\omega_{d1}) = -3,$	$G(\omega_{d2}) = 2,$	$G(\omega_{d3}) = 4$	(force at 1, transducer at 2)
$G(\omega_{d1}) = 5,$	$G(\omega_{d2}) = 2,$	$G(\omega_{d3}) = -2$	(force at 1, transducer at 1)

What are the eigenvectors of the system?

8.6 Calculate the receptance matrix for the two-degree-of-freedom system with

$$M = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}, \qquad K = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}, \qquad D = \begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix}$$

8.7 (**a**) Consider the system of problem 8.6. Using any integration package, numerically solve for the free response of this system to the initial conditions $\mathbf{q}(0) = [1\ 0]^T$ and $\dot{\mathbf{q}}(0) = [0\ 0]^T$.

(**b**) Using the solution to (a), generate the matrices Φ and Φ' of Equations (8.33) and (8.34). Using Φ and Φ' and Equation (8.38), calculate the eigenvalues and eigenvectors of the system. Check to see if they satisfy the eigenvalue problem for this system.

8.8 Repeat problem 8.7 using ERA. This will require the availability of software performing SVD, and the like, such as MATLAB.

8.9 Solve the problem of example 8.1.1 again for the case where the measured frequencies are $\omega_1 = \sqrt{2}$ and $\omega_2 = 2$.

8.10 Solve the model updating problem of example 8.8.1 using the pole placement method of Sections 6.6 and 7.3 to see if it is possible to obtain a symmetric updated stiffness matrix with the required eigenvalues.

9
Distributed-parameter Models

9.1 INTRODUCTION

This chapter presents an informal introduction to the vibrations of systems having distributed mass and stiffness, often called distributed-parameter systems. For lumped-parameter systems, the single-degree-of-freedom system served as a familiar building block with which more complicated multiple-degree-of-freedom structures can be modeled. Similarly, an examination of the vibrations of simple models of strings, beams, and plates provides a set of 'building blocks' for understanding the vibrations of systems with distributed mass, stiffness, and damping parameters. Such systems are referred to as distributed-parameter systems, elastic systems, continuous systems, or flexible systems.

This chapter focuses on the basic methods used to solve the vibration problem of flexible systems. The purpose of this chapter is to list the equations governing the linear vibrations of several distributed-parameter structures, list the assumptions under which they are valid, and discuss some of the possible boundary conditions for such structures. The equations are not derived; they are simply stated with a few representative examples of solutions. References such as Meirovitch (2001) and Magrab (1979) should be consulted for derivations of the equations of motion. These solution methods are made rigorous and discussed in more detail in Chapter 10.

9.2 VIBRATION OF STRINGS

Figure 9.1 depicts a string fixed at both ends and displaced slightly from its equilibrium position. The lateral position of the string is denoted by w. The value of w will depend not only on the time t but also on the position along the string x. This spatial dependency is the essential difference between lumped-parameter systems and distributed-parameter systems – namely the deflection of the string is a function of both x and t and hence is denoted by $w = w(x, t)$.

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

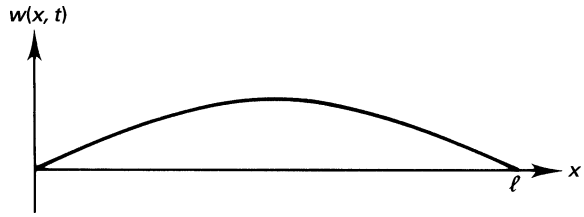

Figure 9.1 String fixed at both ends and displaced from its equilibrium position.

For small deflections and slopes of the string where the restoring forces in the vertical displacement of the string are due entirely to the axial tension T in the string, the equation governing the function $w(x, t)$ can be shown to be

$$w_{xx}(x, t) - \frac{1}{c^2} w_{tt}(x, t) \tag{9.1}$$

where $c = \sqrt{T/\rho}$, ρ is the mass per unit length of the string, w_{xx} denotes the second partial derivative of $w(x, t)$ with respect to x, and w_{tt} denotes the second partial derivative of $w(x, t)$ with respect to t. Alternatively, these partial derivatives are denoted by $\partial^2/\partial x^2$ and $\partial^2/\partial t^2$. The derivation of this equation can be found in many texts (see, for instance, Inman, 2001). Briefly, Equation (9.1) comes from applying a simple force balance on an infinitesimal element of the string. The quantity w_{xx} is the derivative of the slope and hence is proportional to a restoring force. The right-hand side of Equation (9.1) is just the acceleration multiplied by a coefficient and hence the inertia of the string.

The function $w(x, t)$ that satisfies Equation (9.1) must also satisfy two initial conditions because of the second-order time derivatives. The second-order spatial derivative implies that two spatial conditions must also be satisfied. In the cases of interest here, the value of x will vary over a finite range. Physically, if the string is fixed at both ends, then $w(x, t)$ would have to be zero at $x = 0$ and again at $x = \ell$. These conditions are known as boundary conditions. Mathematically, there must be one boundary condition (or one constant of integration) for each spatial derivative and one initial condition for each time derivative.

The problem of finding the lateral vibrations of a string fixed at both ends can then be summarized as follows. Find a function $w(x, t)$ such that

$$\begin{aligned} &w_{xx}(x, t) = \frac{1}{c^2} w_{tt}(x, t), \qquad x \in (0, \ell) \qquad \text{for } t > 0 \\ &w(0, t) = w(\ell, t) = 0, \qquad t > 0 \\ &w(x, 0) = w_0(x) \qquad \text{and} \qquad w_t(x, 0) = \dot{w}_0(x) \qquad \text{at } t = 0 \end{aligned} \tag{9.2}$$

where $w_0(x)$ and $\dot{w}_0(x)$ are specified time-invariant functions representing the initial (at $t = 0$) displacement and velocity distribution of the string. The notation $x \in (0, \ell)$ means that the equation holds for values of x in the interval $(0, \ell)$.

One approach used to solve the system given by Equations (9.2) is to assume that the solution has the form $w(x, t) = X(x)T(t)$. This approach is called the method of *separation of variables* (see, for instance, Boyce and DiPrima, 2000). The method proceeds by

substitution of this assumed separated form into Equation (9.2), which yields the set of equations

$$
\begin{aligned}
&X''(x)T(t) = \frac{1}{c^2}X(x)\ddot{T}(t) \\
&X(0)T(t) = 0 \qquad \text{and} \qquad X(\ell)T(t) = 0 \\
&X(x)T(0) = w_0(x) \qquad \text{and} \qquad X(x)\dot{T}(0) = \dot{w}_0(x)
\end{aligned}
\tag{9.3}
$$

where the primes denote total derivatives with respect to x and the overdots indicate total time differentiation. Rearranging the first equation in system (9.3) yields

$$
\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} \tag{9.4}
$$

Differentiating this expression with respect to x yields

$$
\frac{d}{dx}\left(\frac{X''(x)}{X(x)}\right) = 0
$$

or that

$$
\frac{X''(x)}{X(x)} = \sigma \tag{9.5}
$$

where σ is a constant (independent of t or x) of integration. The next step then is to solve Equation (9.5), i.e.,

$$
X''(x) - \sigma X(x) = 0 \tag{9.6}
$$

subject to the two boundary conditions, which become

$$
X(0) = 0 \qquad \text{and} \qquad X(\ell) = 0 \tag{9.7}
$$

since $T(t) \neq 0$ for most values of t.

The nature of the constant σ needs to be determined next. There are three possible choices for σ: it can be positive, negative, or zero. If $\sigma = 0$, the solution of Equation (9.6) subject to Equation (9.7) becomes $X(x) = 0$, which does not satisfy the condition of a nontrivial solution. If $\sigma > 0$, then the solution to Equation (9.6) is of the form $X(x) = A_1 \cosh(\sigma x) + A_2 \sinh(\sigma x)$. Applying the initial conditions to this form of the solution then yields

$$
\begin{aligned}
0 &= A_1 \\
0 &= A_2 \sinh(\ell\sigma)
\end{aligned}
$$

so that again the only possible solution is the trivial solution, $w(x, t) = 0$.

Thus, the only nontrivial choice for σ is that it must have a negative value. To indicate this, $\sigma = -\lambda^2$ is used. Equations (9.6) and (9.7) become

$$X''(x) + \lambda^2 X(x) = 0$$
$$X(0) = X(\ell) = 0 \tag{9.8}$$

Equations (9.8) are called a *boundary value problem* as the values of the solution are specified at the boundaries. The form of the solution of Equations (9.8) is

$$X(x) = A_1 \sin \lambda x + A_2 \cos \lambda x \tag{9.9}$$

Applying the boundary conditions to the solution (9.9) indicates that the constants A_1 and A_2 must satisfy

$$A_2 = 0 \qquad \text{and} \qquad A_1 \sin \lambda \ell = 0 \tag{9.10}$$

Examination of Equation (9.10) shows that the only nontrivial solutions occur if

$$\sin \lambda \ell = 0 \tag{9.11}$$

or when λ has the value $n\pi/\ell$, where n is any integer value, denoted by

$$\lambda_n = \frac{n\pi}{\ell}, \qquad n = 1, 2, \ldots, \infty$$

Note that the $n = 0$, $\lambda = 0$ is omitted in this case because it results in zero solution. The solution given in Equation (9.9) is thus the infinite set of functions denoted by

$$X_n(x) = A_n \sin\left(\frac{n\pi}{\ell} x\right) \tag{9.12}$$

where n is any positive integer. Equation (9.11) resulting from applying the boundary conditions is called the *characteristic equation* and the values λ_n are called *characteristic values*.

Substitution of Equation (9.12) into Equation (9.4) then shows that the temporal coefficient $T(t)$ must satisfy the infinite number of equations

$$\ddot{T}(t) + \left(\frac{n\pi}{\ell}\right)^2 c^2 T(t) = 0 \tag{9.13}$$

The solution of these equations, one for each n, is given by

$$T_n(t) = A_n^1 \sin\left(\frac{n\pi c}{\ell} t\right) + A_n^2 \cos\left(\frac{n\pi c}{\ell} t\right) \tag{9.14}$$

where A_n^1 and A_n^2 are the required constants of integration and the subscript n has been added to indicate that there are an infinite number of solutions of this form. Thus, the solutions of Equation (9.1), also infinite in number, are of the form

$$w_n(x, t) = a_n \sin\left(\frac{n\pi c}{\ell} t\right) \sin\left(\frac{n\pi}{\ell} x\right) + b_n \cos\left(\frac{n\pi c}{\ell} t\right) \sin\left(\frac{n\pi}{\ell} x\right) \tag{9.15}$$

Here, a_n and b_n are constants representing the product of A_n^1 and A_n^2 of Equation (9.14) and the constants of Equation (9.12).

Since Equation (9.15) is a linear system, the sum of all of these solutions is also a solution, so that

$$w(x,t)=\sum_{n=1}^{\infty}\left[a_n\sin\left(\frac{n\pi c}{\ell}t\right)+b_n\cos\left(\frac{n\pi c}{\ell}t\right)\right]\sin\left(\frac{n\pi}{\ell}x\right) \tag{9.16}$$

This infinite sum may or may not converge, as is discussed in Chapter 11. Next, the constants a_n and b_n need to be calculated. These constants come from the initial conditions. Applying the displacement initial condition to Equation (9.16) yields

$$w(x,0)=w_0(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{\ell}x\right) \tag{9.17}$$

This equation can be solved for the constants b_n by using the (orthogonality) property of the functions $\sin(n\pi/\ell)x$:

$$\int_0^{\ell}\sin\left(\frac{n\pi}{\ell}x\right)\sin\left(\frac{m\pi}{\ell}x\right)dx=\delta_{mn}=\begin{cases}\dfrac{1}{2}, & m=n\\ 0, & m\neq n\end{cases} \tag{9.18}$$

Thus, multiplying Equation (9.17) by $\sin(m\pi x/\ell)$ and integrating yields the desired constants

$$b_n=\frac{2}{\ell}\int_0^{\ell}w_0(x)\sin\left(\frac{n\pi}{\ell}x\right)dx \tag{9.19}$$

since the sum vanishes for each $n\neq m$. Likewise, if Equation (9.16) is differentiated with respect to time, multiplied by $\sin(m\pi x/\ell)$, and integrated, the constants a_n are found from

$$a_n=\frac{2}{n\pi c}\int_0^{\ell}\dot{w}_0(x)\sin\left(\frac{n\pi}{\ell}x\right)dx \tag{9.20}$$

The solution to problem (9.2) is given by the relations (9.16), (9.19), and (9.20).

The problem of solving the most basic distributed-parameter system is much more complicated than solving for the free response of a simple one-degree-of-freedom lumped-parameter system. Also, note that the solution just described essentially yields the theoretical modal analysis solution established for lumped-parameter systems as developed in Section 3.3. The functions $\sin(n\pi x/\ell)$ serve in the same capacity as the eigenvectors of a matrix in calculating a solution. The major difference between the two developments is that the sum in Equation (3.42) is finite, and hence always converges, whereas the sum in Equation (9.16) is infinite and may or may not converge.

Physically, the functions $\sin(n\pi x/\ell)$ describe the configuration of the string for a fixed time and hence are referred to as the *natural modes of vibration* of the system. Likewise, the numbers $(n\pi c/\ell)$ are referred to as the *natural frequencies of vibration*, since they describe the motion periodicity in time. Mathematically, the characteristic values $(n\pi/\ell)$ are also called the *eigenvalues* of the system, and $\sin\,(n\pi x/\ell)$ are called the *eigenfunctions* of the system and form an analogy to what is known about lumped-parameter systems. These quantities are defined more precisely in Section 10.2. For now, note that the eigenvalues

and eigenfunctions defined here serve the same purpose as eigenvalues and eigenvectors defined for matrices.

The basic method of separation of variables combined with the infinite sums and orthogonality as used here forms the basic approach in solving for the response of distributed-parameter systems. This approach, also called modal analysis, is used numerous times in the following chapters to solve a variety of vibration and control problems.

In the case of lumped-parameter systems, a lot was gained by looking at the eigenvalue and eigenvector problem. This information allowed the calculation of the solution of both the free and forced response by using the properties of the eigenstructures. In the following, the same approach (modal analysis) is further developed for distributed-parameter systems.

The fact that the solution (9.16) is a series of sine functions should not be a surprise. In fact, Fourier's theorem states that every function $f(x)$ that is piecewise continuous and bounded on the interval $[a, b]$ can be represented as a Fourier series of Equation (8.1), i.e.,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{\ell} x\right) + b_n \sin\left(\frac{n\pi}{\ell} x\right) \right] \tag{9.21}$$

where $\ell = b - a$. This fact is used extensively in Chapter 8 on vibration testing.

Recall that a function f is said to be *bounded* on the interval $[a, b]$ if there exists a finite constant M such that $|f(x)| < M$ for all x in $[a, b]$. Furthermore, a function $f(x)$ is *continuous* on the interval $[a, b]$ if for every x_1 in $[a, b]$, and for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that $|x - x_1| < \delta$ implies $|f(x) - f(x_1)| < \varepsilon$. A function is *piecewise continuous* on $[a, b]$ if it is continuous on every subinterval of $[a, b]$ (note here that the square brackets indicate that the endpoints of the interval are included in the interval).

Note that in many cases either all of the coefficients a_n or all of the coefficients b_n are zero. Also, note that many other functions $\Theta_n(x)$ besides the functions $\sin(n\pi x/\ell)$ and $\cos(n\pi x/\ell)$ have the property that arbitrary functions of a certain class can be expanded in terms of an infinite series of such functions, i.e., that

$$f(x) = \sum_{n=1}^{\infty} a_n \Theta_n(x) \tag{9.22}$$

This property is called *completeness* and is related to the idea of completeness used with orthogonal eigenvectors (Section 3.3). This concept is discussed in detail in Chapter 10.

Note that Equation (9.22) really means that the sequence of partial sums

$$\left\{ a_1\Theta_1, a_1\Theta_1 + a_2\Theta_2, \ldots, \sum_{i=1}^{m} a_i\Theta_i, \ldots \right\} \tag{9.23}$$

converges to the function $f(x)$, i.e., that

$$\lim_{m\to\infty} \left(\sum_{n=1}^{m} a_n \Theta_n \right) = f(x) \tag{9.24}$$

as defined in most introductory calculus texts.

Example 9.2.1

Now that the formal solution of the string has been examined and the eigenvalues and eigenfunctions have been identified, it is important to realize that these quantities are the physical notions of mode shapes and natural frequencies. To this end, suppose that the string is given the following initial conditions:

$$w(x,0)\sin\left(\frac{\pi}{\ell}x\right), \qquad w_t(x,0)=0 \tag{9.25}$$

Calculation of the expansion coefficients yields [from Equations (9.19) and (9.20)]

$$a_n=\frac{2}{n\pi c}\int_0^{\ell} w_t(x,0)\sin\left(\frac{n\pi}{\ell}x\right)dx=0 \tag{9.26}$$

and

$$b_n=\frac{2}{\ell}\int_0^{\ell}\sin\left(\frac{\pi}{\ell}x\right)\sin\left(\frac{n\pi}{\ell}x\right)dx=\begin{cases}1, & n=1\\ 0, & n>2\end{cases} \tag{9.27}$$

The solution thus becomes

$$w(x,t)=\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi ct}{\ell}\right) \tag{9.28}$$

In Figure 9.2, this solution is plotted versus x for a fixed value of t. This plot is the *shape* that the string would take if it were viewed by a stroboscope blinking at a frequency of $\pi c/\ell$. One of the two curves in Figure 9.2 is the plot of $w(x,t)$ that would result if the string were given an initial displacement of $w(x,0)=\sin(\pi x/\ell)$, the first eigenfunctions. The other curve, which takes on negative values, results from an initial condition of $w(x,0)=\sin(2\pi x/\ell)$, the second eigenfunction. Note that all the eigenfunctions $\sin(n\pi x/\ell)$ can be generated in this fashion by choosing the appropriate initial conditions. Hence, this set of functions is known as the set of *mode shapes* of vibration of the string. These correspond to the mode shapes defined for lumped-parameter systems and are the quantities measured in the modal tests described in Chapter 8.

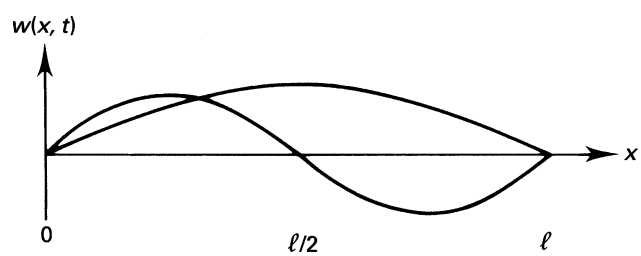

Figure 9.2 First two mode shapes of a vibrating string fixed at both ends.

9.3 RODS AND BARS

Next, consider the longitudinal vibration of a bar – that is, the vibration of a long slender material in the direction of its longest axis, as indicated in Figure 9.3. Again, by summing forces, the equation of motion is found (see, for instance, Timoshenko, Young, and Weaver, 1974) to be

$$[EA(x)w_x(x,t)]_x = \rho(x)A(x)w_{tt}(x,t), \qquad x \in (0,\ell) \tag{9.29}$$

where $A(x)$ is the variable cross-sectional area, $\rho(x)$ represents the variable mass distribution per unit area, E is the elastic modulus, and $w(x,t)$ is the axial displacement (in the x direction).

The form of Equation (9.29) is the same as that of the string if the cross-sectional area of the bar is constant. In fact, the 'stiffness' *operator* (see Section 10.2) in both cases has the form

$$-\alpha \frac{\partial^2}{\partial x^2} \tag{9.30}$$

where α is a constant. Hence, the eigenvalues and eigenfunctions are expected to have the same mathematical form as those of the string, and the solution will be similar. The main difference between these two systems is physical. In Equation (9.29) the function $w(x,t)$ denotes displacements along the long *axis* of the rod, where, as in Equation (9.1), $w(x,t)$ denotes displacements *perpendicular to the axis* of the string.

Several different ways of supporting a rod (and a string) lead to several different sets of boundary conditions associated with Equation (9.29). Some are stated in terms of the displacement $w(x,t)$ and others are given in terms of the strain $w_x(x,t)$ (recall that strain is the change in length per unit length).

Free boundary. If the bar is free or unsupported at a boundary, then the stress at the boundary must be zero, i.e., no force should be present at that boundary, or

$$EA(x)\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0} = 0 \qquad \text{or} \qquad EA(x)\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=\ell} = 0 \tag{9.31}$$

Note that, if $A(\ell) \neq 0$, the strain $w_x(\ell,t)$ must also be zero. The vertical bar in Equation (9.31) denotes that the function is to be evaluated at $x = 0$ or ℓ after the derivative is taken and indicates the location of the boundary condition.

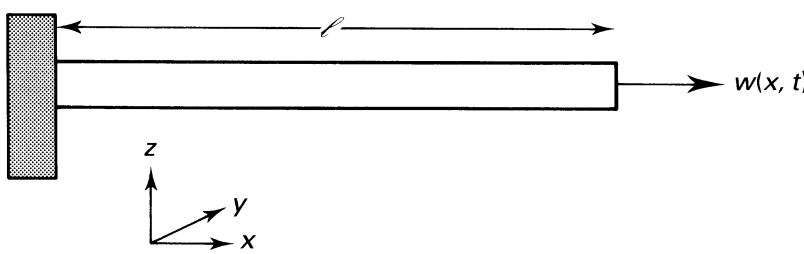

Figure 9.3 Schematic of a rod or bar, indicating the direction of longitudinal vibration.

Clamped boundary. If the boundary is rigidly fixed, or clamped, then the displacement must be zero at that point or

$$w(x,t)|_{x=0}=0 \qquad \text{or} \qquad w(x,t)\,|_{x=\ell}=0 \tag{9.32}$$

Appended boundary. If the boundary is fastened to a lumped element, such as a spring of stiffness k, the boundary condition becomes

$$EA(x)\frac{\partial w(x,t)}{\partial x}|_{x=0}=-kw(x,t)|_{x=0} \qquad \text{or} \qquad EA(x)\frac{\partial w(x,t)}{\partial x}|_{x=\ell}=kw(x,t)|_{x=\ell} \tag{9.33}$$

which expresses a force balance at the boundary. In addition, if the bar has a lumped mass at the end, the boundary condition becomes

$$EA(x)w_x(x,t)\,|_{x=0}=mw_{tt}(x,t)\,|_{x=0} \qquad \text{or} \qquad EA(x)w_x(x,t)\,|_{x=\ell}=-mw_{tt}(x,t)\,|_{x=\ell} \tag{9.34}$$

which also represents a force balance. These types of problems are discussed in detail in Section 12.4. They represent a large class of applications and are also referred to as structures with time-dependent boundary conditions, constrained structures, or combined dynamical systems.

As noted, the equation of motion is mathematically the same for both the string and the bar, so that further discussion of the method of solution is not required. A third problem again has the same mathematical model: the torsional vibration of circular shafts (rods). The derivation of the equation of motion is very similar, comes from a force balance, and can be found in several references (see, for instance, Timoshenko, Young, and Weaver, 1974). If G represents the shear modulus of elasticity of the shaft, I_p is the polar moment of inertia of the shaft, ρ is the mass per unit area, $\theta(x,t)$ is the angular displacement of the shaft from its neutral position, and x is the distance measured along the shaft, then the equation governing the torsional vibration of the shaft is

$$\theta_{tt}(x,t)=\frac{G}{\rho}\theta_{xx}(x,t), \qquad x\in(0,\ell) \tag{9.35}$$

As in the case of the rod or bar, the shaft can be subjected to a variety of boundary conditions, some of which are described in the following.

Free boundary. If the boundaries of the shaft are not attached to any device, there cannot be any torque acting on the shaft at that point, so that

$$GI_p\theta_x\,|_{x=0}=0 \qquad \text{or} \qquad GI_p\theta_x\,|_{x=\ell}=0 \tag{9.36}$$

at that boundary. If G and I_p are constant, then Equation (9.36) becomes simply

$$\theta_x(x,t)\,|_{x=0}=0 \qquad \text{or} \qquad \theta_x(x,t)\,|_{x=\ell}=0 \tag{9.37}$$

Clamped boundary. If a boundary is clamped, then no movement of the shaft at that position can occur, so that the boundary condition becomes

$$\theta(x,t)\,|_{x=0}=0 \qquad \text{or} \qquad \theta(x,t)\,|_{x=\ell}=0 \tag{9.38}$$

Appended boundaries. If a torsional spring of stiffness k is attached at the right end of the shaft (say at $x = \ell$), then the spring force (torque) must balance the internal bending moment. The boundary condition becomes

$$GI_p\theta(x, t)\,|_{x=\ell} = k\theta(x, t)\,|_{x=\ell} \tag{9.39}$$

At the left end this becomes (see Figure 9.4)

$$GI_p\theta_x(x, t)\,|_{x=0} = -k\theta(x, t)\,|_{x=0} \tag{9.40}$$

Quite often a shaft is connected to a disc at one end or the other. If the disc has mass polar moment of inertia I_d at the right end, the boundary condition becomes (see Figure 9.4)

$$GI_p\theta_x(x, t)\,|_{x=\ell} = -I_d\theta_{tt}(x, t)\,|_{x=\ell} \tag{9.41}$$

or

$$GI_p\theta_x(x, t)\,|_{x=0} = I_d\theta_{tt}(x, t)\,|_{x=0} \tag{9.42}$$

if the disc is placed at the left end.

The shaft could also have both a spring and a mass at one end, in which case the boundary condition, obtained by summing forces, is

$$GI_p\theta_x(x, t)\,|_{x=0} = -I_d\theta_{tt}(x, t)\,|_{x=0} - k\theta(x, t)\,|_{x=0} \tag{9.43}$$

As illustrated in the examples and problems, the boundary conditions affect the natural frequencies and mode shapes. These quantities have been tabulated for many common boundary conditions (Gorman, 1975, and Blevins, 2001).

Example 9.3.1

Consider the vibration of a shaft that is fixed at the left end ($x = 0$) and has a disc attached to the right end ($x = \ell$). Let G, I_p, and ρ all have unit values, and calculate the eigenvalue and eigenfunctions of the system.

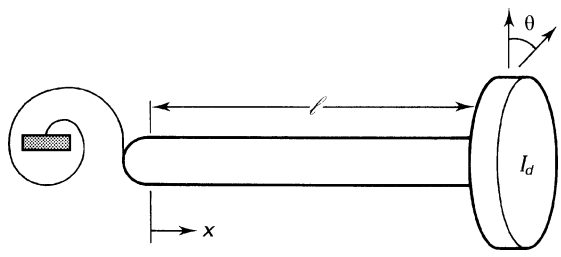

Figure 9.4 Shaft connected to a rotational spring on one end and a disc on the other end.

Following the separation of variables procedure used in the solution of the string problems, a solution of the form $\theta(x, t) = \Theta(x)T(t)$ is assumed and substituted into the equation of motion and boundary conditions, resulting in

$$\Theta''(x) + \lambda^2\Theta(x) = 0$$

$$\Theta(0) = 0$$

$$GI_p\Theta'(\ell)T(t) = I_d\Theta(\ell)\ddot{T}(t)$$

Recall that $T(t)$ is harmonic, so that $\ddot{T}(t) = -\lambda^2 T(t)$, and the last boundary condition can be written as

$$GI_p\Theta'(\ell) = -\lambda^2 I_d\Theta(\ell)$$

which removes the time dependence. The general spatial solution is

$$\Theta(x) = A_1 \sin\lambda x + A_2 \cos\lambda x$$

Application of the boundary condition at $x = 0$ yields

$$A_2 = 0 \qquad \text{and} \qquad \Theta(x) = A_1 \sin\lambda x$$

The second boundary condition yields (for the case $G = I_p = 1$)

$$\lambda A_1 \cos\lambda\ell = -\lambda^2 I_d A_1 \sin\lambda\ell$$

which is satisfied for all values of λ such that

$$\tan(\lambda\ell) = -\frac{\ell}{I_d}\frac{1}{\lambda\ell} \tag{9.44}$$

Equation (9.44) is a transcendental equation for the values of λ, the eigenvalues, and has an infinite number of solutions denoted by λ_n, calculated either graphically or numerically from the points of intersection given in Figure 9.5. The values of λ correspond to the values of $(\lambda\ell)$ at the intersections

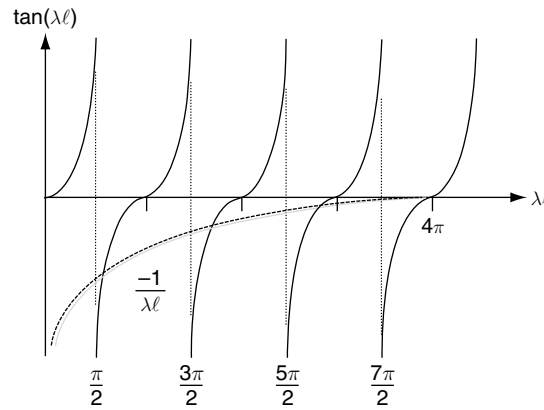

Figure 9.5 Graphical solution of the transcendental equation for $\tan\lambda\ell = -1/\lambda\ell$.

of the two curves of Figure 9.5. Note that the effect of the disc inertia, I_d, is to shift the points of intersection of the two curves. The transcendental equation [Equation (9.44)] is solved numerically near each crossing, using the plot to obtain an initial guess for the numerical procedure. For values of n greater than 3, the crossing points in the plot approach the zeros of the tangent, and hence the sine or $n\pi$.

9.4 VIBRATION OF BEAMS

In this section, the transverse vibration of a beam is examined. A beam is represented in Figure 9.6. The beam has mass density $\rho(x)$, cross-sectional area $A(x)$, and moment of inertia $I(x)$ about its equilibrium axis. The deflection $w(x, t)$ of the beam is the result of two effects, bending and shear. The derivation of equations and boundary conditions for the vibration of the beam can be found in several texts (see, for instance, Timoshenko, Young, and Weaver, 1974).

A beam is a transversely loaded (along the z axis in Figure 9.6), prismatic structural element with length ℓ, which is large in value when compared with the magnitude of the beam cross-sectional area, A (the y–z plane in Figure 9.6). In the previous section, vibration of such a structure in the x direction was considered and referred to as the longitudinal vibration of a bar. In this section the *transverse vibration* of the beam, i.e., vibration in the z direction, perpendicular to the long axis of the beam, is considered.

The equations of motion are not developed here; however, the three basic assumptions used in the derivation are important to bear in mind. The fundamental small displacement assumption of linear vibration theory in this case results in requiring (1) that the material in the y–z plane of the beam remains in the plane during deformation, (2) that the displacements along the y direction are zero (called the *plane strain* assumption), and (3), that, along any cross-section, the displacement in the z direction is the same (i.e., no stretch of material in thickness). These assumptions lead to the *Timoshenko beam equations* in the transverse

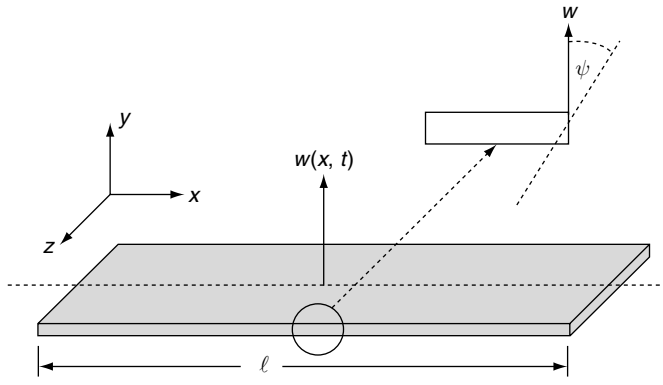

Figure 9.6 Beam indicating the variables used in the transverse vibration model.

deflection $w(x, t)$ and the bending slope $\psi(x, t)$ (see, for example, Reismann and Pawlik, 1974). They are

$$\begin{aligned} -\rho A w_{tt} + \frac{\partial}{\partial x}\left[\kappa^2 AG\left(\frac{\partial w}{\partial x} - \psi\right)\right] + p = 0, \qquad x \in (0, \ell) \\ -\rho I \psi_{tt} + \frac{\partial}{\partial x}\left[EI\frac{\partial \psi}{\partial x}\right] + \kappa^2 AG\left(\frac{\partial w}{\partial x} - \psi\right) = 0, \qquad x \in (0, \ell) \end{aligned} \tag{9.45}$$

where κ^2 is called the shear coefficient, G is the shear modulus, E is the elastic modulus, and $p = p(x, t)$ is an externally applied force along the length of the beam. The shear coefficient κ^2 can be determined in several ways. A summary of values for κ^2 is given by Cowper (1966).

With the additional assumptions that (1) shear deformations are negligible, so that the shear angle is zero and $\psi = -w_x(x, t)$, and (2) that the rotary inertia is negligible, so that $\rho I \psi_{tt} = 0$, the so-called *Euler–Bernoulli* beam equations result. Under these additional assumptions, Equations (9.45) become the single equation (for $p = 0$ and A constant)

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\right] = -m(x) w_{tt}(x, t), \qquad x \in (0, \ell) \tag{9.46}$$

where $m(x) = \rho A$. Equation (9.46) is the *Euler–Bernoulli beam equation* and is more commonly used in applications than the Timoshenko equations [Equation (9.45)] because of its comparative simplicity. If the beam is long and skinny (aspect ratio, say, greater then 10), the Euler–Bernoulli assumptions are appropriate. The rest of the section is devoted to the Euler–Bernoulli model. Again, many boundary conditions are possible, and several common cases are considered here.

Clamped boundary. If the beam is firmly fixed at a boundary $x = 0$, then both the deflection of the beam and the slope of the deflection must be zero, i.e.,

$$w(x, t)|_{x=0} = w_x(x, t)|_{x=0} = 0 \tag{9.47}$$

Simply supported boundary. If the beam is supported by a hinge at $x = 0$, then the bending moment and deflection must both be zero at that point, i.e.,

$$\begin{aligned} EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}|_{x=0} = 0 \\ w(x, t)|_{x=0} = 0 \end{aligned} \tag{9.48}$$

This boundary condition is also referred to as *hinged*.

Free boundary. Again, in the case of no support at a boundary, the bending moment must be zero. In addition, the shearing force at a free end must be zero, i.e.,

$$EI(x)\left.\frac{\partial^2 w(x, t)}{\partial x^2}\right|_{x=0} = 0, \qquad \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x, t)}{\partial x^2}\right]\Bigg|_{x=0} = 0 \tag{9.49}$$

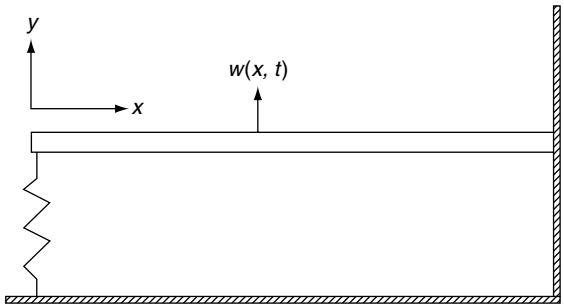

Figure 9.7 Beam connected to a linear spring on one end and clamped on the other end.

Appended boundaries. If a spring, with spring constant k, is attached to one end of the beam as shown in Figure 9.7, a force balance indicates that the shear force must equal the spring-restoring force. In addition, the bending moment must still be zero.

$$EI(x)\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0}=0, \qquad \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]\Bigg|_{x=0}=-kw(x,t)\,|_{x=0} \tag{9.50}$$

Note that the sign again changes if the spring is attached at the other end. If a mass is placed at one end, the shear force must balance the inertial force provided by the mass, and again the bending moment must be zero. Using the same end as indicated in Figure 9.7, the boundary conditions are

$$EI(x)\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0}=0, \qquad \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]\Bigg|_{x=0}=mw_{tt}(x,t)\,|_{x=0} \tag{9.51}$$

where m is the mass of the appended piece.

Example 9.4.1

Consider a cantilevered beam (clamped at $x=0$ and free at the $x=l$) and compute the natural frequencies and mode shapes using separation of variables as described above. Assume that E, I, ρ, and l are constant. The method is basically the same as that used for a string, rod, and bar, but the resulting boundary value problem is slightly more complex. The clamped–free boundary conditions are

$$w(0,t)=w_x(0,t)=0 \qquad \text{and} \qquad w_{xx}(l,t)=w_{xxx}(l,t)=0$$

The equation of motion is

$$\frac{\partial^2 w}{\partial t^2}+\left(\frac{EI}{\rho A}\right)\frac{\partial^4 w}{\partial x^4}=0$$

Using the method of separation of variables, assume the solution is of the form $w(x,t)=X(x)T(t)$ to obtain

$$\left(\frac{EI}{\rho A}\right)\frac{X''''}{X}=-\frac{\ddot{T}(t)}{T(t)}=\omega^2$$

The spatial equation becomes

$$X''''(x) - \left(\frac{\rho A}{EI}\right)\omega^2 X(x) = 0$$

Next, define $\beta^4 = (\rho A\omega^2)/(EI)$ so that the equation of motion becomes $X'' - \beta^4 X = 0$ which has the solution

$$X(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

Applying the boundary conditions in separated form

$$X(0) = X'(0) = 0 \qquad \text{and} \qquad X''(l) = X'''(l) = 0$$

yields four equations in the four unknown coefficients C_i. These four equations are written in matrix form as

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\beta l & -\cos\beta l & \sinh\beta l & \cosh\beta l \\ -\cos\beta l & \sin\beta l & \cosh\beta l & \sinh\beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0$$

For a nonzero solution for the coefficients C_i, the matrix determinant must be zero. The determinant yields the characteristic equation

$$(-\sin\beta l - \sinh\beta l)(\sin\beta l - \sinh\beta l) - (-\cos\beta l - \cosh\beta l)(-\cos\beta l - \cosh\beta l) = 0$$

Simplifying, the characteristic equation becomes $\cos\beta l \cosh\beta l = -1$, or

$$\cos\beta_n l = -\frac{1}{\cosh\beta_n l}$$

This last expression is solved numerically for the values βl. The frequencies are then

$$\omega_n = \sqrt{\frac{\beta_n^4 EI}{\rho A}}$$

The mode shapes given by the solution for each β_n are

$$X_n = C_{1n} \sin\beta_n x + C_{2n} \cos\beta_n x + C_{3n} \sinh\beta_n x + C_{4n} \cosh\beta_n x$$

Using the boundary condition information that $C_4 = -C_2$ and $C_3 = -C_1$ yields

$$-C_1(\sin\beta l + \sinh\beta l) = C_2(\cos\beta l + \cosh\beta l)$$

so that

$$C_1 = -C_2\left(\frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}\right)$$

The mode shapes can then be expressed as

$$X_n = -C_{2n}\left[-\left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sin\beta_n x + \cos\beta_n x + \left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sinh\beta_n x - \cosh\beta_n x\right]$$

Additional examples of the solution to the beam equation can be found in the next chapter which discusses formal methods of solution.

9.5 MEMBRANES AND PLATES

In this section, the equations for linear vibrations of membranes and plates are discussed. These objects are two-dimensional versions of the strings and beams discussed in the preceding sections. They occupy plane regions in space. The membrane represents a two-dimensional version of a string, and a plate can be thought of as a membrane with bending stiffness.

First, consider the equations of motion for a membrane. A membrane is basically a two-dimensional system that lies in a plane when in equilibrium. A common example is a drum head. The structure itself provides no resistance to bending, so that the restoring force is due only to the tension in the membrane. Thus, a membrane is similar to a string and, as was mentioned, is a two-dimensional version of a string. The reader is referred to Timoshenko, Young, and Weaver (1974) for the derivation of the membrane equation. Let $w(x, y, t)$ represent the displacement in the z direction of a membrane lying in the x–y plane at the point (x, y) and time t. The displacement is assumed to be small, with small slopes, and is perpendicular to the x–y plane. Let T be the tensile force per unit length of the membrane, assumed the same in all directions, and ρ be the mass per unit area of the membrane. Then, the equation for free vibration is given by

$$T\nabla^2 w(x, y, t) = \rho w_{tt}(x, y, t), \qquad x, y \in \Omega \tag{9.52}$$

where Ω denotes the region in the x–y plane occupied by the membrane. Here ∇^2 is the *Laplace operator*. In rectangular coordinates this operator has the form

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{9.53}$$

The boundary conditions for the membrane must be specified along the shape of the boundary, not just at points, as in the case of the string. If the membrane is fixed or clamped at a segment of the boundary, then the deflection must be zero along that segment. If $\partial\Omega$ is the curve in the x–y plane corresponding to the edge of the membrane, i.e., the boundary of Ω, then the clamped boundary condition is denoted by

$$w(x, y, t) = 0, \qquad x, y \in \partial\Omega \tag{9.54}$$

If, for some segment of $\partial\Omega$, denoted by $\partial\Omega_1$, the membrane is free to deflect transversely, then there can be no force component in the transverse direction, and the boundary condition becomes

$$\frac{\partial w(x, y, t)}{\partial n} = 0, \qquad x, y \in \partial\Omega_1 \tag{9.55}$$

Here, $\partial w/\partial n$ denotes the derivative of $w(x, y, t)$ normal to the boundary in the reference plane of the membrane.

Example 9.5.1

Consider the vibration of a square membrane, as indicated in Figure 9.8, clamped at all of the edges. With $c^2 = T/\rho$, the equation of motion, [Equation (9.52)] becomes

$$c^2\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] = \frac{\partial^2 w}{\partial t^2}, \qquad x, y \in \Omega \tag{9.56}$$

Assuming that the solution separates, i.e., that $w(x, y, t) = X(x)Y(y)T(t)$, Equation (9.56) becomes

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} + \frac{Y''}{Y} \tag{9.57}$$

Equation (9.57) implies that $\ddot{T}/(Tc^2)$ is a constant (recall the argument used in Section 9.2). Denote the constant by ω^2, so that

$$\frac{\ddot{T}}{Tc^2} = -\omega^2 \tag{9.58}$$

Then Equation (9.57) implies that

$$\frac{X''}{X} = -\omega^2 - \frac{Y''}{Y} \tag{9.59}$$

By the same argument used before, both X''/X and Y''/Y must be constant (that is, independent of t and x or y). Hence

$$\frac{X''}{X} = -\alpha^2 \tag{9.60}$$

and

$$\frac{Y''}{Y} = -\gamma^2 \tag{9.61}$$

where α^2 and γ^2 are constants. Equation (9.59) then yields

$$\omega^2 = \alpha^2 + \gamma^2 \tag{9.62}$$

These expressions result in two spatial equations to be solved

$$X'' + \alpha^2 X = 0 \tag{9.63}$$

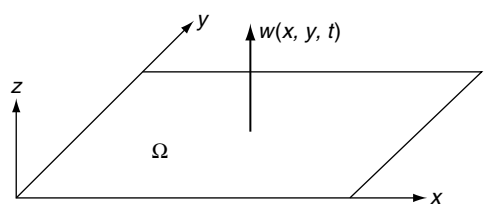

Figure 9.8 Square membrane illustrating vibration perpendicular to its surface.

which have a solution (A and B are constants of integration) of the form

$$X(x) = A \sin \alpha x + B \cos \alpha x \tag{9.64}$$

and

$$Y'' + \gamma^2 Y = 0 \tag{9.65}$$

which yields a solution (C and D are constants of integration) of the form

$$Y(y) = C \sin \gamma y + D \cos \gamma y \tag{9.66}$$

The total spatial solution is the product $X(x)Y(y)$, or

$$\begin{aligned} X(x)Y(y) = & A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y \\ & + A_3 \cos \alpha x \sin \gamma y + A_4 \cos \alpha x \cos \gamma y \end{aligned} \tag{9.67}$$

Here, the constants A_i consist of the products of the constants in Equations (9.64) and (9.66) and are to be determined by the boundary and initial conditions.

Equation (9.67) can now be used with the boundary conditions to calculate the eigenvalues and eigenfunctions of the system. The clamped boundary condition, along $x = 0$ in Figure 9.8, yields

$$T(t)X(0)Y(y) = T(t)B(A_3 \sin \gamma y + A_4 \cos \gamma y) = 0$$

or

$$A_3 \sin \gamma y + A_4 \cos \gamma y = 0 \tag{9.68}$$

Now, Equation (9.68) must hold for any value of y. Thus, as long as γ is not zero (a reasonable assumption, since if it is zero the system has a rigid body motion), A_3 and A_4 must be zero. Hence, the spatial solution must have the form

$$X(x)Y(y) = A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y \tag{9.69}$$

Next, application of the boundary condition $w = 0$ along the line $x = 1$ yields

$$A_1 \sin \alpha \sin \gamma y + A_2 \sin \alpha \cos \gamma y = 0 \tag{9.70}$$

Factoring this expression yields

$$\sin \alpha (A_1 \sin \gamma y + A_2 \cos \gamma y) = 0 \tag{9.71}$$

Now, either $\sin \alpha = 0$ or, by the preceding argument, A_1 and A_2 must be zero. However, if A_1 and A_2 are both zero, the solution is zero. Hence, in order for a nontrivial solution to exist, $\sin \alpha = 0$, which yields

$$\alpha = n\pi, \qquad n = 1, 2, \ldots, \infty \tag{9.72}$$

Using the boundary condition $w = 0$ along the line $y = 1$ results in a similar procedure and yields

$$\gamma = m\pi, \qquad m = 1, 2, \ldots, \infty \tag{9.73}$$

Note that the possibility of $\gamma = \alpha = 0$ is not used because it was necessary to assume that $\gamma \neq 0$ in order to derive Equation (9.69). Equation (9.62) shows that the constant ω in the temporal equation must have the form

$$\begin{aligned} \omega_{mn} &= \sqrt{\alpha_n^2 + \gamma_m^2} \\ &= \pi\sqrt{m^2 + n^2}, \qquad m, n = 1, 2, 3, \ldots, \infty \end{aligned} \tag{9.74}$$

Thus, the eigenvalues and eigenfunctions for the clamped membrane are Equation (9.74) and $\{sin\ n\pi x$ and $\sin\ m\ \pi y\}$ respectively. The solution of Equation (9.56) becomes

$$\begin{aligned} w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\sin m\pi x \sin n\pi y)\{[A_{mn}\sin[\sqrt{n^2 + m^2}c\pi t] \\ + B_{mn}\cos[\sqrt{n^2 + m^2}c\pi t]\} \end{aligned} \tag{9.75}$$

where A_{mn} and B_{mn} are determined by the initial conditions and the orthogonality of the eigenfunctions.

In progressing from the vibration of a string to considering the transverse vibration of a beam, the beam equation allowed for bending stiffness. In somewhat the same manner, a plate differs from a membrane because plates have bending stiffness. The reader is referred to Reismann (1988) or Sodel (1993) for a more detailed explanation and a precise derivation of the plate equation. Basically, the plate, like the membrane, is defined in a plane (x–y) with the deflection $w(x, y, t)$ taking place along the z axis perpendicular to the x–y plane. The basic assumption is again small deflections with respect to the thickness, h. Thus, the plane running through the middle of the plate is assumed not to deform during bending (called a *neutral plane*). In addition, normal stresses in the direction transverse to the plate are assumed to be negligible. Again, there is no thickness stretch. The displacement equation of motion for the free vibration of the plate is

$$-D_E\nabla^4 w(x, y, t) = \rho w_{tt}(x, y, t), \qquad x, y \in \Omega \tag{9.76}$$

where E again denotes the elastic modulus, ρ is the mass density (per unit area), and the constant D_E, the plate flexural rigidity, is defined in terms of Poisson's ratio ν and the plate thickness h as

$$D_E = \frac{Eh^3}{12(1 - \nu^2)} \tag{9.77}$$

The operator ∇^4, called the *biharmonic operator*, is a fourth-order operator, the exact form of which depends on the choice of coordinate systems. In rectangular coordinates, the biharmonic operator becomes

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \tag{9.78}$$

The boundary conditions for a plate are a little more difficult to write, as their form, in some cases, also depends on the coordinate system in use.

Clamped edge. For a clamped edge, the deflection and normal derivative $\partial/\partial n$ are both zero along the edge:

$$w(x, y, t) = 0 \qquad \text{and} \qquad \frac{\partial w(x, y, t)}{\partial n} = 0, \qquad x, y \in \partial\Omega \tag{9.79}$$

Here, the normal derivative is the derivative of w normal to the neutral plane.

Simply supported. For a rectangular plate, the simply supported boundary conditions become

$$w(x, y, t) = 0 \qquad \text{along all edges}$$

$$\frac{\partial^2 w(x, y, t)}{\partial x^2} = 0 \qquad \text{along the edges } x = 0, x = \ell_1 \tag{9.80}$$

$$\frac{\partial^2 w(x, y, t)}{\partial y^2} = 0 \qquad \text{along the edges } y = 0, y = \ell_2 \tag{9.81}$$

where ℓ_1 and ℓ_2 are the lengths of the plate edges and the second partial derivatives reflect the normal strains along these edges.

9.6 LAYERED MATERIALS

The use of layered materials and composites in the design of modern structures has become very popular because of increased strength-to-weight ratios. The theory of vibration of layered materials is not as developed, but does offer some interesting design flexibility.

The transverse vibration of a three-layer beam consisting of a core between two faceplates, as indicated in Figure 9.9, is considered here. The layered beam consists of two faceplates of thickness h_1 and h_3, which are sandwiched around a core beam of thickness h_2. The distance between the center-lines of the two faceplates is denoted by d. The displacement equation of vibration becomes (see, for instance, Sun and Lu, 1995)

$$\frac{\partial^6 w(x, t)}{\partial x^6} - g(1+\beta)\frac{\partial^4 w(x, t)}{\partial x^4} + \frac{\rho}{D_E}\left[\frac{\partial^4 w}{\partial x^2 \partial t^2} - g\frac{\partial^2 w}{\partial t^2}\right] = 0, \qquad x \in (0, \ell) \tag{9.82}$$

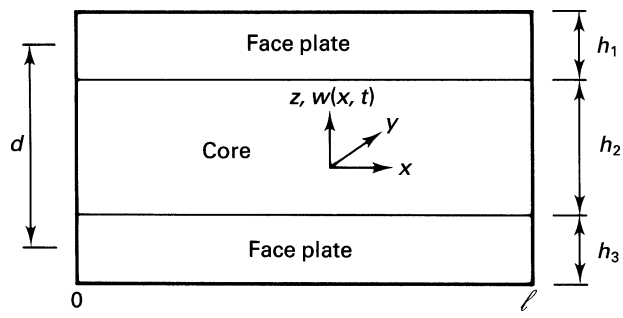

Figure 9.9 Cross-section of a three-layer beam for transverse vibration analysis.

where

$$g = \frac{G}{h_2}\left[\frac{1}{E_1 h_1} + \frac{1}{E_3 h_3}\right]$$

$$G = \text{shear modulus of the core}$$

$$E_i = \text{Young's modulus of the } i\text{th face plate}$$

$$D_E = \frac{E_1 h_1^3 + E_3 h_3^3}{12}$$

$$\beta = \frac{d^2 E_1 h_1 E_3 h_3^3 (E_1 h_1 + E_3 h_3)}{D_E}$$

$$d = h_2 + \frac{h_1 + h_3}{2}$$

$$\rho = \text{mass per unit length of the entire structure}$$

The boundary conditions are again not as straightforward as those of a simple beam. In fact, since the equation for free vibration contains six derivatives, there are six boundary conditions that must be specified. If the beam has both ends clamped, the boundary conditions are

$$\begin{aligned}
w(0,t) &= w(\ell,t) = 0 && \text{(zero displacement)}\\
w_x(0,t) &= w_x(\ell,t) = 0 && \text{(zero rotation)}\\
w_{xxxx}(0,t) - g(1+\beta)w_{xx}(0,t) - \frac{\rho}{D_E}w_{tt}(0,t) &= 0 &&\\
w_{xxxx}(\ell,t) - g(1+\beta)w_{xx}(\ell,t) - \frac{\rho}{D_E}w_{tt}(\ell,t) &= 0 && \text{(zero bending moments)}
\end{aligned} \tag{9.83}$$

Note that the form of this equation is quite different from all the other structures considered in this chapter. In all the previous cases, the equation for linear vibration can be written in the form

$$\begin{aligned}
w_{tt}(\mathbf{x},t) + L_2 w(\mathbf{x},t) &= 0, && \mathbf{x} \in \Omega\\
B(w)(\mathbf{x},t) &= 0, && \mathbf{x} \in \partial\Omega
\end{aligned} \tag{9.84}$$

plus initial conditions, where Ω is a region in three-dimensional space bounded by $\partial\Omega$. Here, L_2 is a linear operator in the spatial variables only, $\boldsymbol{x}$ is a vector consisting of the spatial coordinates x, y, and z, and B represents the boundary conditions. As long as the vibration of a structure fits into the form of Equations (9.84) and the operator L_2 satisfies certain conditions (specified in Chapter 11), separation of variables can be used to solve the problem. However, Equation (9.82) is of the form

$$\begin{aligned}
L_0 w_{tt}(\boldsymbol{x},t) + L_2 w(\mathbf{x},t) &= 0, && \mathbf{x} \in \Omega\\
Bw(\mathbf{x},t) &= 0, && \mathbf{x} \in \partial\Omega
\end{aligned} \tag{9.85}$$

where both L_0 and L_2 are linear operators in the spatial variables. Because of the presence of two operators it is not clear if separation of variables will work as a solution technique. The circumstances and assumptions required for separation of variables to work is the topic of the next chapter. In addition, classifications and further discussion of the operators are contained in Chapter 11.

The boundary conditions associated with the operator B in Equation (9.84) can be separated into two classes. Boundary conditions that arise clearly as a result of the geometry of the structure, such as Equation (9.47), are called *geometric boundary conditions*. Boundary conditions that arise by requiring a force (or moment) balance at the boundary are called *natural boundary conditions*. Equation (9.49) represents an example of a natural boundary condition.

9.7 VISCOUS DAMPING

Only viscous damping, introduced in Section 1.3, is considered in this section. The reason for this consideration is that viscous damping lends itself to analytical solutions for transient as well as steady state response by relatively simple techniques. While there is significant evidence indicating the inaccuracies of viscous damping models (for instance, Snowden, 1968), modeling dissipation as viscous damping represents a significant improvement over the conservative models given by Equations (9.84) or (9.85). In this section, several distributed-parameter models with viscous-type damping are presented.

First, consider the transverse free vibration of a membrane in a surrounding medium (such as air) furnishing resistance to the motion that is proportional to the velocity (i.e., viscous damping). The equation of motion is Equation (9.52) with the addition of a damping force. The resulting equation is

$$\rho w_{tt}(x, y, t) + \gamma w_t(x, y, t) - T\nabla^2 w(x, y, t) = 0, \qquad x, y \in \Omega \tag{9.86}$$

where ρ, T, and ∇^2 are as defined for Equation (9.53) and γ is the viscous damping coefficient. The positive constant γ reflects the proportional resistance to velocity. This system is subject to the same boundary conditions as those discussed in Section 9.5.

The solution method (separation of variables) outlined in example 9.5.1 works equally well for solving the damped membrane equation [Equation (9.86)]. The only change in the solution is that the temporal function $T(t)$ becomes an exponentially decaying sinusoid rather than a constant-amplitude sinusoid, depending on the relative size of γ.

External damping of the viscous type can also be applied to the model of the free flexural vibration of a plate. This problem has been considered by Murthy and Sherbourne (1972). They modeled the damped plate by

$$\rho w_{tt}(x, y, t) + \gamma w_t(x, y, t) + D_E \nabla^4 w(x, y, t) = 0, \qquad x, y, \in \Omega \tag{9.87}$$

subject to the same boundary conditions as Equation (9.86). Here, D_E, ρ, and ∇^4 are as defined for Equation (9.76), and γ again represents the constant viscous damping parameter. The plate boundary conditions given in Section 9.5 also apply to the damped plate equation [Equation (9.87)].

Finally, consider the longitudinal vibration of a bar subject to both internal and external damping. In this case, the equation of vibration for the bar in Figure 9.3 becomes

$$w_{tt}(x,t) + 2\left(\gamma - \beta\frac{\partial^2}{\partial x^2}\right)w_t(x,t) - \frac{EA}{\rho}w_{xx}(x,t) = 0, \qquad x \in \Omega \tag{9.88}$$

subject to the boundary conditions discussed in Section 9.3. The quantities E, A, and ρ are taken to be constant versions of the like quantities defined in Equation (9.29). The constant γ is again a viscous damping factor derived from an external influence, whereas the constant β is a viscous damping factor representing an internal damping mechanism.

Note that the internal model is slightly more complicated than the other viscous damping models considered in this section because of the inclusion of the second spatial derivative. Damping models involving spatial derivatives can cause difficulties in computing analytical solutions and are discussed in Sections 10.4 and 11.6. In addition, damping terms can alter the boundary conditions.

The general model for vibration of distributed-parameter systems with viscous damping can be written as

$$\begin{aligned} L_0 w_{tt}(\mathbf{x},t) + L_1 w_t(\mathbf{x},t) + L_2 w(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \Omega \\ Bw(\mathbf{x},t) &= 0, \qquad \mathbf{x} \in \partial\Omega \end{aligned} \tag{9.89}$$

plus appropriate initial conditions. Here, the operators B, L_0, and L_2 are as defined for Equation (9.85), and the operator L_1 is exemplified by the models illustrated in this section. The operator L_0 is called the mass operator, the operator L_1 is called the damping operator, and the operator L_2 is called the stiffness operator. As illustrated by the examples, the operator L_0 is often the identity operator. The properties of these operators, the nature of the solutions of Equations (9.84) and (9.89), and their relationship to the vibration problem are topics of the next three chapters.

CHAPTER NOTES

This chapter presents a brief introduction to the linear vibration of distributed-parameter systems without regard for mathematical rigor or a proper derivation of the governing equations. The chapter is intended to review and familiarize the reader with some of the basic structural elements used as examples in the study of distributed-parameter systems and points out the easiest and most commonly used method of solution – separation of variables.

Section 9.2 introduces the classic vibrating string. The derivation and solution of the string equation can be found in almost any text on vibration, partial differential equations, or applied mathematics. The vibration of bars covered in Section 9.3 is almost as common and can again be found in almost any vibration text. An excellent detailed derivation of most of the equations can be found in Magrab (1979) and more basic derivations can be found in Inman (2001) or any of the other excellent introductory texts (such as Rao, 2004). The material on membranes and plates of Section 9.5 is also very standard and can be found in most advanced texts, such as Meirovitch (1967, 1997, 2001). Ventsel and Krauthammer (2001) present a complete derivation of the thin plate equations used here. A classic reference for plates is Sodel (1993). The material on layered structures of Section 9.6 is nonstandard, but such

materials have made a significant impact in engineering design and should be considered. Sun and Lu's (1995) book gives a list of useful papers in this area. Blevins (2001) is an excellent reference and tabulates natural frequencies and mode shapes for a variety of basic elements (strings, bars, beams, and plates in various configurations). Elishakoff (2005) gives the solutions of several unusual configurations.

Section 9.7 introduces some simple viscous damping models for distributed-parameter systems. Information on such models in the context of transient vibration analysis is difficult to come by. The majority of work on damping models centers on the steady state forced response of such systems and presents a very difficult problem. Snowden (1968) and Nashif, Jones, and Henderson (1985) present an alternative view on damping and should be consulted for further reading. Banks, Wang, and Inman (1994) discuss damping in beams.

As indicated, most of the material in this chapter is standard. However, this chapter does include some unusual boundary conditions representing lumped-parameter elements appended to distributed-parameter structures. These configurations are very important in vibration design and control. The texts of Gorman (1975) and of Blevins (2001) tabulates the natural frequencies of such systems. Such configurations are analyzed in Section 12.4.

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PROBLEMS

9.1 Consider the bar of Section 9.3. Using the method of separation of variables, calculate the natural frequencies if the bar is clamped at one end and connected to a linear spring of stiffness k at the other end.

9.2 Consider the string equation of Section 9.2, with clamped boundaries at each end. At the midpoint, $x = \ell/2$, the density changes from ρ_1 to ρ_2, but the tension T remains constant. Derive the characteristic equation.

9.3 Calculate the natural frequencies of vibration of the Euler–Bernoulli beam clamped at both ends.

9.4 Consider a nonuniform bar described by

$$\frac{\partial}{\partial x}[EA(x)u_x] = m(x)u_{tt} \text{ on } (0, \ell)$$

with

$$EA(a) = 2EA_0\left(1 - \frac{x}{\ell}\right) \qquad \text{and} \qquad m(x) = 2m_0\left(1 - \frac{x}{\ell}\right)$$

fixed at 0 and free at ℓ. Calculate the free vibration. What are the first three eigenvalues?

9.5 Solve the Timoshenko beam equation [Equation (9.45)]. Assume that the cross-section remains constant and eliminate ψ from the equations to produce a single equation in the displacement. Compare your result with the Euler–Bernoulli equation and identify the shear deformation term and rotary inertia term. What can you conclude?

9.6 Verify the orthogonality condition for the set of functions

$$\left\{\sin\frac{n\pi x}{\ell}\right\}$$

on the interval $[0, \ell]$.

9.7 Calculate the natural frequencies of the system in Figure 9.4 and compare them with those of example 9.3.1.

9.8 Calculate the solution of the internally and externally damped bar given by Equation (9.88) with a clamped boundary at $x = 0$ and a free boundary at $x = \ell$.

9.9 Are there values of the parameters γ and ρ in Equation (9.88) for which the damping for one or more of the terms $T_n(t)$ is zero? A minimum? (Use the clamped boundary conditions of problem 9.8.)

9.10 Consider the damped membrane of Equation (9.86) with clamped boundary conditions. Calculate values of γ, ρ, and T such that the temporal part, $T(t)$, of the separation of variables solution does not oscillate.

9.11 Compute the natural frequencies of a beam clamped at the right end (0) with a tip mass of value M at the left end.

9.12 Compute the characteristic equation of a shaft with a disc of inertia I at each end.

10

Formal Methods of Solution

10.1 INTRODUCTION

This chapter examines various methods of solving for the vibrational response of the distributed-parameter systems introduced in the previous chapter. As in finite-dimensional systems, the response of a given system is made up of two parts: the transient, or free, response, and the steady state, or forced, response. In general the steady state response is easier to calculate, and in many cases the steady state is all that is necessary. The focus in this chapter is the free response. The forced response is discussed in more detail in Chapter 12.

Several approaches to solving distributed-parameter vibration problems are considered. The formal notion of an operator and the eigenvalue problem associated with the operator are introduced. The traditional separation of variables method used in Chapter 9 is compared with the eigenvalue problem. The eigenfunction expansion method is introduced and examined for systems including damping. Transform methods and integral formulations in the form of Green's functions are also introduced in less detail.

10.2 BOUNDARY VALUE PROBLEMS AND EIGENFUNCTIONS

As discussed in Section 9.6, a general formulation of the undamped boundary value problems presented in Chapter 9 can be written as (the subscript on L is dropped here for notational ease)

$$\begin{aligned} w_{tt}(\mathbf{x}, t) + Lw(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \Omega \qquad \text{for } t > 0 \\ Bw &= 0, \qquad \mathbf{x} \in \partial\Omega \qquad \text{for } t > 0 \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}), &\qquad w_t(\mathbf{x}, 0) = \dot{w}_0(\mathbf{x}) \qquad \text{at } t = 0 \end{aligned} \tag{10.1}$$

where $w(\mathbf{x}, t)$ is the deflection, $\mathbf{x}$ is a three-dimensional vector of spatial variables, and Ω is a bounded region in three-dimensional space with boundary $\partial\Omega$. The operator L is

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

a differential operator of spatial variables only. For example, for the longitudinal vibration of a beam (string or rod) the operator L has the form

$$L = -\alpha \frac{\partial^2}{\partial x^2} \tag{10.2}$$

where α is a constant. An *operator*, or *transformation*, is a rule that assigns, to each function $w(\mathbf{x}, t)$ belonging to a certain class, another function ($-aw_{xx}$ in the case of the string operator) belonging to another, perhaps different, class of functions. Note that a matrix satisfies this definition. B is an operator representing the boundary conditions as given, for example, by Equations (9.2). As indicated previously, the equations of (10.1) define a boundary value problem. A common method of solving (10.1) is to use separation of variables, as illustrated by the examples in Chapter 9. As long as the operator L does not depend on time, and if L satisfies certain other conditions (discussed in the next chapter), this method will work.

In many situations, the separation of variables approach yields an infinite set of functions of the form $a_n(t)\phi_n(\mathbf{x})$ that are solutions of Equations (10.1). The most general solution is then the sum, i.e.,

$$w(\mathbf{x}, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(\mathbf{x}) \tag{10.3}$$

A related method, modal analysis, also uses these functions and is described in the next section.

Similar to the eigenvectors of a matrix, some operators have eigenfunctions. A nonzero function $\phi(\mathbf{x})$ that satisfies the relationships

$$L\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), \qquad \mathbf{x} \in \Omega$$
$$B\phi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega$$

is called an *eigenfunction* of the operator L with boundary conditions B. The scalar λ (possibly complex) is called an *eigenvalue* of the operator L with respect to the boundary conditions B. In some cases the boundary conditions are not present, as in the case of a matrix, and in some cases the boundary conditions are contained in the domain of the operator L. The *domain* of the operator L, denoted by $D(L)$, is the set of all functions $u(x)$ for which Lu is defined and of interest.

To see the connection between separation of variables and eigenfunctions, consider substitution of the assumed separated form $w(\mathbf{x}, t) = a(t)\phi(\mathbf{x})$ into Equations (10.1). This yields

$$\frac{\ddot{a}(t)}{a(t)} = \frac{L\phi(\mathbf{x})}{\phi(\mathbf{x})}, \qquad \mathbf{x} \in \Omega, \qquad t > 0 \tag{10.4}$$

$$a(t)B\phi(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad t > 0 \tag{10.5}$$

$$a(0)\phi(\mathbf{x}) = w_0(\mathbf{x}), \qquad \dot{a}(0)\phi(\mathbf{x}) = \dot{w}_0(\mathbf{x}) \tag{10.6}$$

As before, Equation (10.4) implies that each side is constant, so that

$$L\phi(\mathbf{x}) = \lambda\phi(\mathbf{x}), \qquad \mathbf{x} \in \Omega \tag{10.7}$$

where λ is a scalar. In addition, note that, since $a(t) \neq 0$ for all t, Equation (10.5) implies that

$$B\phi(\mathbf{x}) = 0, \qquad x \in \partial\Omega, \qquad t > 0 \tag{10.8}$$

Equations (10.7) and (10.8) are, of course, a statement that $\phi(\mathbf{x})$ and λ constitute an eigenfunction and eigenvalue of the operator L.

Example 10.2.1

Consider the operator formulation of the longitudinal bar equation presented in Section 9.3. The form of the beam operator is

$$L = -\alpha \frac{\partial^2}{\partial x^2}, \qquad x \in (0, \ell)$$

with boundary conditions ($B = 1$ at $x = 0$ and $B = \partial/\partial x$ at $x = \ell$)

$$\phi(0) = 0 \quad \text{(clamped end)} \qquad \text{and} \qquad \phi_x(\ell) = 0 \quad \text{(free end)}$$

and where $\partial\Omega$ consists of the points $x = 0$ and $x = \ell$. Here, the constant a represents the physical parameters of the beam, i.e., $\alpha = EA/\rho$. The eigenvalue problem $L\phi = \lambda\phi$ becomes

$$-\alpha\phi_{xx} = \lambda\phi$$

or

$$\phi_{xx} + \frac{\lambda}{\alpha}\phi = 0$$

This last expression is identical to Equation (9.6) and the solution is

$$\phi(x) = A_1 \sin\left(\sqrt{\frac{\lambda}{\alpha}}x\right) + A_2 \cos\left(\sqrt{\frac{\lambda}{\alpha}}x\right)$$

where A_1 and A_2 are constants of integration. Using the boundary conditions yields

$$0 = \phi(0) = A_2 \qquad \text{and} \qquad 0 = \phi_x(\ell) = A_1\sqrt{\frac{\lambda}{\alpha}}\cos\sqrt{\frac{\lambda}{\alpha}}\ell$$

This requires that $A_2 = 0$ and

$$A_1 \cos\sqrt{\frac{\lambda}{\alpha}}\ell = 0$$

Since A_1 cannot be zero,

$$\sqrt{\frac{\lambda}{\alpha}}\ell = \frac{n\pi}{2}$$

for all odd integers n. Thus, λ depends on n and

$$\lambda_n = \frac{\alpha n^2 \pi^2}{4\ell^2}, \qquad n = 1, 3, 5, \ldots, \infty$$

Thus, there are many eigenvalues λ, denoted now by λ_n, and many eigenfunctions ϕ, denoted by ϕ_n. The eigenfunctions and eigenvalues of the operator L are given by the sets

$$\{\phi_n(x)\} = \left\{A_n \sin \frac{n\pi x}{2\ell}\right\} \qquad \text{and} \qquad \{\lambda_n\} = \left\{\frac{\alpha n^2 \pi^2}{4\ell^2}\right\}, \ n \text{ odd}$$

respectively. Note that, as in the case of a matrix eigenvector, eigenfunctions are determined only to within a multiplicative constant (A_n in this case).

Comparison of the eigenfunctions of the operator for the beam with the spatial functions calculated in Chapter 9 shows that the eigenfunctions of the operator correspond to the mode shapes of the structure. This correspondence is exactly analogous to the situation for the eigenvectors of a matrix.

10.3 MODAL ANALYSIS OF THE FREE RESPONSE

The eigenfunctions associated with the string equation are shown in the example of Section 9.2 to be the mode shapes of the string. Also, by using the linearity of the equations of motion, the solution is given as a summation of mode shapes. This summation of mode shapes, or eigenfunctions, given in Equation (9.16), constitutes the *eigenfunction expansion or modal analysis* of the solution and provides an alternative point of view to the separation of variables technique.

First, as in the case of eigenvectors of a matrix, eigenfunctions are conveniently normalized to fix a value for the arbitrary constant. To this end, let the eigenfunctions of interest be denoted by $A_n\phi_n(\mathbf{x})$. If the constants A_n are chosen such that

$$\int_\Omega A_n^2 \phi_n(\mathbf{x})\phi_n(\mathbf{x})\, d\Omega = 1 \tag{10.9}$$

then the eigenfunctions $\Theta_n = A_n\phi_n$ are said to be *normalized*, or normal. If, in addition, they satisfy

$$\int_\Omega \Theta_n \Theta_m d\Omega = \delta_{mn} \tag{10.10}$$

the eigenfunctions are said to be *orthonormal*, exactly analogous to the eigenvector case. The method of modal analysis assumes that the solution of Equations (10.1) can be represented as the series

$$w(\mathbf{x}, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(\mathbf{x}) \tag{10.11}$$

where $\Theta_n(\mathbf{x})$ are the normalized eigenfunctions of the operator L. Substitution of Equation (10.11) into Equations (10.1), multiplying by $\Theta_m(\mathbf{x})$, and integrating (assuming uniform convergence) over the domain Ω reduces Equations (10.1) to an infinite set of uncoupled ordinary differential equations of the form

$$\ddot{a}_n(t) + \lambda_n a_n(t) = 0, \qquad n = 1, 2, \ldots, \infty \tag{10.12}$$

Equation (10.12) can then be used along with the appropriate initial conditions to solve for each of the temporal functions. Here, λ_n is the eigenvalue associated with the nth mode, so that

$$\int_\Omega \Theta_n L \Theta_n d\Omega = \lambda_n \int_\Omega \Theta_n \Theta_n d\Omega = \lambda_n \tag{10.13}$$

where Equation (10.7) and (10.10) are used to evaluate the integral.

Example 10.3.1

Consider the transverse vibration of a Euler–Bernoulli beam with hinged boundary conditions. Calculate the eigenvalues and eigenfunctions for the associated operator.

The stiffness operator for constant mass, cross-sectional area, and area moment of inertia is given by (see Equation 9.46)

$$L = \frac{EI}{m}\frac{\partial^4}{\partial x^4} = \beta \frac{\partial^4}{\partial x^4}$$

$$\Theta(0) = \Theta_{xx}(0) = 0$$

$$\Theta(\ell) = \Theta_{xx}(\ell) = 0$$

The eigenvalue problem $Lu = \lambda u$ then becomes

$$\beta \Theta_{xxxx} = \lambda \Theta$$

which has a solution of the form

$$\Theta(x) = C_1 \sin \mu x + C_2 \cos \mu x + C_3 \sinh \mu x + C_4 \cosh \mu x$$

where $\mu^4 = \lambda/\beta$. Applying the four boundary conditions to this expression yields the four simultaneous equations

$$\Theta(0) = C_2 + C_4 = 0$$

$$\Theta_{xx}(0) = -C_2 + C_4 = 0$$

$$\Theta(L) = C_1 \sin \mu\ell + C_2 \cos \mu\ell + C_3 \sin \mu\ell + C_4 \cos \mu\ell = 0$$

$$\Theta_{xx}(\ell) = -C_1 \sin \mu\ell - C_2 \cos \mu\ell + C_3 \sin \mu\ell + C_4 \cos \mu\ell = 0$$

These four equations in the four unknown constants C_i can be solved by examining the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin \mu\ell & \cos \mu\ell & \sinh \mu\ell & \cosh \mu\ell \\ -\sin \mu\ell & -\cos \mu\ell & \sinh \mu\ell & \cosh \mu\ell \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Recall from Chapter 3 that, in order for a nontrivial vector $\mathbf{c} = [C_1 \quad C_2 \quad C_3 \quad C_4]^T$ to exist, the coefficient matrix must be singular. Thus, the determinant of the coefficient matrix must be zero. Setting the determinant equal to zero yields the characteristic equation

$$4 \sin(\mu\ell) \sinh(\mu\ell) = 0$$

This, of course, can be true only if $\sin(\mu\ell) = 0$, leading to

$$\mu = \frac{n\pi}{\ell}, \qquad n = 1,\ 2, \ldots, \infty$$

Here, $n = 0$ is excluded because it results in the trivial solution. In terms of the physical parameters of the structure, the eigenvalues become (here n is an integer and m is the mass per unit length of the beam)

$$\lambda_n = \frac{n^4\pi^4 EI}{m\ell^4}$$

Solving for the four constants C_i yields $C_2 = C_3 = C_4 = 0$ and that is C_1 arbitrary. Hence, the eigenfunctions are of the form

$$\left[A_n \sin\left(\frac{n\pi x}{\ell}\right)\right]$$

The arbitrary constants A_n can be fixed by normalizing the eigenfunctions

$$\int_0^\ell A_n^2 \sin^2\left(\frac{n\pi x}{\ell}\right) dx = 1$$

so that $A_n^2\ell/2 = 1$, or $A_n = \sqrt{2/\ell}$. Thus, the normalized eigenfunctions are the set

$$\{\Theta_n\} = \left\{\sqrt{\frac{2}{\ell}} \sin\left(\frac{n\pi x}{\ell}\right)\right\}_{n=1}^{\infty}$$

Hence, the temporal coefficient in the series expansion of the solution (10.11) will be determined from the initial conditions and the finite number of equations

$$\ddot{a}_n(t) + \frac{n^4\pi^4 EI}{m\ell^4} a_n(t) = 0, \qquad n = 1, \ldots, \infty$$

Equation (10.11) then yields the total solution.

10.4 MODAL ANALYSIS IN DAMPED SYSTEMS

As in the matrix case for lumped-parameter systems, the method of modal analysis (and separation of variables) can still be used for certain types of viscous damping modeled in a distributed structure. Systems that can be modeled by partial differential equations of the form

$$w_{tt}(\mathbf{x}, t) + L_1 w_t(\mathbf{x}, t) + L_2 w(\mathbf{x}, t) = 0, \qquad \mathbf{x} \in \Omega \tag{10.14}$$

(where L_1 and L_2 are operators, with similar properties to L and such that L_1 and L_2 have the same eigenfunctions) can be solved by the method of modal analysis illustrated in Equation (10.11) and example 10.3.1. Section 9.7 lists some examples.

To see this solution method, let L_1 have eigenvalues $\lambda_n^{(1)}$ and L_2 have eigenvalues $\lambda_n^{(2)}$. Substitution of Equation (10.11) into Equation (10.14) then yields (assuming convergence)

$$\sum_{n=1}^{\infty}\left[\ddot{a}_n\Theta_n(\mathbf{x})+\lambda_n^{(1)}\dot{a}_n\Theta_n(\mathbf{x})+\lambda_n^{(2)}a_n\Theta_n(\mathbf{x})\right]=0 \tag{10.15}$$

Multiplying by $\Theta_n(\mathbf{x})$, integrating over Ω, and using the orthogonality conditions (10.10) yields the decoupled set of n ordinary differential equations

$$\ddot{a}_n(t)+\lambda_n^{(1)}\dot{a}_n(t)+\lambda_n^{(2)}a_n(t)=0, \qquad n=1,2,\ldots,\infty \tag{10.16}$$

subject to the appropriate initial conditions.

The actual form of damping in distributed-parameter systems is not always clearly known. In fact, the form of L_1 is an elusive topic of current research and several texts (see, for instance, Nashif, Jones, and Henderson, 1985, or Sun and Lu, 1995). Often, the damping is modeled as being proportional, i.e., $L_1=\alpha I+\beta L_2$, where α and β are arbitrary scalars and L_1 satisfies the same boundary conditions as L_2. In this case, the eigenfunctions of L_1 are the same as those of L_2. Damping is often estimated using equivalent viscous proportional damping of this form as an approximation.

Example 10.4.1

As an example of a proportionally damped system, consider the transverse free vibration of a membrane in a surrounding medium, such as a fluid, providing resistance to the motion that is proportional to the velocity. The equation of motion given by Equation (9.86) is Equation (10.14), with

$$L_1=2\frac{\gamma}{\rho}$$
$$L_2=-\frac{T}{\rho}\nabla^2$$

where γ, T, ρ, and ∇^2 are as defined for Equation (9.86). The position $\mathbf{x}$ in this case is the vector $[x \quad y]$ in two-dimensional space. If $\lambda_1^{(2)}$ is the first eigenvalue of L_2, then the solutions to Equation (10.16) are of the form

$$a_n(t)=\mathrm{e}^{-\frac{\gamma}{\rho}t}\left[A_n\sin\sqrt{\lambda_n^{(2)}-\frac{\gamma}{\rho^2}^2}\,t+B_n\cos\sqrt{\lambda_n^{(2)}-\frac{\gamma}{\rho^2}^2}\,t\right]$$

where A_n and B_n are determined by the initial conditions (see Section 11.9).

Not all damped systems have this type of damping. Systems that have proportional damping are called *normal mode systems*, since the eigenfunctions of the operator L_2 serve to 'decouple' the system. Decouple, as used here, refers to the fact that Equations(10.16) depends only on n and not on any other index. This topic is considered in more detail in Section 11.9.

10.5 TRANSFORM METHODS

An alternative to using separation of variables and modal analysis is to use a transform to solve for the vibrational response. As with the Laplace transform method used on the temporal variable in state-space analysis for lumped-parameter systems, a Laplace transform can also be used in solving Equations (10.1). In addition, a Fourier transform can be used on the spatial variable to calculate the solution. These methods are briefly mentioned here. The reader is referred to a text such as Churchill (1972) for a rigorous development.

The Laplace transform taken on the temporal variable of a partial differential equation can be used to solve for the free or forced response of Equations (10.1) and (10.14). This method is best explained by considering an example.

Consider the vibrations of a beam with constant force F_0 applied to one end and fixed at the other. Recall that the equation for longitudinal vibration is

$$w_{tt}(x, t) = \alpha^2 w_{xx}(x, t) \tag{10.17}$$

with boundary conditions

$$w(0, t) = 0, \qquad EAw_x(\ell, t) = F_0\delta(t) \tag{10.18}$$

Here, $a^2 = EA/\rho$, as defined in Section 9.3. Assuming that the initial conditions are zero, the Laplace transform of Equation (10.17) yields

$$s^2 W(x, s) - \alpha^2 W_{xx}(x, s) = 0 \tag{10.19}$$

and the Laplace transform of Equation (10.18) yields

$$\begin{aligned} W_x(\ell, s) &= \frac{F_0}{EAs} \\ W(0, s) &= 0 \end{aligned} \tag{10.20}$$

Here, W denotes the Laplace transform of w. The solution of Equation (10.19) is of the form

$$W(x, s) = A_1 \sinh \frac{sx}{\alpha} + A_2 \cosh \frac{sx}{\alpha}$$

Applying the boundary condition at $x = 0$, gives $A_2 = 0$. Differentiating with respect to x and taking the Laplace transform yields the boundary condition at $x = \ell$. The constant A_1 is then determined to be

$$A_1 = \left(\frac{\alpha F_0}{EA}\right)\left(\frac{1}{s^2 \cosh(s\ell/\alpha)}\right)$$

The solution in terms of the transform variable s then becomes

$$W(x, s) = \frac{\alpha F_0 \sinh(sx/\alpha)}{EAs^2 \cosh(s\ell/\alpha)} \tag{10.21}$$

By taking the inverse Laplace transform of Equation (10.21) using residue theory, the solution in the time domain is obtained. The inverse is given by Churchill (1972) to be

$$w(x,t) = \frac{F_0}{E}x + \left(\frac{8\ell}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin\frac{(2n-1)\pi x}{2\ell} \cos\frac{(2n-1)\pi a t}{2\ell} \tag{10.22}$$

A text on transforms should be consulted for the details. Basically, the expansion comes from the zeros in the complex plane of $s^2\cosh(s\ell/a)$, i.e., the poles of $W(x,s)$.

This same solution can also be obtained by taking the finite Fourier sine transform of Equations (10.17) and (10.18) on the spatial variable x rather than the Laplace transform of the temporal variable (see, for instance, Meirovitch, 1967). Usually, transforming the spatial variable is more productive because the time dependence is a simple initial value problem.

When boundary conditions have even-order derivatives, a finite sine transformation (Fourier transform) is appropriate. The sine transform is defined by

$$W(n,t) = \int_0^{\ell} w(x,t) \sin\frac{n\pi x}{\ell}\, dx \tag{10.23}$$

Note here that the transform in this case is over the spatial variable.

Again, the method is explained by example. To that end, consider the vibration of a string clamped at each end and subject to nonzero initial velocity and displacement, i.e.,

$$\begin{gathered} w_{xx} = \frac{1}{c^2} w_{tt}(x,t) \\ w(0,t) = w(\ell,t) = 0, \qquad w(x,0) = f(x), \qquad w_t(x,0) = g(x) \end{gathered} \tag{10.24}$$

The finite sine transform of the second derivative is

$$W_{xx}(n,t) = \frac{n\pi}{\ell}\left|(-1)^{n+1}W(\ell,t) + W(0,t)\right| - \left(\frac{n\pi}{\ell}\right)^2 W(n,t) \tag{10.25}$$

which is calculated from integration by parts of Equation (10.23). Substitution of the boundary conditions yields the transformed string equation

$$W_{tt}(n,t) + \left(\frac{n\pi}{\ell}\right)^2 W(n,t) = 0 \tag{10.26}$$

This equation is subject to the transform of the initial conditions, which are

$$W(n,0) = \int_0^{\ell} f(x) \sin\frac{n\pi x}{\ell}\, dx \tag{10.27}$$

and

$$W_t(n,0) = \int_0^{\ell} g(x) \sin\frac{n\pi x}{\ell}\, dx$$

Thus

$$W(n,t) = W(n,0)\cos\frac{n\pi c t}{\ell} + W_t(n,0)\frac{\ell}{n\pi c}\sin\frac{n\pi c t}{\ell} \tag{10.28}$$

Again, Equation (10.28) has to be inverted to obtain the solution $w(x, t)$. The inverse finite Fourier transform is given by

$$w(x, t) = \frac{2}{\ell} \sum_{n=1}^{\infty} W(n, t) \sin\left(\frac{n\pi x}{\ell}\right) \tag{10.29}$$

so that

$$w(x, t) = \frac{2}{\ell} \sum_{n=1}^{\infty} \left[\left\{ W(n, 0) \cos\left(\frac{n\pi ct}{\ell}\right) + \frac{W_t(n, 0)\ell}{n\pi c} \sin\left(\frac{n\pi ct}{\ell}\right) \right\} \sin\frac{n\pi x}{\ell} \right] \tag{10.30}$$

Transform methods are attractive for problems defined over infinite domains and for problems with odd boundary conditions. The transform methods yield a quick 'solution' in terms of the transformed variable. However, the inversion back into the physical variable can be difficult and may require as much work as using separation of variables or modal analysis. However, in some instances, the only requirement may be to examine the solution in its transformed state, such as is done in Section 8.5.

10.6 GREEN'S FUNCTIONS

Yet another approach to solving the free vibration problem is to use the integral formulation of the equations of motion. The basic idea here is that the free response is related to the eigenvalue problem

$$\begin{aligned} Lw &= \lambda w \\ Bw &= 0 \end{aligned} \tag{10.31}$$

where L is a differential operator and B represents the boundary conditions. The inverse of this operator will also yield information about the free vibrational response of the structure. If the inverse of L exists, Equation (10.31) can be written as

$$L^{-1} w = \frac{1}{\lambda} w \tag{10.32}$$

where L^{-1} is the inverse of the differential operator or an integral operator.

The problem of solving for the free vibration of a string fixed at both ends by working essentially with the inverse operator is approached in this section. This is done by introducing the concept of a Green's function. To this end, consider again the problem of a string fixed at both ends and deformed from its equilibrium position. This time, however, instead of looking directly at the vibration problem, the problem of determining the static deflection of the string owing to a transverse load concentrated at a point is first examined. This related problem is called the *auxiliary problem*. In particular, if the string is subject to a point load of unit value at x_0, which is somewhere in the interval (0,1), the equation of the deflection $w(x)$ for a string of tension T is

$$-T \frac{d^2 w(x)}{dx^2} = \delta(x - x_0) \tag{10.33}$$

where $\delta(x - x_0)$ is the Dirac delta function. The delta function is defined by

$$\delta(x - x_0) \begin{cases} 0, & x \neq x_0 \\ \infty, & x = x_0 \end{cases} \tag{10.34}$$

and

$$\int_0^1 \delta(x - x_0)dx = \begin{cases} 0 & \text{if } x_0 \text{ is not in (0,1)} \\ 1 & \text{if } x_0 \text{ is in (0,1)} \end{cases} \tag{10.35}$$

If $f(x)$ is a continuous function, then it can be shown that

$$\int_0^1 f(x)\delta(x - x_0)dx = f(x_0) \tag{10.36}$$

for x_0 in (0,1). Note that the Dirac delta function is not really a function in the strict mathematical sense (see, for instance, Stakgold, 1979).

Equation (10.33) can be viewed as expressing the fact that the force causing the deflection is applied only at the point x_0. Equation (10.33) plus boundary conditions is now viewed as the auxiliary problem of finding a function $g(x, x_0)$, known as *Green's function* for the operator $L = -Td^2/dx^2$, with boundary conditions $g(0, x_0) = 0$ and $g(1, x_0) = 0$. In more physical terms, $g(x, x_0)$ represents the deflection of the string from its equilibrium position at point x owing to a unit force applied at point x_0. Green's function thus defined is also referred to as an *influence function*. The following example is intended to clarify the procedure for calculating a Green's function.

Example 10.6.1

Calculate Green's function for the string of Figure 10.1. Green's function is calculated by solving the equation on each side of the point x_0 and then matching up the two solutions. Thus, since $g'' = 0$ for all x not equal to x_0, integrating yields

$$g(x, x_0) = \begin{cases} Ax + B, & 0 < x < x_0 \\ Cx + D, & x_0 < x < 1 \end{cases}$$

where A, B, C, and D are constants of integration. Applying the boundary condition at $x = 0$ yields

$$g(0, x_0) = 0 = B$$

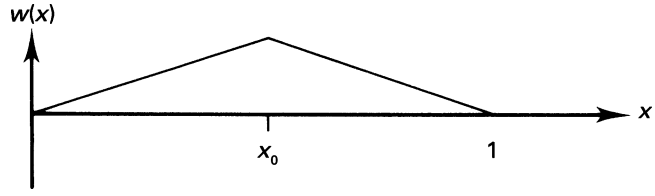

Figure 10.1 Statically deflected string fixed at both ends.

and the boundary condition at 1 yields

$$g(1, x_0) = 0 = C + D$$

Hence, Green's function becomes

$$g(x, x_0) = \begin{cases} Ax, & 0 < x < x_0 \\ C(x-1), & x_0 < x < 1 \end{cases}$$

Since the string does not break at x_0, $g(x, x_0)$ must be continuous at x_0, and this allows determination of one more constant. In particular, this continuity condition requires that

$$Ax_0 = C(x_0 - 1)$$

Green's function now becomes

$$g(x, x_0) \begin{cases} Ax & 0 < x < x_0 \\ A\dfrac{x_0(x-1)}{x_0 - 1}, & x_0 < x < 1 \end{cases}$$

The remaining constant, A, can be evaluated by considering the magnitude of the applied force required to produce the deflection. In this case, a unit force is applied, so that integration of Equation (10.33) along a small interval containing x_0, say, $x_0 - \varepsilon < x < x_0 + \varepsilon$, yields

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{d^2 g}{dx^2}\, dx = -\frac{1}{T} \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\, dx$$

or

$$\left.\frac{dg}{dx}\right|_{x_0-\varepsilon}^{x_0+\varepsilon} = -\frac{1}{T}$$

Denoting the derivative by a subscript and expanding yields

$$g_x(x_0 + \varepsilon, x_0) - g_x(x_0 - \varepsilon, x_0) = -\frac{1}{T}$$

This last expression is called a *jump discontinuity* in the derivative. Upon evaluating the derivative, the above expression becomes

$$A\frac{x_0}{x_0 - 1} - A = -\frac{1}{T}$$

Solving this for the value of A yields

$$A = \frac{1 - x_0}{T}$$

Green's function, with all the constants of integration evaluated, is thus

$$g(x, x_0) = \begin{cases} \dfrac{(1 - x_0)x}{T}, & 0 < x < x_0 \\ \dfrac{(1 - x)x_0}{T}, & x_0 < x < 1 \end{cases}$$

Green's function actually defines the inverse operator (when it exists) of the differential operator L and can be used to solve for the forced response of the string. Consider the equations (for the string operator of the example)

$$\begin{aligned} Lu &= f(x) \\ Bu &= 0 \end{aligned} \tag{10.37}$$

where $f(x)$ is a piecewise continuous function, and L is a differential operator that has an inverse. Let G denote the operator defined by

$$Gf(x) = \int_0^1 g(x, x_0) f(x_0)\, dx_0$$

The operator G defined in this way is called an *integral operator*. Note that the function

$$u(x) = \int_0^1 g(x, x_0) f(x_0)\, dx_0 \tag{10.38}$$

satisfies Equation (10.37), including the boundary conditions, which follows from a straightforward calculation. Equation (10.38) can also be written as

$$u(x) = \int_0^{x-\varepsilon} g(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g(x, x_0) f(x_0)\, dx_0$$

where the integration has been split over two separate intervals for the purpose of treating the discontinuity in g_x. Using the rules for differentiating an integral (Leibnitz's rule) applied to this expression yields

$$\begin{aligned} u_x(x) &= \int_0^{x-\varepsilon} g_x(x, x_0) f(x_0)\, dx_0 + g(x, x-\varepsilon) f(x-\varepsilon) \\ &\quad + \int_{x+\varepsilon}^1 g_x(x, x_0) f(x_0)\, dx_0 - g(x, x+\varepsilon) f(x+\varepsilon) \\ &= \int_0^{x-\varepsilon} g_x(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g_x(x, x_0) f(x_0)\, dx_0 \end{aligned}$$

Taking the derivative of this expression for u_x yields

$$\begin{aligned} u_{xx}(x) &= \int_0^{x-\varepsilon} g_{xx}(x, x_0) f(x_0)\, dx_0 + g_x(x, x-\varepsilon) f(x-\varepsilon) \\ &\quad + \int_{x+\varepsilon}^1 g_{xx}(x, x_0) f(x_0)\, dx_0 - g_x(x, x+\varepsilon) f(x+\varepsilon) \end{aligned}$$

The discontinuity in the first derivative yields

$$g_x(x, x-\varepsilon) f(x-\varepsilon) - g_x(x, x+\varepsilon) f(x+\varepsilon) = \frac{f(x)}{T}$$

Hence

$$u_{xx} = \int_0^{x-\varepsilon} g_{xx}(x, x_0) f(x_0)\, dx_0 + \int_{x+\varepsilon}^1 g_{xx}(x, x_0) f(x_0)\, dx_0 - \frac{f(x)}{T} \tag{10.39}$$

However, $g_{xx} = 0$ in the intervals specified in the two integrals. Thus, this last expression is just the equation $Lu = f$. The function $u(x)$ then becomes

$$u(x) = \int_0^1 g(x, y) f(y)\, \mathrm{d}y$$

which satisfies Equation (10.37) as well as the boundary condition.

Now note that $Gf = u$, so that G applied to $Lu = f$ yields $G(Lu) = Gf = u$, and hence $GLu = u$. Also, L applied to $Gf = u$ yields $LGf = Lu = f$, so that $LGf = f$. Thus, the operator G is clearly the inverse of the operator L.

In the same way, the Green's function can also be used to express the eigenvalue problem for the string. In fact,

$$\int_0^1 g(x, x_0)\theta(x_0)\, dx_0 = \mu\theta(x) \tag{10.40}$$

yields the eigenfunctions $\theta(x)$ for the operator L as defined in Equation (10.2), where $\mu = 1/\lambda$ and Equation (10.32) is defined by G.

To summarize, consider the slightly more general operator given by

$$Lw = a_0(x)w_{xx}(x) + a_1(x)w_x(x) + a_2(x)w(x) = 0 \tag{10.41}$$

with boundary conditions given by

$$B_1 w|_{x=0} = 0 \qquad \text{and} \qquad B_2 w|_{x=1} = 0 \tag{10.42}$$

Green's function for the operator given by Equations (10.41) and (10.42) is defined as the function $g(x, x_0)$ such that:

- $0 < x < 1, 0 < x_0 < 1$;
- $g(x, x_0)$ is continuous for any fixed value of x_0 and satisfies the boundary conditions in Equation (10.42);
- $g_x(x, x_0)$ is continuous except at $x = x_0$;
- as a function of x, $Lg = 0$ everywhere except at $x = x_0$;
- the jump discontinuity $g_x(x, x_0 + \varepsilon) - g_x(x, x_0 - \varepsilon) = 1/a_0(x)$ holds.

Green's function defines the inverse of the operators (10.41) and (10.42). Furthermore, the eigenvalue problem associated with the vibration problem can be recast as an integral equation as given by Equation (10.40). The Green's function approach can be extended to other operators. Both of these concepts are capitalized upon in the following chapters.

CHAPTER NOTES

The majority of this chapter is common material found in a variety of texts, only some of which are mentioned here. Section 10.2 introduces eigenfunctions and makes the connection between eigenfunctions and separation of variables as methods of solving boundary value problems arising in vibration analysis. The method of separation of variables is discussed in

most texts on vibration as well as those on differential equations, such as the text by Boyce and DiPrima (2000). Eigenfunctions are also discussed in texts on operator theory, such as Naylor and Sell (1982). Few texts make an explicit connection between the two methods. The procedure is placed on a firm mathematical base in Chapter 11.

Section 10.3 examines, in an informal way, the method of modal analysis, a procedure made popular by the excellent texts by Meirovitch (1967, 2001). Here, however, the method is more directly related to eigenfunction expansions. Section 10.4 introduces damping as a simple velocity-proportional operator commonly used as a first attempt, as described in Section 9.7. Damping models represent a discipline by themselves. Here, a model of mathematical convenience is used. A brief look at using transform methods is provided in Section 10.5 for the sake of completeness. Transform methods have been developed by Yang (1992). Most transform methods are explained in detail in basic text, for instance by Churchill (1972). The last section on Green's functions follows closely the development in Stakgold (1979, 2000). Green's function methods provide a strong basis for the theory to follow in the next chapter. Most texts on applied mathematics in engineering discuss Green's functions.

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PROBLEMS

10.1 Compute the eigenvalues and eigenfunctions for the operator

$$L = -\frac{d^2}{dx^2}$$

with boundary conditions $u(0) = 0$, $u_x(1) + u(1) = 0$.

10.2 Normalize the eigenfunctions of problem 10.1.

10.3 Show that $u_{nm}(x, y) = A_{nm} \sin n\pi x \sin m\pi y$ is an eigenfunction of the operator

$$L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

with boundary conditions $u(x, 0) = u(x, 1) = u(1, y) = u(0, y) = 0$. This is the membrane operator for a unit square.

10.4 Normalize the eigenfunctions of a membrane (clamped) of problem 10.3 and show that they are orthonormal.

10.5 Calculate the temporal coefficients, $a_n(t)$, for the problem of example 10.3.1.

10.6 Calculate the initial conditions required in order for $a_n(t)$ to have the form given in example 10.4.1.

10.7 Rewrite Equation (10.16) for the case where the eigenfunctions of L_1 are not the same as those of L_2.

10.8 Solve for the free longitudinal vibrations of a clamped-free bar in the special case where the damping is approximated by the operator $L_1 = 0.1I$, $EA = \rho = 1$, and the initial conditions are $w_t(x, 0) = 0$ and $w(x, 0) = 10^{-2}$.

10.9 Calculate Green's function for the operator given by $L = 10^6 \partial^2/\partial x^2$, $u(0) = 0$, $u_x(\ell) = 0$. This corresponds to a clamped bar.

10.10 Calculate Green's function for the operator $L = \partial^4/\partial x^4$, with boundary conditions $u(0) = u(1) = u_{xx}(1) = 0$. (*Hint*: The jump condition is $g_{xxx}(x, x + \varepsilon) - g_{xxx}(x, x - \varepsilon) = 1$.)

10.11 Normalize the eigenfunctions of example 9.31 and discuss the orthogonality conditions.

11
Operators and the Free Response

11.1 INTRODUCTION

Just as knowledge of linear algebra and matrix theory is helpful in the study of lumped-parameter vibration problems, so a working knowledge of functional analysis and operator theory is useful in the study of the vibrations of distributed-parameter systems. A complete introduction to these topics requires a sound background in mathematics and is not possible within the limited space here. However, this chapter introduces some of the topics of relevance in vibration analysis. One of the main concerns of this section is to consider the convergence of the series expansions of eigenfunctions used in the separation of variables and modal analysis methods introduced in the previous chapters. The intent of this chapter is similar to that of Chapter 3, which introduced linear algebra as needed for discussing the free response of lumped-mass systems (also called finite-dimensional systems). The goal of this chapter is to provide a mathematical analysis of the methods used in Chapter 10.

In addition, some results are presented that examine the qualitative nature of the solution of linear vibration problems. In many instances, the describing differential equations cannot be solved in closed form. In these situations, knowing the nature of the solution rather than its details may be satisfactory. For example, knowing that the first natural frequency of a structure is bounded away from a driving frequency, rather than knowing the exact numerical value of the natural frequency, may be sufficient. Also, knowing whether a given structure will oscillate or not without computing the solution is useful. All this qualitative behavior is discussed in this chapter. As indicated in Chapter 6, qualitative results are very useful in design situations.

11.2 HILBERT SPACES

The definition of integration familiar to most engineers from introductory calculus is *Riemann integration*. Riemann integration can be defined on the interval $[a, b]$ as

$$\int_a^b f(x)\,dx = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i \tag{11.1}$$

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

where Δx_i is a small interval of the segment $b - a$, for instance, $(b - a)/n$. In most engineering applications, this type of integration is adequate. However, quite often a sequence of functions $\{f_n(x)\}$, defined for values of x in the interval $[a, b]$, converges to a function $f(x)$, i.e.,

$$\{f_n(x)\} \to f(x) \tag{11.2}$$

as n approaches infinity. Then, being able to conclude that

$$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) \to \int_a^b f(x)\,dx \tag{11.3}$$

is important. For example, when using modal analysis, this property is required. Unfortunately, Equation (11.3) is not always true for Riemann integration. The *Lebesgue integral* was developed to force Equation (11.3) to be true.

Lebesgue integration was developed using the concept of measurable sets, which is beyond the scope of this text. Thus, rather than defining Lebesgue integration directly, the properties are listed below:

1. If $f(x)$ is Riemann integrable, then it is Lebesgue integrable, and the two integrations yield the same value (the reverse is *not* true).
2. If α is a constant and $f(x)$ and $g(x)$ are Lebesgue integrable, then $\alpha f(x)$ and $g(x) + f(x)$ are Lebesgue integrable.
3. If $f^2(x)$ is Lebesgue integrable, then $\int f^2(x)\,dx = 0$ if and only if $f(x) = 0$ *almost everywhere* (i.e., everywhere except at a few points).
4. If $f(x) = g(x)$ almost everywhere, then $\int f^2(x)\,dx = \int g^2(x)\,dx$.
5. If $f_n(x)$ is a sequence of functions that are Lebesgue integrable over the interval $[a, b]$, if the sequence $\{f_n(x)\}$ converges to $f(x)$, and if for sufficiently large n there exists a function $F(x)$ that is Lebesgue integrable and $|f_n(x)| < F(x)$, then

   (a) $f(x)$ is Lebesgue integrable;
   (b) $\lim_{n\to\infty}\left\{\int_a^b f_n(x)\,dx\right\} = \int_a^b f(x)\,dx$

Any function $f(\mathbf{x}), \mathbf{x} \in \Omega$, such that$f^2(\mathbf{x})$is Lebesgue integrable, is called square integrable in the Lebesgue sense, denoted by $f \in \mathcal{L}_2^{\mathrm{R}}(\Omega)$, and, as will be illustrated, defines an important class of functions. In fact, the set of functions that are $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ make up a linear space (see Appendix B or Naylor and Sell, 1982). In this notation, the subscript 2 denotes square integrable, the superscript R denotes the functions are all real, and Ω denotes the domain of integration.

Another important concept is that of linearly independent sets of functions. An arbitrary set of functions is said to be *linearly independent* if every finite subset of elements is linearly independent. Note that this definition is consistent with the notion of linear independence introduced in Chapter 3 and, in fact, is based on that definition.

The concept of a linear space, or vector space, is sufficient for the study of matrices and vectors. However, more mathematical structure is required to discuss operators. In particular, a linear space is defined to be an *inner product* space if, with every pair of elements u and

v in the space, there exists a unique complex number (u, v), called the inner product of u and v, such that:

- $(u, v) = (v, u)^*$, where the asterisk indicates the complex conjugate;
- $(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v)$;
- $(u, u) > 0$;
- $(u, u) = 0$ if and only if $u = 0$.

The inner product most often used in vibrations is that defined by the Lebesgue integral given in the form

$$(u, v) = \int_{\Omega} u(\mathbf{x}) v^*(\mathbf{x})\, d\Omega \tag{11.4}$$

Next, a few more definitions and results are required in order to mimic the structure used in linear algebra to analyze vibration problems. The *norm* of an element in a linear space is denoted by $||u||$ and is defined, in the case of interest here, by

$$||u(\mathbf{x})|| = \sqrt{(u(\mathbf{x}), u(\mathbf{x}))} \tag{11.5}$$

Note that $||u(\mathbf{x})|| = 0$ if and only if $u(\mathbf{x}) = 0$ almost everywhere. In addition, the norm satisfies the following conditions:

- $||u|| > 0$;
- $||\alpha u|| < |\alpha|\ ||u||$, where α is a scalar;
- $||u + v|| \leq ||u|| + ||v||$ with equality if and only if $v = \alpha u > 0$ (referred to as the *triangle inequality*).

This last set of conditions introduces even more structure on a linear space. A linear space with such a norm is called a *normed linear space*. The set $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ can be shown to form an inner product space with the preceding definition of an inner product and a normed linear space with the preceding definition of a norm.

Note that, in the case of vectors, the scalar product $\mathbf{x}^{\mathrm{T}}\mathbf{x}$ satisfies the preceding definition of inner product. An important property of sets of elements is that of orthogonality. Just as in the case of vectors, two elements, u and v, of an inner product space are said to be *orthogonal* if $(u, v) = 0$. Orthogonality is used extensively in Section 10.3 with respect to modal analysis.

Based on the definitions of convergence used in calculus for scalars, a definition of convergence for elements of a normed linear space can be stated. Let $\{u_n(x)\}$ be a set of elements in a normed linear space, denoted by V. The sequence $\{u_n(x)\}$ *converges* to the element $u(x)$ in V if, for all $\varepsilon > 0$, there exists a number $N(\varepsilon)$ such that

$$\|u_n(x) - u(x)\| < \varepsilon$$

whenever $n > N(\varepsilon)$. This form of convergence is denoted by any of the following:

- $u_n \to u \qquad \text{as } n \to \infty;$
- $\lim_{n\to\infty} u_n(x) = u(x)$
- $\lim_{n\to\infty} \|u_n(x) - u(x)\| = 0$

In particular, the notation

$$u(x) = \sum_{n=1}^{\infty} u_n(x)$$

implies that the sequence of partial sums converges to $u(x)$, i.e.,

$$\lim_{k\to\infty} \sum_{n=1}^{k} u_n(x) = u(x)$$

Convergence defined this way is referred to as *strong convergence*, or *norm convergence*. As pointed out briefly in Section 9.3, this convergence is required when writing the modal expansion of Equation (10.11).

In the case of vectors, writing a series of weighted sums of eigenvectors is sufficient. The resulting sum is always finite, since the sum in the modal expansion contains a finite number of terms. However, since the sums in general normed linear spaces may have an infinite number of terms, convergence to some finite-valued function is not obvious.

As an aid in considering the convergence of a sequence of functions, a Cauchy sequence is used. A sequence $\{u_n(x)\}$ is defined to be a *Cauchy sequence* if, for all numbers $\varepsilon > 0$, there exists a number $N(\varepsilon)$ such that $||u_n(x) - u_m(x)|| < \varepsilon$ for every index m and n larger than $N(\varepsilon)$. An immediate consequence of this definition, and the triangle inequality, is that every convergent sequence is a Cauchy sequence. However, in general, not every Cauchy sequence converges. Requiring Cauchy sequences to converge leads to the concept of completeness. A normed linear space is defined as *complete* if every Cauchy sequence in that space converges in that space.

Note that, in a complete normed linear space, convergent sequences and Cauchy sequences are identical. The concept of completeness means that the limits of all of the sequences in the space that converge are also in that space. Comparing the set of real numbers with the set of rational numbers is analogous to the concept of completeness. The set of rational numbers is not complete, since one can construct a sequence of rational numbers that converges to an irrational number (such as the square root of two), which is not rational and hence is not in the set.

A complete normed linear space is called a *Banach space*. The set of all vectors of dimension n with real elements and with the norm defined by $||\mathbf{x}||^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x}$ is a familiar example of a Banach space. The major difference in working with lumped-parameter vibration problems versus working with distributed-parameter vibration problems is based on the fact that every finite-dimensional normed linear space is complete and many common infinite-dimensional spaces are not. This possibility requires some concern over issues of convergence when using mode summation methods.

Example 11.2.1

Show by example that the space defined by the set of all continuous functions defined on the interval $[-1, 1]$ with norm

$$||u||^2 = \int_{-1}^{1} |u|^2 \, dx$$

is not complete and also that a Cauchy sequence in the space does not converge to something in the space. Consider the sequence of continuous functions $u_n(x)$, where

$$u_n(x) = \begin{cases} 0, & -1 \le x \le 0 \\ nx, & 0 < x \le \dfrac{1}{n} \\ 1, & \dfrac{1}{n} < x \le 1 \end{cases}$$

A quick computation verifies that the sequence is a Cauchy sequence, but the sequence $\{u_n\}$ converges to the function $u(x)$ given by

$$u(x) = \begin{cases} 0, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$$

which is *discontinuous* and hence is not an element in the space defined on the set of continuous functions.

In this last example, note that both $u_n(x)$ and $u(x)$ are square integrable in the Lebesgue sense. In other words, if, instead of requiring the linear space in the example to be the set of continuous functions, the set of square integrable functions in the Lebesgue sense is used, the space *is* a complete normed linear space. Since using modal analysis is desirable, mimicking the procedure used in finite dimensions, the natural choice of linear spaces to work in is $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ or, when appropriate, $\mathcal{L}_2^{\mathrm{C}}(\Omega)$

Again, motivated by the method of modal analysis, it is desirable to equip the linear space with an inner product. Gathering all this mathematical structure together yields the class of functions most useful in vibration analysis. This space is called a *Hilbert space*, denoted by H. A Hilbert space is defined as a complete inner product space. Again, the set of real vectors with inner product $\mathbf{x}^T\mathbf{x}$ is an example of a Hilbert space. The Hilbert spaces of interest here are $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ and $\mathcal{L}_2^{\mathrm{C}}(\Omega)$.

A further requirement placed on Hilbert spaces used in vibration applications is the assumption that the space is separable. A *separable Hilbert space* is a Hilbert space that contains a countable set of elements $\{f_n(x)\}$ such that, for any element f in the space and any positive real number ε, there exists an index N and a set of constants $\{\alpha_i\}$ such that

$$\left\| f(x) - \sum_{n=1}^{N} \alpha_n f_n(x) \right\| < \varepsilon \tag{11.6}$$

Here, the set $\{f_n\}$ is called a *spanning set* for the Hilbert space. These concepts and their significance are discussed in the next section. However, note again that finite-dimensional vector spaces are separable and that this property is useful in writing modal expansions. Both spaces $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ and $\mathcal{L}_2^{\mathrm{C}}(\Omega)$ are separable Hilbert spaces. The symbol $\mathcal{H}$ is used to denote a general separable Hilbert space.

11.3 EXPANSION THEOREMS

In this section, the expansion theorem, or modal expansion, used informally in the last chapter is placed in a more rigorous setting by generalizing the Fourier series. Let $\{\phi_k\}$ be an infinite set of orthonormal functions in $\mathcal{H}$, i.e., $(\phi_k, \phi_i) = \delta_{ki}$. For any element $u \in \mathcal{H}$, the scalar (u, ϕ_k) is called the *Fourier coefficient* of u relative to the set $\{\phi_k\}$. Furthermore, the sum

$$\sum_{k=1}^{\infty}(u, \phi_k)\phi_k(x)$$

is called the *Fourier series* of $u(x)$ relative to the set $\{\phi_k\}$.

The following results (stated without proof) are useful in extending the Fourier theorem and in understanding how to use modal expansions in applications. Let $\{\phi_n\}$ again be an orthonormal set of elements in $\mathcal{H}$, let u also be an element in $\mathcal{H}$, and let $\{\lambda_k\}$ denote a set of complex scalars. Then, the following relationships hold

$$\left\| u - \sum_{k=1}^{n} \lambda_k \phi_k \right\| \geq \left\| u - \sum_{k=1}^{n} (u, \phi_k)\phi_k \right\| \tag{11.7}$$

$$\sum_{k=1}^{\infty} |(u, \phi_k)|^2 < \|u(x)\|^2 \quad \text{(called Bessel's inequality)} \tag{11.8}$$

and

$$\lim_{k\to\infty} (u, \phi_k) = 0 \tag{11.9}$$

This last result is referred to as the *Riemann–Lebesgue lemma*.

Furthermore, a famous result, the *Riesz–Fischer theorem*, relates the convergence of a modal expansion to the convergence of the coefficients in that expansion:

1. If $\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$, then $\sum_{k=1}^{\infty} \lambda_k \phi_k(x)$ converges to an element, u, in $\mathcal{H}$ and $\lambda_k = (u, \phi_k)$, i.e., the λ_k are the Fourier coefficients of the function $u(x)$.
2. If $\sum_{k=1}^{\infty} |\lambda_k|^2$ diverges, then so does the expansion $\sum_{k=1}^{\infty} \lambda_k \phi_k$.

A set of orthonormal functions satisfying (1) is called a *complete set* of functions. Recall that a complete space is a space in which Cauchy sequences converge. A complete set, on the other hand, is a set of elements in a space such that every element in the space can be represented as a series expansion of elements of the set. In general, an orthonormal set of functions $\{\phi_k\}$ in $\mathcal{H}$ is complete if any of the following relationships hold:

$$u = \sum_{k=1}^{\infty}(u, \phi_k)\phi_k \qquad \text{for each in } \mathcal{H} \tag{11.10}$$

$$||u||^2 = \sum_{k=1}^{\infty} |(u, \phi_k)|^2 \qquad \text{for each continuous } u \text{ in } \mathcal{H} \tag{11.11}$$

$$\text{If } (u, \phi_n) = 0 \text{ for each index } n, \text{ then } u = 0 \tag{11.12}$$

The second condition [Equation (11.11)] is referred to as *Parseval's equality*.

The preceding theorems generalize the concept of a Fourier series expansion of a function and provide a framework for modal expansions.

11.4 LINEAR OPERATORS

The idea of a linear operator was introduced in Chapter 10 as related to vibration problems. Here, the definition of a linear operator is formalized, and some properties of a linear operator are developed. The eigenfunctions and eigenvalues of operators are also discussed in detail, and the concept of adjoint is introduced.

A *subspace* of a linear space is a subset of elements of that space that again has the structure of a linear space. A linear operator is briefly defined in Section 10.2 as a mapping from one set of functions to another. Then the subspace $D(L)$ of the space $\mathcal{H}$ denotes the *domain* of the operator L and is the space of elements in $\mathcal{H}$ that the operator L is defined to act upon. A rule L is defined to be an operator if, for each $u \in D(L)$, there is a uniquely determined element Lu that lies in $\mathcal{H}$. An operator L is linear if, for every complex scalar α and β as well as for u and v in $D(L)$, the following is true:

$$L(\alpha u + \beta v) = \alpha Lu + \beta Lv \tag{11.13}$$

The operator L defines two other spaces of interest. The first is the *range space*, or *range*, of the operator L, denoted by $R(L)$; it is defined as the set of all functions $\{Lu\}$, where $u \in D(L)$. The domain and range of an operator are exactly analogous to the domain and range of a function. Another very important space associated with an operator is the *null space*. The null space of an operator L is the set of all functions $u \in D(L)$ such $Lu = 0$. This space is denoted by $N(L)$ and corresponds to rigid body modes in a structure. The spaces $R(L)$ and $N(L)$ are in fact subspaces of $\mathcal{H}$.

Obvious examples of these various spaces associated with an operator are the transformation matrices of Chapter 3. In fact, linear operator theory is an attempt to generalize the theory of matrices and linear algebra.

An important difference between matrices and operators can be pointed out by defining *equality* of two operators. Two operators L_1 and L_2 are equal if and only if $D(L_1) = D(L_2)$ and $L_1 u = L_2 u$ for all $u(x) \in D(L_1)$.

Example 11.4.1

Consider the linear operator associated with the string equation with fixed ends. Define this operator's domain and null space. The operator is

$$L = -\frac{\partial^2}{\partial x^2}$$

The domain, $D(L)$, consists of all functions in $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ having two derivatives and satisfying the boundary conditions $u(0) = u(\ell) = 0$. The null space of L is the set of all functions u in $D(L)$ such that $u'' = 0$. Integrating $u'' = 0$ requires that $u = ax + b$, where a and b are constants

of integration. However, elements in $D(L)$ must also satisfy $a(0) + b = 0$ and $a\ell + b = 0$, so that $a = b = 0$. Thus, the null space of this operator consists of only the zero function.

An operator L is called *one-to-one* if $Lu_1 = Lu_2$ holds if and only if $u_1 = u_2$. A linear operator L has an *inverse*, denoted by L^{-1} and defined by $L^{-1}u = v$, if and only if $u = Lv$. Operator L is said to be nonsingular in this case and singular if L^{-1} does not exist. Note that for nonsingular operators

$$D(L^{-1}) = R(L)$$
$$R(L^{-1}) = D(L)$$
$$LL^{-1}u = u \qquad \text{for all } u \text{ in } D(L^{-1}) = R(L)$$

and

$$L^{-1}Lu = u \qquad \text{for all } u \text{ in } D(L) = R(L^{-1})$$

Also, L^{-1} is one-to-one. The inverse operator, as shown in Section 10.6, of a differential operator often turns out to be an integral operator defined by a Green's function.

The concept of null space and inverse are also closely related to the eigenvalues of an operator. In particular, suppose zero is an eigenvalue of L_1. Then $L_1 u = 0$ for $u \neq 0$ (since eigenfunctions are never zero). Thus, there exists a nonzero function u in the null space of the operator L. Actually, a stronger statement holds true. If L is a one-to-one operator, L has an inverse, and $N(L)$ contains only zero if and only if zero is not an eigenvalue of the operator L.

An operator is bounded if there exists a finite constant $c > 0$ such that

$$\|Lu\| < c\,\|u\|$$

for each function $u(x)$ in $D(L)$. A related operator property is continuity. An operator L is *continuous* at $u \in D(L)$ if, whenever $\{u_n\}$ is a sequence of functions in $D(L)$ with limit u, $Lu_n \to Lu$. This definition of continuity is often abbreviated as $u_n \to u \Rightarrow Lu_n \to Lu$. For linear operators, this definition is equivalent to requiring L to be continuous at every element in $D(L)$. In addition, a linear operator is continuous if and only if it is a bounded operator.

Example 11.4.2

Differential operators are not bounded. Consider $L = d/dx$ with $D(L)$ consisting of all functions in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$ such that Lu is in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$. This operator is not bounded since the element $u(x) = \sin(n\pi x) \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$. Then $||u|| = \text{constant} = a$, independent of n, and

$$||Lu|| = n\left(\int \cos^2 n\pi x\,dx\right)^{1/2} = nb$$

where b is a constant. Hence, there is a function $u \in D(L)$ such that $||Lu|| > ||u||$, and the operator cannot satisfy the definition of a bounded operator because n may be arbitrarily large.

Example 11.4.3

Integral operators are bounded. Let $u \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$ and define the operator L by

$$Lu(x) = \int_0^x u(s)\, ds = f(x)$$

Next, consider $f(x)$ as defined earlier and square its modulus to obtain

$$|f(x)|^2 = \left| \int_0^x u(s)\, ds \right|^2$$

However, the Cauchy–Schwarz inequality yields

$$\left| \int_0^x u(s)\, ds \right|^2 < \int_0^1 |1|^2 ds \int_0^1 |u(s)|^2\, ds = \|u\|$$

Hence, $||Lu|| < ||u||$, and this operator is bounded.

The operators used to describe the linear vibrations of common distributed-parameter structures are unbounded. However, the differential operators for strings, beams, membranes, and so on, all have inverses defined by Green's functions. Green's functions define integral operators, which are bounded. This connection of vibration equations to a bounded operator is significant in verifying convergence and eigenfunction expansions.

Another important operator is called the *adjoint operator*, which is defined in the following paragraphs and is basically the generalization of the transpose of a real matrix. First consider a linear operator T, which maps elements in $\mathcal{H}$ into a set of complex numbers. Such an operator is called a *linear functional*. In particular, consider the linear functional defined by an inner product. Let the linear functional T_v be defined as

$$T_v u = (u, v) = \int_\Omega u v^*\, d\Omega \tag{11.14}$$

where $u \in \mathcal{H}$, and v is a fixed element in $\mathcal{H}$. The action of T_v defines a bounded operator.

The following result, called the *Riesz representation theorem*, allows the adjoint operator for a bounded operator to be defined. Let T be a bounded linear functional defined on $\mathcal{H}$. Then there exists a unique element, f, in $\mathcal{H}$ such that

$$Tu = (u, f) \tag{11.15}$$

The significance of the Riesz representation theorem is as follows. Suppose that the operator L is bounded, and u and v are in $\mathcal{H}$. Then

$$Tu = (Lu, v) \tag{11.16}$$

defines a bounded linear functional. However, via the Riesz representation theorem, there exists a unique element f in $\mathcal{H}$ such that

$$Tu = (u, f) \tag{11.17}$$

Hence, Equation (11.16) yields

$$(Lu, v) = (u, f) \tag{11.18}$$

where $f \in \mathcal{H}$ is unique. The unique element f is denoted as $f = L^* v$, where L^* is defined as the *adjoint* of the operator L. Then, Equation (11.18) becomes

$$(Lu, v) = (u, L^* v) \tag{11.19}$$

The adjoint defined this way is linear and unique. The adjoint operator L^* is also bounded if L is bounded.

Unfortunately, the Riesz representation theorem and the preceding discussion of adjoints only hold for bounded operators. The equations of interest in vibrations, however, yield unbounded operators. Since the inverses of these unbounded operators are in fact bounded, the idea of an adjoint operator for unbounded operators is still used. However, to denote that a formal proof of existence of the adjoint operator for unbounded operators does not exist, the adjoint of an unbounded operator is often referred to as a *formal adjoint*.

Let L be an unbounded differential operator with domain $D(L)$. The operator L^* defined on a domain $D(L^*)$ is the formal adjoint of L if

$$(Lu, v) = (u, L^* v), \qquad u \in D(L), \qquad v \in D(L^*) \tag{11.20}$$

Note that $D(L)$ is characterized by certain boundary conditions. The domain $D(L^*)$ will also be characterized by possibly different boundary conditions called *adjoint boundary conditions*.

An operator (possibly unbounded) is defined to be *formally self-adjoint* if

$$D(L) = D(L^*) \qquad \text{and} \qquad Lu = L^* u \qquad \text{for } u \in D(L) \tag{11.21}$$

The prefix *formal* is not always used, however. It is important to note that, if L is formally self-adjoint, then, for all $u, v \in D(L)$,

$$(Lu, v) = (u, Lv) \tag{11.22}$$

which follows from Equation (11.21). This last relationship is used to define a related class of operators. An operator L is *symmetric* if Equation (11.22) holds for all pairs u and v in $D(L)$.

Note that the difference between the definitions of symmetric and formally self-adjoint is the domain requirement of Equation (11.21). That is, if the form of the operator and its adjoint are the same, then the operator is symmetric. If, in addition, both the operator and the adjoint operator have the same boundary conditions, then the operator is self-adjoint. If L is a bounded operator, then $D(L) = \mathcal{H}$, and formally self-adjoint and symmetric mean the same. If L is unbounded, however, an operator may be symmetric but not self-adjoint [i.e., $D(L) \neq D(L^*)$]. A formally self-adjoint operator, however, is a symmetric operator.

Example 11.4.4

Consider the operator for a fixed–fixed string and determine its adjoint. Here, L is of the form

$$L = -\alpha \frac{\partial^2}{\partial x^2}$$

and its domain is defined as

$$D(L) = \left\{ u | u(0) = u(\ell) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \right\}$$

The right-hand side of this expression denotes the set of all functions $u(x)$ such that the boundary conditions are satisfied (i.e., $u(0) = u(\ell) = 0$) and the functions u, u', and u'' belong in the set $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$. Calculating the linear product (Lu, v) for $u \in D(L)$ and $v \in D(L^*)$ yields

$$(Lu, v) = -\alpha \int_0^\ell u'' v \, dx$$

Integrating by parts twice yields

$$(Lu, v) = [-\alpha u'(\ell) v(\ell) + \alpha u'(0) v(0) + \alpha u(\ell) v'(\ell) - \alpha u(0) v'(0)] - \alpha \int_0^\ell u v'' \, dx$$

Since $u \in D(L)$, the above reduces to

$$(Lu, v) = [-\alpha u'(\ell) v(\ell) + \alpha u'(0) v(0)] + (u, Lv) \tag{11.23}$$

Now, if both u and v are in $D(L)$, then $v(\ell) = v(0) = 0$ and Equation (11.23) shows that L is a symmetric operator. To see that L is in fact formally self-adjoint, however, consider Equation (11.23) with $v \in D(L^*)$. Since $v \in D(L^*)$, the integration by parts requires v, v', and v'' to be in $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$. In order for $(Lu, v) = (u, Lv)$ to be satisfied, the term in brackets in Equation (11.23) must be zero, i.e.,

$$-\alpha u'(\ell) v(\ell) + \alpha u'(0) v(0) = 0$$

However, since $u'(\ell)$ and $u'(0)$ are arbitrary [i.e., there is no restriction on these values in $D(L)$], $v(\ell)$ and $v(0)$ must both be zero. Thus

$$D(L^*) = \left\{ v | v(0) = v(\ell) = 0, v, v', v'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \right\} = D(L)$$

and this operator is in fact formally self-adjoint. The boundary conditions $v(0) = v(\ell) = 0$ are the adjoint boundary conditions.

In order for a differential operator to be symmetric, it must be of even order (problem 11.20). Most of the physical structures examined so far are self-adjoint with respect to most boundary conditions. The following example illustrates a symmetric, nonself-adjoint operator.

Example 11.4.5

Consider the operator $L = \partial^2/\partial x^2$ on the domain

$$D(L) = \{u | u(0) = u'(0) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell)\}$$

Equation (11.23) still holds with $a = -1$. However, the term in brackets is now

$$[u'(\ell)v(\ell) - u(\ell)v'(\ell)] = 0$$

Since $u'(\ell)$ and $u(\ell)$ are arbitrary, this last expression requires that $v(\ell) = v'(\ell) = 0$. Hence, $D(L^*) = \left\{v | v, v', v'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell) \text{ and } v(\ell) = v'(\ell) = 0\right\} \neq D(L)$ and the operator is not self-adjoint. However, this operator is symmetric.

Symmetric operators (and thus self-adjoint operators) have special properties, as do symmetric matrices, which are useful in the study of vibrations. If L is symmetric, the following are true:

1. (Lu, u) is real.
2. If L has eigenvalues, they are all real.
3. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Furthermore, if L is self-adjoint, it is called *positive definite* if $(Lu, u) > 0$ for all nonzero $u \in D(L)$ and *positive semidefinite* if $(Lu, u) \geq 0$ for all nonzero $u \in D(L)$.

Example 11.4.6

Consider the operator of example 11.4.4 and show that it is positive definite. Start by calculating (Lu, u) for an arbitrary element $u \in D(L)$. Integration by parts yields

$$\begin{aligned}(Lu, u) &= -(u', u)\Big|_0^\ell + \int_0^\ell (u')^2\,dx \\ &= \int_0^\ell (u')^2\,dx \geq 0\end{aligned}$$

where the boundary conditions eliminate the constant terms. This calculation shows that the operator L is positive semidefinite. To see that the operator is, in fact, strictly positive definite, note that, if $u' = 0$, then $u = c$, a constant. However, $u(0) = u(\ell) = 0$ must be satisfied, so that u must be zero, contradicting the semidefinite condition. Thus, $(Lu, u) > 0$ and L is in fact positive definite.

In the case of matrices, a symmetric matrix is positive definite if and only if its eigenvalues are all positive real numbers. For an operator, a weaker version of this statement holds.

A positive definite operator has positive eigenvalues (assuming it has eigenvalues). With further assumptions, which are clarified in the next section, a stronger statement can be made that is more in line with the matrix result.

11.5 COMPACT OPERATORS

In this section, the last major mathematical requirement for eigenfunction expansions is considered. Compact operators are defined and some of their principal properties examined. Self-adjoint compact operators have essentially the same eigenstructure as symmetric matrices and hence are ideal for modal analysis.

The notion of a compact operator is based on restricting a bounded operator. Let $\{u_n\}$ be a uniformly bounded, infinite sequence in $\mathcal{H}$. A *uniformly bounded sequence* is one for which $\|u_n\| < M$ for all values of n (i.e., M does not depend on n). Let L be a bounded linear operator defined on $\mathcal{H}$. The operator L is defined to be *compact* if, from the sequence $\{Lu_n\}$, one can extract a subsequence, denoted by $\{(Lu_n)_k\}$, that is a Cauchy sequence. Another way to describe a compact operator is to note that a compact operator maps bounded sets into compact sets. A compact set is a set such that each sequence of elements in the set contains a convergent subsequence. In this way, the notion of a compact operator is related to a compact set.

The idea of a compact operator is stronger than that of a bounded operator. In fact, if an operator is compact, it is also bounded. However, bounded operators are not necessarily compact. Since bounded operators are continuous operators and compactness requires more of the operator, compact operators are also called *completely continuous* operators.

The identity operator is an example of a bounded operator, i.e., $\|Iu\| < a\|u\|$, which is not necessarily compact. Every bounded operator defined on a finite-dimensional Hilbert space is compact, however. Bounded operators are compact because every set containing a finite number of elements is compact.

Consider next the integral operator defined by Green's function. Such operators are compact. In fact, if $g(x, y)$ is any continuous function where x and y are both in the interval $(0, \ell)$, then the integral operator defined by

$$Lu = \int_0^\ell g(x, y)u(y)\, dy \tag{11.24}$$

is a compact operator on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$.

Self-adjoint compact operators have the desired expansion property – namely, if L is compact and self-adjoint on $\mathcal{H}$, then L can be shown to have nonzero eigenvalues $\{\mu_n\}$ and orthonormal eigenfunctions $\{\Theta_n\}$. Next, let $u(x)$ be any element in $\mathcal{H}$; $u(x)$ can then be represented as the generalized Fourier series

$$u(x) = \sum_{n=1}^{\infty} (u, \Theta_n)\Theta_n + u_0(x) \tag{11.25}$$

where $u_0(x)$ lies in the null space of L. Furthermore, the function Lu can also be represented as

$$Lu(x) = \sum_{n=1}^{\infty} \mu_n (u, \Theta_n)\Theta_n \tag{11.26}$$

Note that, if L is nonsingular, $u_0(x) = 0$. Also note that, from comparing Equation (11.10) with Equation (11.25), the eigenfunctions of a compact self-adjoint operator are complete in $\mathcal{H}$. These two powerful results form the backbone of modal analysis of distributed-parameter systems.

Expression (11.26) also allows a more concrete statement about the relationship between the eigenvalues of an operator and the definiteness of an operator. In fact, a compact self-adjoint operator L is positive definite if and only if each of its eigenvalues are positive real numbers.

11.6 THEORETICAL MODAL ANALYSIS

In this section, the idea of a compact operator, along with the associated expansion, is applied to a generic model of a differential equation describing the linear vibration of a distributed-parameter system. This theory results in the modal analysis of such structures and provides a firm mathematical foundation for the material presented in Sections 10.3 and 10.4.

Consider again Equations (10.1), repeated here for convenience:

$$\begin{aligned} w_{tt}(x,t) + L_2 w(x,t) &= 0, & x &\in \Omega, & t &> 0 \\ Bw &= 0, & x &\in \partial\Omega, & t &> 0 \\ w(x,0) &= w_0(x), & w_t(x,0) &= \dot{w}_0(x) & t &= 0 \end{aligned} \tag{11.27}$$

With the additional assumptions that the (nonsingular) operator L_2 is self-adjoint and has a compact inverse, the following shows that the sum of Equation (10.3), i.e., the modal expansion of the solution, converges.

Since L_2^{-1} is compact, the eigenfunctions of L_2^{-1} are complete. As noted before, these eigenfunctions are also those of L_2, so that the eigenfunctions of L_2 are complete. As a result, the solution $w(x,t)$, considered as a function of x defined on the Hilbert space $\mathcal{H}$ for a fixed value of t, can be written as

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{11.28}$$

where it is anticipated that the Fourier coefficient a_n will depend on the fixed parameter t, $t > 0$. From Equation (11.10) the coefficient $a_n(t)$ is

$$a_n(t) = \int_\Omega w(x,t)\Theta_n \, d\Omega = (w, \Theta_n). \tag{11.29}$$

Multiplying Equations (11.27) by $\Theta_n(x)$ and integrating over Ω yields

$$\int_\Omega w_{tt}\Theta_n(x)\, d\Omega + \int_\Omega L_2 w \Theta_n(x)\, d\Omega = 0 \tag{11.30}$$

Equation (11.30) can be rewritten as

$$\frac{\partial^2}{\partial t^2}[(w, \Theta_n)] + (L_2 w, \Theta_n) = 0 \tag{11.31}$$

Using the self-adjoint property of L_2 and the fact that Θ_n is an eigenfunction of L_2 with corresponding eigenvalue λ_n yields

$$\frac{\partial^2}{\partial t^2}[(w, \Theta_n)] + \lambda_n(w, \Theta_n) = 0$$

or, from Equation (11.29),

$$\ddot{a}_n(t) + \lambda_n a_n(t) = 0, \qquad t > 0 \tag{11.32}$$

This expression, along with the appropriate initial conditions, can be used to calculate $a_n(t)$. In particular, if L_2 is positive definite, $\lambda_n > 0$ for all n, then

$$a_n(t) = \frac{\dot{a}_n(0)}{\omega_n} \sin \omega_n t + a_n(0) \cos \omega_n t \tag{11.33}$$

where

$$a_n(0) = \int_\Omega w(x, 0)\Theta_n(x)\, d\Omega$$

$$\dot{a}_n(0) = \int_\Omega w_t(x, 0)\Theta_n(x)\, d\Omega$$

$$\omega_n = \sqrt{\lambda_n}$$

These formulae are, of course, consistent with those developed formally in Section 10.3. Here, however, the convergence of Equation (11.28) is guaranteed by the compactness theorem of Section 11.5.

As indicated in Section 10.4, this procedure can be repeated for damped systems if the operators L_1 and L_2 commute on a common domain. This result was first pointed out formally by Caughey and O'Kelly (1965), but the development did not concern itself with convergence. The approach taken by Caughey and O'Kelley, as well as in Section 10.4, is to substitute the series of Equation (11.28) into Equation (10.14). Unfortunately, this substitution raises the issue of convergence of the derivative of the series. The convergence problem is circumvented by taking the approach described in Equations (11.30) through (11.32).

Repeating this procedure for the damped system, as described by Equation (10.14), requires that L_1 and L_2 have the same eigenfunctions and that L_2 is self-adjoint with compact inverse. Multiplying Equation (10.14) by $\Theta_n(x)$ and integrating yields

$$\frac{d^2}{dt^2}[(w, \Theta_n)] + \frac{d}{dt}[(L_1 w, \Theta_n)] + (L_2 w, \Theta_n) = 0 \tag{11.34}$$

Using the property that L_1 and L_2 are self-adjoint, and denoting the eigenvalues of L_1 by $\lambda_n^{(1)}$ and those of L_2 by $\lambda_n^{(2)}$, Equation (11.34) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)} a_n(t) = 0 \tag{11.35}$$

Equations (11.32) and (11.35) constitute a theoretical modal analysis of a distributed-mass system. Equation (11.34) is solved using the methods of Section 1.3 from initial conditions determined by using mode orthogonality.

11.7 EIGENVALUE ESTIMATES

As indicated previously, knowledge of the eigenvalues (natural frequencies) of a system provides knowledge of the dynamic response of a structure. Unfortunately, the eigenvalue problem for the operators associated with many structures cannot be solved. Having no solution requires the establishment of estimates and bounds on the eigenvalues of operators. Note that, while this section is an extension of Section 3.7 on eigenvalue estimates for matrices, the operator problem is more critical because in the matrix case estimates are primarily used as an analytical tool, whereas in the operator case the estimates are used where solutions do not even exist in closed form.

Consider first the conservative vibration problems of Equation (10.1) for the case where the operator L is self-adjoint and positive definite with compact inverse. As noted in Section 11.6, these assumptions guarantee the existence of a countable infinite set of positive real eigenvalues $\{\lambda_i\}$, which can be ordered as

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \tag{11.36}$$

with corresponding orthonormal eigenfunctions, $\{\Theta_n(x)\}$, complete in $\mathcal{L}_2^{\mathrm{R}}(\Omega)$.

Next, further assume that the operator L is *coercive*, i.e., that there exists a constant c such that

$$c\|u\|^2 < (Lu, u) \tag{11.37}$$

for all $u \in D(L)$. Thus

$$\frac{(Lu, u)}{\|u\|^2} > c \tag{11.38}$$

for all $u \in D(L)$. Thus, the quantity $(Lu, u)/\|u\|^2$ is bounded below, and therefore there is a *greatest lower bound*, denoted by glb, of this ratio. Define the functional $R(u)$ by

$$R(u) = \frac{(Lu, u)}{\|u\|^2} \tag{11.39}$$

The functional $R(u)$ is called the *Rayleigh quotient* of the operator L.

Since the eigenfunctions of L are complete, Equation (11.11) yields

$$\|u\|^2 = \sum_{n=1}^{\infty} |(u, \Theta_n)|^2$$

and

$$(Lu, u) = \sum_{n=1}^{\infty} \lambda_n |(u, \Theta_n)|^2 \tag{11.40}$$

Hence, the Rayleigh quotient can be written as

$$R(u) = \frac{\sum \lambda_n |(u, \Theta_n)|^2}{\sum |(u, \Theta_n)|^2} > \lambda_1 \tag{11.41}$$

since Equation (11.7) implies that

$$\sum \lambda_n |(u, \Theta_n)|^2 > \lambda_1 \sum |(u, \Theta_n)|^2 \tag{11.42}$$

Here, the summation limits have been suppressed. Also, note that $R(\Theta_1) = \lambda_1$, so that

$$\lambda_1 = \min(\lambda_i) = \underset{\substack{u \neq 0 \\ u \in D(L)}}{\text{glb}} \; [R(u)] \tag{11.43}$$

As in the matrix case, the Rayleigh quotient yields a method of finding an upper bound to the eigenvalues of a system.

Example 11.7.1

Consider the operator $L = -\partial^2/\partial x^2$ defined on the domain $D(L) = \{u | u(0) = u(1) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, 1)\}$. Estimate the first natural frequency by using the Rayleigh quotient.

Note that the function $u(x) = x(1 - x)$ is in $D(L)$. Calculation of the Rayleigh quotient then yields

$$R(u) = \frac{-(u'', u)}{(u, u)} = \frac{1/3}{1/30} = 10$$

A calculation of the exact value of λ_1 for this operator yields

$$\lambda_1 = (\pi^2) < 10$$

so that the Rayleigh quotient in this case provides an upper bound to the lowest eigenvalue.

Bounds are also available for the other eigenvalues of an operator. In fact, with the previously mentioned assumptions on the operator L, the domain of L can be split into two subspaces M_k and $M_k^{\perp}$ (read 'M_k perp') defined by the eigenfunctions of L. Let $M_k = \{\Theta_1, \Theta_2, \ldots, \Theta_k\}$ be the set of the first k eigenfunctions of L, and $M_k^{\perp}$ be the set of remaining eigenfunctions, i.e., $M_k^{\perp} = \{\Theta_{k+1}, \Theta_{k+2}, \ldots\}$. From these considerations (see, for instance, Stakgold, 1967, 2000a) the following holds:

$$\lambda_k = \min_{u \in M_{k-1}^{\perp}} [R(u)] = \max_{u \in M_k} [R(u)] \tag{11.44}$$

This formulation of the eigenvalues of an operator as a minimum or a maximum over sets of eigenfunctions can be further extended to extremals over arbitrary subspaces. Equation (11.44) is called the *Courant minimax principle*. Again, with L satisfying the assumptions of this section, let E_k be any k-dimensional subspace of $D(L)$. Then

$$\lambda_k = \min_{E_k \in D(L)} \left[\max_{\substack{u \in E_k \\ u \neq 0}} (R(u)) \right] = \max_{E_k \in D(L)} \left[\max_{\substack{u \in E_{k-1}^{\perp} \\ u \neq 0}} (R(u)) \right] \tag{11.45}$$

where the value of $u(x)$ satisfying Equation (11.45) becomes Θ_k. The minimum over $E_k \in D(L)$ refers to the minimum over all the subspaces E_k of dimension k contained in

$D(L)$. The maximum value of $R(u)$, $u \in E_k$, refers to the maximum value of $R(u)$ for each element u in the set E_k. The difference between Equations (11.44) and (11.45) is that Equation (11.44) is restricted to subspaces generated by the eigenfunctions of L, whereas in Equation (11.45) the subspaces are any subspace of $D(L)$.

11.8 ENCLOSURE THEOREMS

The Rayleigh quotient and formulations of the eigenvalue problem of the previous section provide a means of estimating an upper bound of the eigenvalues of an operator by examining sets of arbitrary functions in the domain of the operator. In this section, lower bounds and enclosure bounds are examined. Furthermore, bounds in terms of related operators are examined. In this way, eigenvalues of operators that are difficult or impossible to calculate are estimated in terms of operators with known eigenvalues.

The first two results follow from the definition of the definiteness of an operator. This definition can be used to build a partial ordering of linear operators and to provide an eigenvalue estimate. For two self-adjoint operators L_1 and L_2, the operator inequality denoted by

$$L_1 \leq L_2 \tag{11.46}$$

is defined to mean that

$$\mathcal{H} \supset D(L_1) \supset D(L_2) \tag{11.47}$$

and

$$(L_1 u, u) \leq (L_2 u, u) \qquad \text{for all } u \in D(L_2) \tag{11.48}$$

where Equation (11.47) denotes that $D(L_2)$ is a subset of $D(L_1)$, and so on.

If Equation (11.48) holds with strict equality, i.e., if L_1 and L_2 have the same form, with L_1 defined on a subspace of L_2, and if L_1 and L_2 are positive definite with compact inverses, then

$$\lambda_i^{(1)} \leq \lambda_i^{(2)}, \qquad i = 1, 2, \ldots \tag{11.49}$$

Here, $\lambda_i^{(1)}$ denotes the eigenvalues of L_1, and $\lambda_i^{(2)}$ denotes those of the operator L_2. This inequality is called the first *monotonicity principle*.

Example 11.8.1

Consider the two operators defined by $L_1 = L_2 = -\partial^2/\partial x^2$ with domains $D(L_1) = \{u \mid u(0) = u(\ell_1) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell_1)\}$ and $D(L_2) = \{u \mid u(0) = u(\ell_2) = 0, u, u', u'' \in \mathcal{L}_2^{\mathrm{R}}(0, \ell_2)\}$. Consider the case with $\ell_1 \leq \ell_2$. Then redefine the domain $D(L_1)$ to be the completion of the set of functions that vanish outside $D(L_1)$, i.e., for $x > \ell_1$. Then, $D(L_2) \supset D(L_1)$ and expression (11.47) and inequality (11.49) are satisfied so that

$$L_2 < L_1$$

Thus, inequality (11.49) yields (note the indices are interchanged)

$$\lambda_i^{(2)} \le \lambda_i^{(1)}$$

To see that this is true, note that $\lambda_i^{(2)} = \sqrt{i\pi/\ell_2} \le \sqrt{i\pi/\ell_1} = \lambda_i^{(1)}$. This example shows that shrinking the domain of definition of the problem increases the eigenvalues, as expected.

The trick in using the first monotonicity principle is the ability to extend the domain $D(L_2)$ so that it can be considered as a subspace of $D(L_1)$. Extending the domain works in the example because the boundary condition is $u = 0$ along the boundary. The method fails, for instance, for the membrane equation with clamped boundaries and a hole removed (Weinberger, 1974).

The preceding example was chosen to illustrate that the principle works. The use of this principle is more interesting, however, in a situation where one of the boundaries is such that the eigenvalues cannot be calculated, i.e., as in the case of an odd-shaped membrane. In this case, the unknown eigenvalues, and hence natural frequencies, can be bracketed by two applications of the first monotonicity theorem.

For instance, if the eigenvalues of a membrane of irregular shape are required, the eigenvalues of inscribed and circumscribed rectangles can be used to provide both upper and lower bounds of the desired eigenvalues. Thus, the monotonicity principle can also be thought of as an *enclosure theorem* (see problem 11.17).

If the operators L_1 and L_2 are positive definite and of different form, i.e., if the equality in expression (11.48) does not hold, then inequality (11.49) is known as the *second monotonicity theorem*. The following example illustrates how the monotonicity results can be used to create enclosures for the eigenvalues of certain operators.

Example 11.8.2

Consider the membrane operator $L = -\nabla^2$, as defined by Equation (9.53). In particular, consider the three operators

$$L_1 = -\nabla^2, \qquad D(L_1) = \left\{ u \,\middle|\, \frac{\partial u}{\partial n} + k_1 u = 0 \qquad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

$$L_2 = -\nabla^2, \qquad D(L_2) = \left\{ u \,\middle|\, \frac{\partial u}{\partial n} + k_2 u = 0 \qquad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

$$L_3 = -\nabla^2, \qquad D(L_3) = \left\{ u | u = 0 \qquad \text{on } \partial\Omega,\ u, u_x, u_y, u_{xy}, u_{xx}, u_{yy} \in \mathcal{L}_2^{\mathrm{R}}(\Omega) \right\}$$

where $k_1 > k_2 > 0$. Compare the eigenvalues of these three operators.

With $k_1 > k_2$, integration yields

$$(u, L_2 u) < (u, L_1 u)$$

For all $u \in D(L_2)$ and using the second monotonicity principle we have

$$\lambda_i^{(2)} \le \lambda_i^{(1)}$$

Comparing operators L_3 and L_1, note that $D(L_3) \supset D(L_1)$, so that application of the first monotonicity principle yields

$$\lambda_i^{(2)} \leq \lambda_i^{(1)} \leq \lambda_i^{(3)}$$

11.9 OSCILLATION THEORY

In this section, the damped system of Equation (10.14) with the assumption stated in Section 11.6 is considered. Under these assumptions, the solution of Equation (10.14) is expressible as the convergent series

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{11.50}$$

where the coefficients $a_n(t)$ satisfy Equations (11.35), i.e.,

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = 0, \qquad n = 1,\ 2,\ 3, \ldots \tag{11.51}$$

The topic of interest in this section is the nature of the temporal solution $a_n(t)$ subjected to arbitrary initial conditions. This section is an extension of the oscillation results presented in Section 3.6 for lumped-mass systems to those distributed-mass structures described by Equation (10.14).

Further, assume that the operators L_1 and L_2 are both positive definite. Then the sets of eigenvalues $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ are all positive real numbers. Hence, the solutions of Equations (11.51), for arbitrary initial conditions, take one of three forms depending on the sign of the discriminant

$$d_n = \left[\lambda_n^{(1)}\right]^2 - 4\lambda_n^{(2)} \tag{11.52}$$

of Equation (11.51). These forms are described next.

If, for each value of the index n, $d_n > 0$, the solution of Equation (10.14) does not oscillate. In this situation, the structure is said to be *overdamped* and the temporal solutions $a_n(t)$ are all of the form

$$a_n(t) = C_n\, e^{r_{1n}t} + D_n e^{r_{2n}t} \tag{11.53}$$

where C_n and D_n are constants of integration determined by the initial conditions and r_{1n} and r_{2n} are the positive real roots of the characteristic equation associated with Equation (11.51). In this case, the solution $w(x, t)$ of Equation (10.14) does not oscillate in time.

Also, if the eigenvalues of the operators L_1 and L_2 are such that $d_n = 0$ for each index n, the solution of Equation (11.51) does not oscillate. In this case, the structure is said to be *critically damped*, and the solutions, $a_n(t)$, are all of the form

$$a(t) = (C_n t + D_n)\, e^{-(r_n/2)t} \tag{11.54}$$

where, as before, C_n and D_n are constants of integration determined by the initial conditions and r_n is the positive real repeated root of the characteristic equation associated with Equation (11.51). As in the overdamped case, the solution $w(x, t)$ of Equation (10.14) does not oscillate in time.

If, for each value of the index n, the eigenvalues of L_1 and L_2 are such that $d_n < 0$, then the solution of Equation (11.15) oscillates with decaying amplitude. In this case, the structure is said to be *underdamped*, and the solutions $a_n(t)$ are all of the form

$$a_n(t) = e^{-(\lambda_n/2)t}(C_n \sin \omega_n t + D_n \cos \omega_n t) \tag{11.55}$$

where C_n and D_n are constants of integration determined by the initial conditions. Here, λ_n and ω_n are positive real numbers determined from the roots of the characteristic equation, which appear in complex conjugate pairs. In this case, the solution $w(x, t)$ of Equation (10.14) oscillates in time with decaying amplitude.

An additional possibility is that, for a given structure, d_n takes on different signs. That is, there exists an n for which $d_n < 0$ and at least one value of the index n such that $d_n > 0$. In this situation, the structure is said to be *mixed damped*. The solution $w(x, t)$ will contain one oscillatory term and at least one nonoscillatory term.

The preceding four conditions can be checked and the oscillatory nature of the solution determined without calculating the eigenvalues of the operators L_1 and L_2. The definiteness of the operator $(L_1^2 - 4L_2)$ can be determined by simple integration and, as illustrated by Inman and Andry (1982), the definiteness determines the oscillatory nature of the solution $w(x, t)$. In particular, if L_1 and L_2 commute on $D(L_1) = D(L_2)$, then:

1. The operator $L_1^2 - 4L_2$ is positive definite on $D(L_2)$ if and only if the structure is overdamped.
2. The operator $L_1^2 = 4L_2$ on $D(L_2)$ if and only if the structure is critically damped.
3. The operator $4L_2 - L_1^2$ is positive definite on $D(L_2)$ if and only if the structure is underdamped.
4. The operator $L_1^2 - 4L_2$ is indefinite on $D(L_2)$ if and only if the structure is mixed damped.

These conditions specify the oscillatory nature of the solution $w(x, t)$ of Equation (10.14). In addition to providing a criterion for oscillation, they also lead to simple inequalities in the parameters of the structure, which can be used in design and control applications. The following example illustrates this point.

Example 11.9.1

Consider the longitudinal vibration of a clamped bar with both internal and external damping. Under the assumption of linearity, the equation of motion is written as

$$u_{tt}(x, t) + 2\left[\gamma - b\frac{\partial^2}{\partial x^2}\right]u_t(x, t) - a\frac{\partial^2}{\partial x^2}u(x, t) = 0, \qquad x \in \Omega = (0, 1)$$

where $u(x, t)$ is the displacement of the bar, and the positive constants γ, b, and a reflect the relevant physical parameters. Thus, the operator forms are

$$L_1 = 2\left[\gamma - b\frac{\partial^2}{\partial x^2}\right] \qquad \text{and} \qquad L_2 = -a\frac{\partial^2}{\partial x^2}$$

The boundary conditions are taken to be

$$u(0, t) = u(1, t) = 0$$

with $D(L_2) = \{u(\cdot, t) \in \mathcal{L}_2^{\mathrm{R}}(0, 1)$ such that all partial derivatives up to order 4 are in $\mathcal{L}_2^{\mathrm{R}}(0, 1)$ and $u(0, t) = u(1, t) = 0\}$. Note that L_1 and L_2 are positive definite and commute on $D(L_2)$. Also, calculation of L_1^2 and $L_1^2 - 4L_2$ yields

$$L_1^2 = 4\left(\gamma^2 - 2\gamma b\frac{\partial^2}{\partial x^2} + b^2\frac{\partial^4}{\partial x_4}\right)$$

and

$$L_1^2 = 4L_2 = 4\left(\gamma^2 - 2\gamma b\frac{\partial^2}{\partial x^2} + b^2\frac{\partial^4}{\partial x_4} + a\frac{\partial^2}{\partial x^2}\right)$$

The eigenvalues of the operator $L_1^2 - 4L_2$ are then

$$4\left(\gamma^2 - 2\gamma\frac{b}{a}\lambda_n + \frac{b^2}{a^2}\lambda_n^2 - \lambda_n\right) = 4\left[\left(\gamma - \frac{b}{a}\lambda_n\right)^2 - \lambda_n\right]$$

where $\lambda_n = an^2\pi^2$. Demanding that the sign of this expression be positive, negative, or zero is equivalent to characterizing the definiteness of the operator $4L_2 - L_1^2$. Note that the only possibilities for this problem are either overdamping for each mode or mixed damping with all higher modes overdamped.

CHAPTER NOTES

The material of this chapter is a much condensed, and somewhat oversimplified, version of the contents of a course in applied functional analysis. Several texts are recommended and were used to develop the material here. The books by Stakgold (1967, 1968) present most of the material here in two volumes. This text, republished in 2000 (Stakgold, 2000a, 2000b), is recommended because it makes a very useful comparison between finite-dimensional systems (matrices) and infinite-dimensional systems. The book by Hocstadt (1973) presents an introduction to Hilbert spaces and compact operators in fairly short order. A good fundamental text on functional analysis, such as Bachman and Narici (1966), or on operator theory, such as Naylor and Sell (1982), presents the information of Sections 11.2 through 11.5 in rigorous detail. The material of Section 11.5 is usually not presented, except formally in vibration texts. MacCluer (1994) presents conditions for the existence of convergent eigenfuction expansions and connects this to operator properties that result in the successful application of the method of separation of variables.

The eigenvalue estimate methods given in Section 11.7 are the most common eigenvalue estimates and can be found in most texts, including Stakgold (1967, 2000a). The literature is full of improvements and variations of these estimates. The short and excellent text by Weinberger (1974) is summarized in Section 11.8 on enclosure theorems. These theorems are quite useful, but apparently have not been taken advantage of by the engineering community. The oscillation results of Section 11.9 paraphrase the paper by Inman and Andry (1982) and present a direct extension of Section 3.6.

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PROBLEMS

11.1 Show that $\mathcal{L}_2^{\mathrm{R}}(\Omega)$ is a linear space.

11.2 Show that the sequence of example 11.2.1 is in fact a Cauchy sequence.

11.3 Show that the sequence of example 11.2.1 converges to $u(x)$ as claimed.

11.4 Show that the operator of example 11.4.1 is linear.

11.5 For the operator of example 11.4.1, show that $N(L), R(L)$, and $D(L)$ are linear spaces and hence subspaces of $\mathcal{H}$.

11.6 Consider the operator $-\partial^2/\partial x^2$, defined on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ such that $u_x(0) = u_x(\ell) = 0$. Calculate the null space of the operator and show that it is a subspace of $\mathcal{H}$. Is this operator equal to the operator of example 11.4.1?

11.7 Show that the linear functional defined by Equation (11.14) is in fact bounded.

11.8 Show that the operator defined by the Green's function of example 10.6.1 is a bounded operator.

11.9 Calculate the adjoint of the operator in problem 11.6. Is the operator formally self-adjoint? Positive definite?

11.10 Show that the operator $-\partial^4/\partial x^4$ defined on $\mathcal{L}_2^{\mathrm{R}}(0, \ell)$ with boundary conditions $u(0) = u(\ell) = u'(0) = u'(\ell) = 0$ is formally self-adjoint.

11.11 Consider the transverse vibrations of a beam with variable stiffness $(EI(x))$ and of dimensions compatible with the Euler–Bernoulli assumptions and with cantilevered boundary conditions. Show that the corresponding operator is symmetric, positive definite, and self-adjoint.

11.12 Calculate the adjoint of the operator $L = \partial/\partial x$ with boundary conditions $u(0) = \alpha u(1)$, α constant. [Do not forget to calculate $D(L^*)$.]

11.13 Suppose that A is a compact linear operator and B is a bounded linear operator such that AB is defined. Show that AB is compact.

11.14 Show that, if the linear self-adjoint operators L_1 and L_2 commute and if L_2 has a compact inverse and L_1 is nonsingular, then L_1 and L_2 have a common set of eigenfunctions.

11.15 Show that the identity operator is not compact.

11.16 Prove that, if L has a compact inverse, it is positive definite if and only if each of its eigenvalues is positive.

11.17 Calculate some estimates of the eigenvalues of the operator for the transverse vibration of a simply supported, nonuniform beam with $EI(x) = (1.1 - x)$. Compare the results of your estimates to the exact values for $EI = 1$.

11.18 Calculate the eigenvalues of a square membrane clamped along its boundary on each of the following:

(a) the square defined by the axis and the line $x = 1$, $y = 1$;
(b) the square defined by the axis and the line $x = 2$, $y = 2$;
(c) the square defined by the axis and the line $x = 3$, $y = 3$.

Compare the results of these three operators using the enclosure result of example 11.8.2 and illustrate that the monotonicity results hold.

11.19 Consider the transverse vibrations of three beams all of dimensions compatible with the Euler–Bernoulli assumptions and all with cantilevered boundary conditions. Suppose two of the beams have constant stiffness denoted by E_1I_1 and E_2I_2 respectively and that the third beam has a variable stiffness denoted by $EI(x)$. Show that, if $E_1I_1 < EI(x) < E_2I_2$, then the eigenvalues of the variable-stiffness beam fall in between those of the constant-stiffness beams.

11.20 Consider the damped plate described by Equation (9.87) with simply supported boundary conditions. Calculate inequalities in the constants ρ, γ, and D_E such that the free response is (a) overdamped, (b) critically damped, and (c) underdamped. What can you conclude from your calculation?

11.21 Consider the problem of example 11.9.1. Can this system be designed to be underdamped if a mass is attached to one end and fixed at the other?

11.22 Show that a differential operator must be of even order for it to be symmetric. (*Hint*: Use integration of parts to show that an odd-order differential operator is not symmetric.)

12

Forced Response and Control

12.1 INTRODUCTION

This chapter considers the response of distributed-parameter structures that are under some external influence. This includes consideration of the response of distributed-mass structures to applied external forces, the response of distributed-mass structures connected to lumped-mass elements, and the response of distributed-mass structures under the influence of both passive and active control devices.

If the equations of motion can be decoupled, then many of the results used for lumped-mass systems described in Chapter 5 can be repeated for the distributed-mass case. However, because of the infinite-dimensional nature of distributed-mass systems, convergence of solutions occasionally preempts the use of these methods. Convergence issues are especially complicated if the structure is subjected to control forces or unknown disturbances.

12.2 RESPONSE BY MODAL ANALYSIS

This section considers the forced response of damped distributed-parameter systems of Equation (10.14) of the form

$$w_{tt}(x,t) + L_1 w_t(x,t) + L_2 w(x,t) = f(x,t), \qquad x \in \Omega \tag{12.1}$$

with appropriate boundary and initial conditions. Here, the operators L_1 and L_2 are self-adjoint, positive definite operators; L_2 has a compact inverse, and L_1 shares the set of eigenfunctions $\{\Theta_n(x)\}$ with L_2 (i.e., L_1 and L_2 commute). For the moment, the only assumption made of $f(x,t)$ is that it lies in $\mathcal{L}_2^{\mathrm{R}}(\Omega)$.

Since $f(x,t) \in \mathcal{L}_2^{\mathrm{R}}(\Omega)$, Equation (12.1) can be multiplied by the function $\Theta_n(x)$ and then integrated over Ω. This integration yields

$$(w_{tt}, \Theta_n) + (L_1 w_t, \Theta_n) + (L_2 w, \Theta_n) = (f, \Theta_n) \tag{12.2}$$

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

The left-hand side of this equation is identical to Equation (10.16). Applying the analysis of Section 11.6, Equation (12.2) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = f_n(t), \qquad n = 1, 2, 3 \tag{12.3}$$

where $f_n(t)$ has the form

$$f_n(t) = \int_\Omega f(x, t)\Theta_n(x)\, d\Omega, \qquad n = 1, 2, 3 \tag{12.4}$$

This scalar equation in the function $a_n(t)$ can be solved and analyzed using the single-degree-of-freedom model of Section 1.4.

Equation (12.3) is essentially the same as Equation (5.38), and the solution is thus given by Equation (5.39). That is, if the system is underdamped ($4L_2 - L_1^2 > 0$), then for zero initial conditions

$$a_n(t) = \frac{1}{\omega_{dn}} \int_0^t e^{-\zeta_n \omega_n \tau} f_n(t - \tau) \sin(\omega_{dn}\tau)\, d\tau \tag{12.5}$$

where for each value of the index n

$$\omega_n = \sqrt{\lambda_n^{(2)}}, \quad \text{the } n\text{th natural frequency} \tag{12.6}$$

$$\zeta_n = \frac{\lambda_n^{(1)}}{2\sqrt{\lambda_n^{(2)}}}, \quad \text{the } n\text{th modal damping ratio} \tag{12.7}$$

$$\omega_{dn} = \omega_n\sqrt{1 - \zeta_n^2}, \quad \text{the } n\text{th damped natural frequency} \tag{12.8}$$

Thus, in the solution where the operators L_1 and L_2 commute, the temporal coefficients in the series solution are determined by using results from single-degree-of-freedom theory discussed in Chapter 1. The solution to Equation (12.1) is the sum

$$w(x, t) = \sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{12.9}$$

where the $a_n(t)$ are determined by Equation (12.5) for the case where the initial conditions are set to zero and the set $\{\Theta_n(x)\}$ consists of the eigenfunctions of the operator L_2. Since the set of functions $\{\Theta_n(x)\}$ consists of the modes of free vibration, the procedure just described is referred to as a modal analysis solution of the forced response problem.

Example 12.2.1

Consider the hinged–hinged beam of example 10.3.1. Assuming the beam is initially at rest ($t = 0$), calculate the response of the system to a harmonic force of $\sin(t)$ applied at $x = \ell/2$, where ℓ is the length of the beam. Assume the damping in the beam is of the form $2\alpha w_t(x, t)$, where α is a constant. First, note that the operator

$$L_2 = \frac{EI}{m}\frac{\partial^4}{\partial x^4}$$

has a compact inverse and is self-adjoint and positive definite with respect to the given boundary conditions. Furthermore, the eigenfunctions of the operator L_2 serve as eigenfunctions for the operator $L_1 = 2\alpha I$. Thus, the eigenvalues of the operator $4L_2 - L_1^2$ are (for the given boundary conditions)

$$4\left[n^4\pi^4\frac{EI}{m\ell^4} - \alpha^2\right]$$

which are greater than zero for every value of the index n if

$$\frac{\pi^2}{\ell^2}\sqrt{\frac{EI}{m}} > \alpha$$

Hence, each coefficient, $a_n(t)$, is underdamped in this case. The solution given by Equation (12.5) then applies.

The forcing function for the system is described by

$$f(x,t) = \delta\left(x - \frac{\ell}{2}\right)\sin t$$

where $\delta(x - \ell/2)$ is the Dirac delta function. Substitution of this last expression into Equation (12.4) along with the normalized eigenfunctions of example 10.3.1 yields

$$\begin{aligned} f_n(t) &= \sqrt{\tfrac{2}{\ell}}\sin t\int_0^\ell \sin\left(\tfrac{n\pi x}{\ell}\right)\delta\left(x - \tfrac{\ell}{2}\right)dx \\ &= \sqrt{\tfrac{2}{\ell}}\sin t\sin\tfrac{n\pi}{2} \end{aligned}$$

In addition, the natural frequency, damping ratio, and damped natural frequency become

$$\begin{aligned} \omega_n &= \left(\frac{n\pi}{\ell}\right)^2\sqrt{\frac{EI}{m}} \\ \zeta_n &= \frac{\alpha}{\omega_n} \\ \omega_{dn} &= \omega_n\sqrt{1 - \frac{\alpha^2}{\omega_n^2}} \end{aligned}$$

With these modal damping properties determined, specific computation of Equation (12.5) can be performed. Note that the even modes are not excited in this case, since $f_{2n} = 0$ for each n. Physically, these are zero because the even modes all have nodes at the point of excitation, $x = \ell/2$.

If the damping in the system is such that the system is overdamped or mixed damped, the solution procedure is the same. The only difference is that the form of $a_n(t)$ given by Equation (12.5) changes. For instance, if there is zero damping in the system, then $\zeta_n \to 0$, $\omega \to \omega_n$, and the solution of Equation (12.5) becomes

$$a_n(t) = \frac{1}{\omega_n}\int_0^t f_n(t-\tau)\sin(\omega_n\tau)\,d\tau \tag{12.10}$$

12.3 MODAL DESIGN CRITERIA

The previous section indicates how to calculate the forced response of a given structure to an external disturbance by modal analysis. This modal approach is essentially equivalent to decoupling a partial differential equation into an infinite set of ordinary differential equations. This section examines some of the traditional design formulae for single-degree-of-freedom oscillators applied to the modal coordinates of a distributed-parameter structure of the form given in Equation (12.9). This modal design approach assumes that the summation of Equation (12.9) is uniformly convergent and the set of eigenfunctions $\{\Theta_n(x)\}$ is complete. Hence, there is a value of the index n, say $n = N$, for which the difference between $w(x, t)$ and the partial sum

$$\sum_{n=1}^{N} a_n(t)\Theta_n(x)$$

is arbitrarily small. Physically, observation of certain distributed-mass systems indicates that some key modes seem to dominate the response, $w(x, t)$, of the system. Both the mathematics and the physics in this case encourage the use of these dominant modes in the design criteria.

As an illustration of modal dominance, consider again the problem of example 12.2.1. With zero initial conditions, the response $a_n(t)$ is of the same form as Equation (1.18) multiplied by 0, 1, or -1, depending on the value of n (i.e., $\sin n\pi/2$). In fact, integration of Equation (12.5) for the case $f_n(t) = f_{n0} \sin \omega t$ yields

$$a_n(t) = X_n \sin(\omega t + \beta_n) \tag{12.11}$$

The coefficient X_n is determined (see Section 1.4) to be

$$X_n = \frac{f_{n0}}{\sqrt{(\lambda_n^{(2)} - \omega^2)^2 + (\lambda_n^{(1)}\omega)^2}} \tag{12.12}$$

and the phase shift β_n becomes

$$\beta_n = \tan^{-1} \frac{\lambda_n^{(1)}\omega}{\lambda_n^{(2)} - \omega^2} \tag{12.13}$$

The quantity X_n can be thought of as a *modal participation factor* in that it is an indication of how dominant the nth mode is. For a fixed value of the driving frequency ω, the values of X_n steadily decrease as the index n increases. The modal participation factor decreases unless the driving frequency is close to the square root of one of the eigenvalues of the operator L_2. In this case the modal participation factor for that index may be a maximum. By examining the modal participation factors or modal amplitudes, the designer can determine which modes are of interest or which modes are most important. The following example illustrates this point.

Example 12.3.1

Calculate the modal amplitudes for the clamped beam of example 12.2.1. Note that in this case the driving frequency is 1, i.e., $\omega = 1$. For the sake of simplicity, let $EI = m = \alpha = 1$and $\ell = 2$, so that the system is underdamped. From example 12.2.1, $f_{n0} = 1$ for each n. Also, $\lambda_n^{(1)} = 2$ for each n and

$$\lambda_n^{(2)} = 6.088n^4$$

for each value of the index n. In this case, Equation (12.12) yields

$$X_1 = 0.183, \qquad X_2 = 0.010, \qquad X_3 = 0.002$$

Note that the modal participation factor, X_n, decreases rapidly with increasing n.

Next, consider the same problem with the same physical parameters, except with a new driving frequency of $\omega = 22$. In this case the modal participation factors are

$$X_1 = 0.002, \qquad X_2 = 0.003$$
$$X_3 = 0.022, \qquad X_4 = 0.0009$$
$$X_5 = 0.0003, \qquad X_6 = 0.0001$$

This example illustrates that, if the driving frequency is close to a given mode frequency (X_3 in this case), the corresponding modal amplitude will increase in absolute value.

By examining the solution $w(x, t)$ mode by mode, certain design criteria can be formulated and applied. For example, the magnification curve of Figure 1.9 follows directly from Equation (12.12) on a per mode basis. Indeed, all the design and response characterizations of Section 1.4, such as bandwidth and overshoot, can be applied per mode. However, all the design procedures become more complicated because of coefficient coupling between each of the mode equations given by Equation (12.11). While the equations for $a_n(t)$ are decoupled in the sense that each $a_n(t)$ can be solved for independently of each other, the coefficients in these equations will depend on the same physical parameters (i.e., E, I, ρ, m, and so on). This is illustrated in the following example.

Example 12.3.2

Consider the step response of the clamped beam of example 12.2.1. A modal time to peak can be defined for such a system by using Equation (1.27) applied to Equation (12.3). With a proper interpretation of ζ_n and ω_n, the *modal time to peak*, denoted by t_{pn}, is

$$t_{pn} = \frac{\pi}{\omega_n\sqrt{1-\zeta_n^2}}$$

where $\omega_n = (n^2\pi^2/\ell^2)\sqrt{EI/m}$ and $\zeta_n = \alpha/\omega_n$. Examination of this formula shows that, if E, I, m, and α are chosen so that t_{p2} has a desired value, then $t_{p3}, t_{p4}, \ldots$ are fixed. Thus, the peak time, overshoot, and so on, of a distributed-mass system cannot be independently chosen on a per mode basis even though the governing equations decouple.

12.4 COMBINED DYNAMICAL SYSTEMS

Many systems are best modeled by combinations of distributed-mass components and lumped-mass components. Such systems are called *hybrid systems*, distributed systems with lumped appendages, or *combined dynamical systems*. This section discusses the natural frequencies and mode shapes of such structures and the use of the eigensolution to solve for the forced response of such structures.

As an example of such a system, consider the free vibration of a beam of length ℓ connected to a lumped mass and spring as illustrated in Figure 12.1. The equation of motion of the beam with the effect of the oscillator modeled as an external force, $f(t)\delta(x - x_1)$, is

$$EIw_{xxxx} + \rho A w_{tt} = f(t)\delta(x - x_1), \qquad x \in (0, \ell) \tag{12.14}$$

The equation of motion of the appended system is given by

$$m\ddot{z}(t) + kz(t) = -f(t) \tag{12.15}$$

where m is the appended mass and k is the associated stiffness. Here, the coordinate, $z(t)$, of the appended mass is actually the displacement of the beam at the point of attachment, i.e.,

$$z(t) = w(x_1, t) \tag{12.16}$$

Combining Equations (12.14) and (12.15) yields

$$\left[EI\frac{\partial^4}{\partial x^4} + k\delta(x - x_1)\right] w(x, t) + [\rho A + m\delta(x - x_1)]\, w_{tt}(x, t) = 0 \tag{12.17}$$

The solution $w(x, t)$ is now assumed to separate, i.e., $w(x, t) = u(x)a(t)$. Following the method of separation of variables, substitution of the separated form into Equation (12.17) and rearrangement of terms yields

$$\frac{EIu''''(x) + k\delta(x - x_1)u(x)}{[\rho A + m\delta(x - x_1)]u(x)} = -\frac{\ddot{a}(t)}{a(t)} \tag{12.18}$$

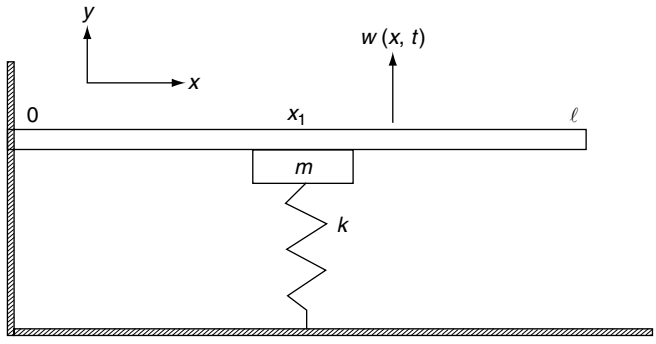

Figure 12.1 A beam with an attached lumped mass–spring system.

As before (see Section 9.2), each side of the equality must be constant. Taking the separation constant to be ω^2, the temporal function $a(t)$ has the harmonic form

$$a(t) = A\sin(\omega t) + B\cos(\omega t) \tag{12.19}$$

where A and B are constants of integration determined by the initial conditions. The spatial equation becomes

$$ELu''''(x) + \left[(k - m\omega^2)\delta(x - x_1) - \rho A\omega^2\right]u(x) = 0 \tag{12.20}$$

subject to the appropriate boundary conditions.

Solution of Equation (12.20) yields the generalized eigenfunctions, $\Theta_n(x)$, and eigenvalues, ω_n^2, for the structure. These are called generalized eigenfunctions because Equation (12.20) does not formally define an operator eigenvalue problem, as specified in Section 10.2. Hence, the procedure and modal analysis are performed formally. Note, however, that, if $k/m = \omega_n^2$, i.e., if the appended spring–mass system is tuned to a natural frequency of the beam, the related eigenfunction becomes that of the beam without the appendage. The solution of Equation (12.20) can be constructed by use of a Green's function for the vibrating beam.

The Green's function $g(x, x_1)$ for a beam satisfies

$$g'''' - \beta^4 g = \delta(x - x_1) \tag{12.21}$$

where $\beta^4 = \rho A\omega^2/(EI)$ and g satisfies the appropriate boundary conditions. Following the development of Section 10.6, Equation (12.21) has the solution

$$g(x, x_1) = -\frac{1}{2\beta^3 \sin\beta\ell \sinh\beta\ell}\begin{cases} y(x, x_1), & 0 < x < x_1 \\ y(x_1, x), & x_1 < x < \ell \end{cases} \tag{12.22}$$

where the function $y(x, x_1)$ is symmetric in x_1 and x and has the form

$$\begin{aligned} y(x, x_1) = {} & \sin(\beta\ell - \beta x_1)\sin(\beta x)\sinh(\beta\ell) \\ & - \sinh(\beta\ell - \beta x_1)\sinh(\beta x)\sin(\beta\ell) \end{aligned} \tag{12.23}$$

In terms of the Green's function just defined, the solution to Equation (12.20) for the simply supported case can be written as (see Nicholson and Bergman, 1986)

$$u(x) = \frac{1}{EI}(m\omega^2 - k)g(x, x_1)u(x_1) \tag{12.24}$$

If $u(x_1)$ were known, then Equation (12.24) would specify the eigenfunctions of the system. Fortunately, the function $u(x_1)$ is determined by writing Equation (12.24) for the case $x = x_1$, resulting in

$$\left[EI - \left\{m\left(\omega^2 - \frac{k}{m}\right)\right\} g(x_1, x_1)\right] u(x_1) = 0 \tag{12.25}$$

which yields the characteristic equation for the system. In order to allow $u(x_1)$ to be nonzero, the coefficient in Equation (12.25) must vanish, yielding an expression for computing the

natural frequencies, ω. This transcendental equation in ω contains terms of the form $\sin \beta\ell$ and hence has an infinite number of roots, denoted by ω_n. Thus, Equation (12.24) yields an infinite number of eigenfunctions, denoted by $u_n(x)$.

Both the free and forced response of a combined dynamical system, such as the one described in Figure 12.1, can be calculated using a modal expansion for a cantilevered beam. Following Section 9.4, the eigenfunctions are, from Equation (12.20), those of a nonappended cantilevered beam, i.e.,

$$\Theta_i(x) = \cosh \beta_i x - \cos \beta_i x - \alpha_i(\sinh \beta_i x - \sin \beta_i x) \tag{12.26}$$

Here, the constants α_i are given by

$$\alpha_i = \frac{\cosh \beta_i \ell + \cos \beta_i \ell}{\sinh \beta_i \ell + \sin \beta_i \ell} \tag{12.27}$$

and the eigenvalues β_i are determined from the transcendental equation

$$1 + \cosh \beta_i \ell \cos \beta_i \ell = 0 \tag{12.28}$$

Note that in this case the arguments of Section 11.6 hold and the functions $\Theta_i(x)$ form a complete orthogonal set of functions. Hence, the spatial solution $u(x)$ can be written as

$$u(x) = \sum_{i=1}^{\infty} b_i \Theta_i(x) \tag{12.29}$$

with the set $\{\Theta_i\}$ normalized so that

$$(\Theta_i, \Theta_j) = \ell \delta_{ij} \tag{12.30}$$

Substitution of Equation (12.29) for $u(x)$ in Equation (12.20), multiplying by $\Theta_j(x)$, using the property

$$\Theta_j''''(x) = \beta_j^4 \Theta_j(x)$$

and integrating over the interval $(0,\ell)$ yields

$$\begin{aligned} EI\ell\beta_i^4 b_i + (k - m\omega^2)\Theta_i(x_1) b_i \Theta_i(x_1) - \rho A \ell \omega^2 b_i = 0 \qquad &\text{for } i = j \\ (k - m\omega^2)\Theta_j(x_1) b_i \Theta_i(x_1) = 0 \qquad &\text{for } i \neq j \end{aligned} \tag{12.31}$$

Dividing this last expression by $\rho A \ell$ and defining two new scalars, A_{ij} and B_{ij}, by

$$A_{ij} = \frac{k\Theta_i(x_1)\Theta_j(x_1)}{\rho A \ell} + \frac{EI\beta_i^4}{\rho A}\delta_{ij} \tag{12.32}$$

$$B_{ij} = \frac{1}{\rho A \ell} m \Theta_i(x_1)\Theta_j(x_1) + \delta_{ij} \tag{12.33}$$

allows Equation (12.31) to be simplified. Equation (12.31) can be rewritten as

$$\sum_{j=1}^{\infty} A_{ij} b_j = \omega^2 \sum_{j=1}^{\infty} B_{ij} b_j \tag{12.34}$$

This last expression is in the form of a generalized infinite matrix eigenvalue problem for ω^2 and the generalized Fourier coefficients b_j. The elements b_j are the modal participation factors for the modal expansion given by Equation (12.29).

The orthogonality relationship for the eigenfunctions $\{\Theta_n(x)\}$ is calculated from Equation (12.20) and rearranged in the form

$$\Theta_n''''(x) - \left(\frac{\rho A \omega^2}{EI}\right) \Theta_n(x) = \frac{1}{EI}(\omega_n^2 m - k)\delta(x - x_1)\Theta_n(x) \tag{12.35}$$

Premultiplying Equation (12.35) by $\Theta_m(x)$ and integrating yields (see problem 12.5)

$$\rho A \int_0^{\ell} \Theta_m(x)\Theta_n(x)\,dx = -\int_0^{\ell} m\delta(x - x_1)\Theta_m(x)\Theta_n(x)\,dx$$

or

$$\int_0^{\ell} \left[1 + \frac{m}{A\rho}\delta(x - x_1)\right] \Theta_m(x)\Theta_n(x)\,dx = \delta_{nm} \tag{12.36}$$

The preceding characteristic equation and orthogonality relationship completes the modal analysis of a cantilevered Euler–Bernoulli beam connected to a spring and lumped mass.

Equipped with the eigenvalues, eigenfunctions, and the appropriate orthogonality condition, a modal solution for the forced response of a damped structure can be carried out for a proportionally damped beam connected to a lumped spring–mass dashpot arrangement following these procedures. Bergman and Nicholson (1985) showed that the modal equations for a damped cantilevered beam attached to a spring–mass dashpot appendage have the form

$$\ddot{a}_n(t) + \sum_{m=1}^{\infty} \left\{ \varepsilon_b \delta_{nm} + \mu \left(\frac{\varepsilon \alpha_m^4 \alpha_n^4}{\alpha_0^8} - \varepsilon_b \right) A_m A_n \Theta_m(x_1)\Theta_n(x_1)\dot{a}_m(t) \right\} + \alpha_n^4 a_n(t) = f_n(t) \tag{12.37}$$

Here

$$f_n(t) = \gamma \int_0^L \Theta_n(x) f(x, t)\,dx$$

$$f(x, t) = \text{externally applied force}$$

$$\varepsilon_b = \text{distributed damping coefficient}$$

$$\varepsilon = \text{lumped damping rate}$$

$$\mu = \text{lumped mass}$$

$$\alpha_n^2 = \text{system natural frequencies}$$

$$\alpha_0 = \text{lumped stiffness}$$

$$A_n = \frac{\alpha_0^4}{\alpha_0^4 - \alpha_n^4}$$

With the given parameters and orthogonality conditions, the combined system has a modal solution given by

$$w(x,t)=\sum_{n=1}^{\infty} a_n(t)\Theta_n(x) \tag{12.38}$$

where $a_n(t)$ satisfies Equation (12.37) and the appropriate initial conditions. Note from Equation (12.37) that proportional damping results if $\varepsilon\alpha_m^4\alpha_n^4=\varepsilon_b\alpha_0^8$.

12.5 PASSIVE CONTROL AND DESIGN

The lumped appendage attached to a beam of the previous section can be viewed as a passive control device, much in the same way that the absorber of Section 6.2 can be thought of as a passive control element. In addition, the layered materials of Section 9.6 can be thought of as either a passive control method or a redesign method. In either case, the desired result is to choose the parameters of the system in such a way that the resulting structure has improved vibration response.

First, consider a single absorber added to a cantilevered beam. The equations of motion as discussed in the previous section have a temporal response governed by Equation (12.37). Thus, the rate of decay of the transient response is controlled by the damping terms:

$$\sum_{m=1}^{\infty}\left\{\varepsilon_b\delta_{nm}+\mu\frac{\varepsilon\alpha_m^4\alpha_n^4}{\alpha_0^8}-\varepsilon_b A_m A_n\Theta_m(x_1)\Theta_n(x_1)\right\}\dot{a}_m(t) \tag{12.39}$$

The design problem becomes that of choosing x_1, ε, μ, and α_0 so that Equation (12.39) has the desired value. With only four parameters to choose and an infinite number of modes to effect, there are not enough design parameters to solve the problem. In addition, the summation in Equation (12.39) effectively couples the design problem so that passive control cannot be performed on a per mode basis. However, for specific cases the summation can be truncated, making the design problem more plausible.

Next, consider the layered material of Section 9.6. Such materials can be designed to produce both a desired elastic modulus and a desired loss factor (Nashif, Jones, and Henderson, 1985). Consider the problem of increasing the damping in a beam so that structural vibrations in the beam decay quickly. Researchers in the materials area often approach the problem of characterizing the damping in a material by using the concept of loss factor, introduced as η in Section 1.4, and the concept of *complex modulus* introduced next.

For a distributed-mass structure, it is common practice to introduce damping in materials by simply replacing the elastic modulus for the material, denoted by E, with a complex modulus of the form

$$E(1+\mathrm{i}\eta) \tag{12.40}$$

where η is the experimentally determined loss factor for the material and i is the square root of (-1). The rationale for this approach is based on an assumed temporal solution of the form $A\,e^{i\omega t}$. If $A\,e^{i\omega t}$ is substituted into the equation of motion of a damped structure, the velocity term yields a coefficient of the form $\mathrm{i}\omega$, so that the resulting equation may be viewed as

having a complex stiffness. This form of damping is also called the *Kimball–Lovell complex stiffness* (see Bert, 1973).

The loss factor for a given structure made of standard metal is usually not large enough to suppress unwanted vibrations in many applications. One approach to designing more highly damped structures is to add a layer of damping material to the structure, as indicated in Figure 12.2. The new structure then has different elastic modulus (frequencies) and loss factor. In this way, the damping material can be thought of as a passive control device used to change the poles of an existing structure to more desirable locations. Such a treatment of structures is called *extensional damping*. Sometimes it is referred to as *unconstrained layer damping*, or *free layer damping*.

Let E and η denote the elastic modulus and loss factor of the combined system of Figure 12.2. Let E_1 and η_1 denote the modulus and loss factor of the original beam, and let E_2 and η_2 denote the modulus and loss factor of the added damping material. In addition, let H_2 denote the thickness of the added damping layer and H_1 denote the thickness of the original beam. Let $e_2 = E_2/E_1$ and $h_2 = H_2/H_1$. The design formulae relating the 'new' modulus and loss factor to those of the original beam and added damping material are given in Nashif, Jones, and Henderson (1985) as

$$\frac{EI}{E_1 I_1} = \frac{1 + 4e_2h_2 + 6e_2h_2^2 + 4e_2h_2^3 + e_2^2h_2^4}{1 + e_2h_2} \tag{12.41}$$

and

$$\frac{\eta}{\eta_1} = \frac{e_2h_2(3 + 6h_2 + 4h_2^2 + 2e_2h_2^3 + e_2^2h_2^4)}{(1 + e_2h_2)(1 + 4e_2h_2 + 6e_2h_2^2 + 4e_2h_2^3 + e_2^2h_2^4)} \tag{12.42}$$

where $(e_2h_2)^2$ is assumed to be much smaller than e_2h_2.

Equations (12.41) and (12.42) can be used to choose an appropriate damping material to achieve a desired response.

The preceding complex modulus approach can also be used to calculate the response of a layered structure. Note that the response of an undamped uniform beam can be written in the form

$$w(x, t) = \sum_{n=1}^{\infty} g(E)a_n(t, E)\Theta_n(x, E) \tag{12.43}$$

where $g(E)$ is some function of the modulus E. This functional dependence is usually not explicitly indicated but rather is contained in the eigenvalues of the eigenfunctions

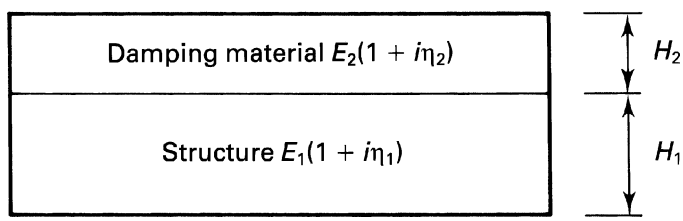

Figure 12.2 Passive vibration control by using a damping layer.

$\Theta_n(x, E)$ and in the temporal coefficients $a_n(t, E)$. One approach used to include the effects of damping in a layered beam is simply to substitute the values of $E + i\eta$ obtained from Equations (12.41) and (12.42) into g, a_n, and Θ_n in Equation (12.43). Each term of the series (12.43) is complex and of the form $g(E(1+i\eta)a_n(t, E(1+i\eta))\Theta_n(x, E(1+i\eta))$, so some manipulation is required to calculate the real and imaginary parts. This approach should be treated as an approximation, as it is not rigorous.

12.6 DISTRIBUTED MODAL CONTROL

In this section, the control of systems governed by partial differential equations of the form of Equation (12.1) is considered. The control problem is to find some function $f(x, t)$ such that the response $w(x, t)$ has a desired form. If $f(x, t)$ is a function of the response of the system, then the resulting choice of $f(x, t)$ is called active control. If, on the other hand, $f(x, t)$ is thought of as a change in the design of the structure, it is referred to as passive control (or redesign). Modal control methods can be used in either passive or active control. Any control method that uses the eigenfunctions, or modes, of the system in determining the control law $f(x, t)$ is considered to be a modal control method.

Repeating the analysis of Section (12.2) yields the modal control equations. For the control problem, the functions $f_n(t)$ of Equation (12.4) are thought of as modal controls, or inputs, in the jargon of Chapter 7. As indicated in Chapter 7, there are many possible control techniques to apply to Equation (12.3). Perhaps the simplest and most physically understood is state feedback. Viewed by itself, Equation (12.3) is a two-state model, with the states being the generalized velocity, $\dot{a}_n(t)$, and position, $a_n(t)$. If $f_n(t)$ is chosen in this way, Equation (12.3) becomes

$$\ddot{a}_n(t) + \lambda_n^{(1)}\dot{a}_n(t) + \lambda_n^{(2)}a_n(t) = -c_n^p a_n(t) \; - \; c_n^v \dot{a}_n(t) \tag{12.44}$$

where c_n^p and c_n^v are modal position and velocity gains respectively. Obviously, the choice of the position and velocity feedback gains completely determines the nth temporal coefficient in the free response given by Equation (12.10). In theory, c_n^p and c_n^v can be used to determine such performance criteria as the overshoot, decay rate, speed of response, and so on. These coefficients can be chosen as illustrated for the single-degree-of-freedom problem of example 12.6.1.

The question arises, however, about the convergence of $f(x, t)$. Since

$$f_n(t) = -c_n^p a_n(t) - c_n^v \dot{a}_n(t) = \int_\Omega f(x, t)\Theta_n(x)\, d\Omega \tag{12.45}$$

the series

$$f(x, t) = \sum_{n=1}^{\infty} [-c_n^p a_n(t) \; - \; c_n^v \dot{a}_n(t)]\, \Theta_n(x)\, d\Omega \tag{12.46}$$

must converge. Furthermore, it must converge to some function $f(x, t)$ that is physically realizable as a control. Such controls $f(x, t)$ are referred to as *distributed controls* because they are applied along the spatial domain Ω.

Example 12.6.1

Consider the problem of controlling the first mode of a flexible bar. An internally damped bar clamped at both ends has equations of motion given by Equation (12.1) with

$$L_1 = -2b\frac{\partial^2}{\partial x^2}, \qquad L_2 = -\alpha\frac{\partial^2}{\partial x^2}$$

and boundary conditions $w(x, t) = 0$ at $x = 0$ and $x = 1$. Here, b is the constant denoting the rate of internal damping, and α denotes a constant representing the stiffness in the bar (EI/ρ). Solution of the eigenvalue problem for L_1 and L_2 and substitution of the appropriate eigenvalues into Equation (12.3) yields

$$\ddot{a}_n(t) + 2bn^2\pi^2\dot{a}_n(t) + \alpha n^2\pi^2 a_n(t) = f_n(t)$$

For the sake of illustration, assume that $\alpha = 400\pi^2$ and $b = 1$ in the appropriate units.

Suppose it is desired to control only the lowest mode. Furthermore, suppose it is desired to shift the frequency and damping ratio of the first mode. Note that the equation for the temporal coefficient for the first mode is

$$\ddot{a}_1(t) + 2\pi^2\dot{a}_n(t) + 400\pi^4 a_1(t) = f_1(t)$$

so that the first mode has an undamped natural frequency of $\omega_1 = 20\pi^2$ and a damping ratio of $\zeta_1 = 0.05$.

The control problem is taken to be that of calculating a control law, $f(x, t)$, that raises the natural frequency to $25\pi^2$ and the damping ratio to 0.1. This goal will be achieved if the displacement coefficient, after control is applied, has the value

$$(25\pi^2)^2 = 624\pi^4$$

and the velocity coefficient of the closed-loop system has the value

$$2\zeta_1\omega_1 = 2(0.1)(25\omega^2) = 5\omega^2$$

Using Equation (12.44) with $n = 1$ yields

$$\ddot{a}_1(t) + 2\pi^2\dot{a}_n(t) + 400\pi^4 a_1(t) = -c_1^p a_1(t) - c_1^v\dot{a}_1(t)$$

Combining position coefficients and then velocity coefficients yields the following two simple equations for the control gains:

$$c_1^p + 400\pi^4 = 625\pi^4$$
$$c_1^v + 2\pi^2 = 5\pi^2$$

Thus, $c_1^p = 225\pi^4$ and $c_1^v = 3\pi^2$ will yield the desired first mode values. The modal control force is thus

$$f_1(t) = -225\pi^4 a_1(t) - 3\pi^2\dot{a}_1(t)$$

In order to apply this control law only to the first mode, the control force must be of the form

$$f(x, t) = f_1(t)\Theta_1(x)$$

For this choice of $f(x, t)$ the other modal controls, $f_n(t)$, $n > 1$, are all zero, which is very difficult to achieve experimentally because the result requires $f(x, t)$ to be distributed along a single mode.

Unfortunately, designing distributed actuators is difficult in practice. The design of actuators that produce a spatial distribution along a given mode, as required by the example, is even more difficult. Based on the availability of actuators, the more practical approach is to consider actuators that act at a point, or points, in the domain of the structure. The majority of control methods used for distributed-parameter structures involve using finite-dimensional models of the structure. Such models are often obtained by truncating the series expansion of the solution of Equation (12.9). The methods of Chapter 7 are then used to design a vibration control system for the structure. The success of such methods is tied to the process of truncation (Gibson, 1981). Truncation is discussed in more detail in Chapter 13.

12.7 NONMODAL DISTRIBUTED CONTROL

An example of a distributed actuator that provides a nonmodal approach to control is the use of a piezoelectric polymer. Piezoelectric devices offer a convenient source of distributed actuators. One such actuator has been constructed and used for vibration control of a beam (Bailey and Hubbard, 1985) and is presented here.

Consider the transverse vibrations of a cantilevered beam of length ℓ with a piezoelectric polymer bonded to one side of the beam. The result is a two-layer material similar to the beam illustrated in Figure 12.2. Bailey and Hubbard (1985) have shown that the equation governing the two-layer system is

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right]+\rho A\frac{\partial^2}{\partial t^2}=0, \qquad x\in\Omega \tag{12.47}$$

with boundary conditions

$$\begin{aligned} w(0,t)&=w_x(0,t)=0\\ EIw_{xx}(\ell,t)&=-cf(t) \qquad \text{and} \qquad w_{xxx}(\ell,t)=0 \end{aligned} \tag{12.48}$$

where it is assumed that the voltage applied by the polymer is distributed evenly along x, i.e., that its spatial dependence is constant. Here, EI reflects the modulus and inertia of both the beam and the polymer, ρ is the density, and A is the cross-sectional area. The constant c is the bending moment per volt of the material and $f(t)$ is the voltage applied to the polymer. This distributed actuator behaves mathematically as a *boundary control*.

One approach to solving this control problem is to use a Lyapunov function, $V(t)$, for the system, and choose a control, $f(t)$, to minimize the time rate of change of the Lyapunov function.

The chosen Lyapunov function is

$$V(t)=\frac{1}{2}\int_0^{\ell}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right]dt \tag{12.49}$$

which is a measure of how far the beam is from its equilibrium position. Minimizing the time derivative of this functional is then equivalent to trying to bring the system to

rest (equilibrium) as fast as possible. Differentiating Equation (12.49) and substitution of Equation (12.47) yields

$$\frac{\partial V}{\partial t} = \int_0^{\ell} \left[1 - \frac{EI}{\rho A}\right] w_{xxt} w_{xx}\, dx + \frac{c}{\rho A} f(t) w_{xx}(\ell, t) \tag{12.51}$$

The voltage $f(t)$ is thus chosen to minimize this last quantity. Bailey and Hubbard (1985) showed that $f(t)$, given by

$$f(t) = -\mathrm{sgn}(c w_{xt}(\ell, t)) f_{\max} \tag{12.52}$$

is used as a minimizing control law. Here, sgn denotes the signum function.

Not only is the control law of Equation (12.52) distributed and independent of the modal description of the structure but it also allows the control force to be magnitude limited, i.e., $|f(t)| < f_{\max}$. These are both very important practical features. In addition, the control law depends only on feeding back the velocity of the tip of the beam. Hence, this distributed control law requires that a measurement be taken at a single point at the tip ($x = \ell$).

12.8 STATE-SPACE CONTROL ANALYSIS

This section examines the control problem for distributed-parameter structures cast in the state-space formulation. Considering Equation (12.1), define the two-dimensional vector $z(x, t)$ by

$$z(x, t) = [\boldsymbol{w}(x, t) \quad \boldsymbol{w}_t(x, t)]^T \tag{12.53}$$

The state equation for the system of Equation (12.2) then becomes

$$z_t = Az + bu \tag{12.54}$$

where the matrix of operators A is defined by

$$A = \begin{bmatrix} 0 & I \\ -L_2 & -L_1 \end{bmatrix} \tag{12.55}$$

the vector b is defined by $b = [0 \quad 1]^T$, and $u = u(x, t)$ is now used to denote the applied force, which in this case is a control. As in the lumped-mass case, there needs to be an observation equation, denoted here as

$$y(x, t) = Cz(x, t) \tag{12.56}$$

In addition, the state vector z is subject to boundary conditions and initial conditions and must have the appropriate smoothness (i.e., the elements z belong to a specific function space). Equations (12.55) and (12.56) form the state-space equations for the control of distributed-mass systems and are a direct generalization of Equations (7.1) and (7.2) for the control of lumped-mass systems.

As discussed in the preceding sections, the input or control variable $u(x, t)$ can be either distributed or lumped in nature. In either case the general assumption is that the function $u(x, t)$ separates in space and time. The most common form taken by the control $u(x, t)$ describes the situation in which several time-dependent control forces are applied at various points in the domain of the structure. In this case

$$u(x, t) = \sum_{i=1}^{m} \delta(x - x_i) u_i(t) \tag{12.57}$$

where the m control forces of the form $u_i(t)$ are applied to the m locations x_i. Note that this formulation is consistent with the development of combined dynamical systems of Section 12.4.

The output, or measurement, of the system is also subject to the physical constraint that most devices are lumped in nature. Measurements are most often proportional to a state or its derivatives. In this case, $y(x, t)$ takes the form

$$y(x, t) = \sum_{j=1}^{p} c_j \delta(x - x_j) z(x, t) \tag{12.58}$$

where c_j are measurement gains of each of the p sensors located at the p points, x_j, in the domain of the structure.

The concepts of controllability and observability are of course equally as critical for distributed-mass systems as they are for lumped-mass systems. Unfortunately, a precise definition and appropriate theory is more difficult to develop and hence is not covered here. Intuitively, however, the actuators and sensors should not be placed on nodes of the vibrational modes of the structures. If this practice is adhered to, then the system will be controllable and observable. This line of thought leads to the idea of modal controllability and observability (see, for instance, Goodson and Klein, 1970).

An optimal control problem for a distributed-parameter structure can be formulated following the discussion in Section 7.4 by defining various cost functionals. In addition, pole placement and state feedback schemes can be devised for distributed-parameter systems, generalizing the approaches used in Chapter 7. Note, however, that not all finite-dimensional control methods have direct analogs in distributed-parameter systems. This lack of analogy is largely due to the difference between functional analysis and linear algebra.

CHAPTER NOTES

This chapter discusses the analysis of the forced response and control of structures with distributed mass. Section 12.2 presents standard, well-known modal analysis of the forced response of a distributed-mass system. Such an approach essentially reduces the distributed-mass formulation to a system of single-degree-of-freedom models that can be analyzed by the methods of Chapter 1. Section 12.3 examines some design specifications for distributed-mass systems in modal coordinates. One cannot assign design criteria to each mode independently, as is sometimes suggested by using modal coordinates.

Section 12.4 examines a method of calculating the response of hybrid systems, i.e., systems composed of both distributed-mass elements and lumped-mass elements. Several authors

have approached this problem over the years. Most recently, Nicholson and Bergman (1986) produced a series of papers (complete with computer code) discussing combined dynamical systems based on beam equations. Their paper is essentially paraphrased in Section 12.4. Banks *et al.* (1998) clarify the existence of normal modes in combined dynamical systems. The papers by the Bergman group also contain an excellent bibliography of this area. The importance of the analysis of combined dynamical systems is indicated in Section 12.5 on passive control. The most common passive vibration suppression technique is to use an absorber or an isolator. The theory by Bergman *et al.* provides an excellent analytical tool for the design of such systems. An alternative and very useful approach to vibration design and passive control is to use layers of material and high damping (loss factor). This approach is discussed extensively in the book by Nashif, Jones, and Henderson (1985), which provides a complete bibliography.

Section 12.6 discusses the concept of modal control for distributed-mass structures with distributed actuators and sensors. This material is expanded in a paper by Inman (1984). The details of using a modal control method are outlined by Meirovitch and Baruh (1982) and were originally introduced by Gould and Murray-Lasso (1966). Gibson (1981) discusses some of the problems associated with using finite-dimensional state models in designing control laws for distributed-mass systems. This result has sparked interest in nonmodal control methods, an example of which is discussed in Section 12.7. The material of Section 12.7 is taken from the paper by Bailey and Hubbard (1985).

Section 12.8 presents a very brief introduction to formulating the control problem in the state space. Several books, notably Komkov (1970) and Lions (1972), discuss this topic in more detail. A more practical approach to the control problem is presented in the next chapter. Tzou and Bergman (1998) present a collection of works on the vibration and control of distributed-mass systems.

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PROBLEMS

12.1 Calculate the response of the first mode of a clamped membrane of Equation (9.86) subject to zero initial conditions and an applied force of

$$f(x, y, t) = 3 \sin t \delta(x - 0.5)\delta(y - 0.5)$$

12.2 Derive the modal response [i.e., $a_n(t)$] for the general system given by Equation (12.1) and associated assumptions if, in addition to $f(x, t)$, the system is subject to initial conditions of the form

$$w(x, 0) = w_0, \qquad w_t(x, 0) = w_{t0}$$

Use the notation of Equations (12.3) through (12.8).

12.3 Calculate an expression for the modal participation factor for problem (12.1).

12.4 Define a modal logarithmic decrement for the system of Equation (12.1) and calculate a formula for it.

12.5 Derive Equation (12.36) from Equation (12.35) by performing the suggested integration. Integrate the term containing Θ_n'''' 4 times using the homogeneous boundary conditions and again using Equation (12.35) to evaluate Θ_m''''.

12.6 Discuss the possibility that the sum in Equation (12.37) can be truncated because

$$\varepsilon \alpha_m^4 \alpha_n^4 = \alpha_0^8 \varepsilon_b$$

for some choices of m and n.

12.7 Show that the complex stiffness is a consistent representation of the equation of motion by substituting the assumed solution $A e^{i\omega t}$ into Equation (10.14). What assumption must be made on the operators L_1 and L_2?

12.8 **(a)** Calculate the terms $g(E)$, $a_n(t, E)$, and $\Theta_n(t, E)$ explicitly in terms of the modulus E for a damped free beam of unit length.

(b) Next, substitute $e(1 + i\eta)$ for E in your calculation and compare your result with the same beam having a damping operator of $L_1 = 2\eta I$, where I is the identity operator.

12.9 Formulate an observer equation for a beam equation using the state-space formulation of Section 12.8.

12.10 Consider the transverse vibration of a beam of length ℓ, modulus E, and mass density ρ. Suppose an accelerometer is mounted at the point $x = \ell/2$. Determine the observability of the first three modes.

13

Approximations of Distributed-parameter Models

13.1 INTRODUCTION

This chapter is devoted to examining approximations of distributed-parameter systems with lumped-parameter models. Since the solutions of distributed-parameter systems are often given in terms of an infinite series, and since only a few configurations have closed-form solutions, there is a need to cast distributed-parameter systems into finite-dimensional systems that can easily be solved numerically. In addition, control and design are well developed for lumped-parameter systems, providing further motivation to approximate distributed systems with the more easily manipulated lumped systems. From the experimentalist point of view, most common measurement methods only 'see' a finite (dimensional) number of points.

In this chapter, several common methods of approximating distributed-mass structures by lumped-mass models are presented. Most of these methods eliminate the spatial dependence in the solution technique by discretizing the spatial variable in some way, effectively approximating an eigenfunction with an eigenvector. This chapter ends with a discussion of the effects of active control of distributed-mass structures and the accuracy of the approximation.

13.2 MODAL TRUNCATION

Since the solution of the vibration problem given by

$$w_{tt}(x,t) + L_1 w_t(x,t) + L_2 w(x,t) = f(x,t), \qquad x \in \Omega \tag{13.1}$$

plus appropriate boundary conditions is of the form

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \tag{13.2}$$

Vibration with Control D. J. Inman
 ISBN: 0-470-01051-7

which converges uniformly, it is possible to approximate the solution by

$$w_N(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x) \tag{13.3}$$

where N is finite. This finite sum approximation ignores the sum given by

$$w_R(x,t) = \sum_{n=N+1}^{\infty} a_n(t)\phi_n(x) \tag{13.4}$$

called the *residual*. The modes in this sum are called the *truncated modes*, i.e., the functions $\phi_n(x)$ for values of the index $n = N+1 \to \infty$. The assumption is that the residual solution is small, i.e., that $\| w_R(x,t) \| < \varepsilon$. This assumption is often satisfied by physical structures, giving rise to the statement that structures behave like low-pass filters.

Substitution of Equation (13.3) into Equation (13.1) yields

$$\sum_{n=1}^{N} [\ddot{a}_n(t)\phi_n(x) + \dot{a}_n L_1 \phi_n(x) + a_n(t) L_2 \phi_n(x)] = \sum_{n=1}^{N} b_n(t)\phi_n(x) \tag{13.5}$$

where $f(x,t)$ has also been expanded in terms of the functions $\phi_n(x)$ with coefficients $b_n(t)$. Premultiplying Equation (13.5) by $\phi_m(x)$ and integration over Ω yields two possibilities. Note that the sum is now finite, so that convergence is not a problem. First, if $L_1 L_2 = L_2 L_1$ on the appropriate domain, then Equation (13.5) becomes N decoupled ordinary differential equations of the form

$$\ddot{a}_n(t) + \lambda_n^{(1)} \dot{a}_n(t) + \lambda_n^{(2)} a_n(t) = b_n(t) \tag{13.6}$$

In matrix form this becomes

$$I\ddot{\mathbf{a}} + \Lambda_D \dot{\mathbf{a}} + \Lambda_K \mathbf{a} = \mathbf{f} \tag{13.7}$$

which can then be analyzed by the methods of Chapter 5. Here, Λ_D and Λ_K are diagonal matrices and $\mathbf{a}$ and $\mathbf{f}$ are N vectors of obvious definition.

If the commutivity condition does not hold, then Equation (13.6) becomes

$$I\ddot{\mathbf{a}} + D\dot{\mathbf{a}} + \Lambda_K \mathbf{a} = \mathbf{f} \tag{13.8}$$

where the elements of D are

$$d_{ij} = \int_{\Omega} \phi_i L_1 \phi_j \, d\Omega \tag{13.9}$$

and the functions $\phi_j(x)$ are the eigenfunctions of L_2. The boundary conditions are incorporated in the matrices D, Λ_D, and Λ_K automatically by virtue of the integration. The initial conditions on $\mathbf{a}(t)$ are defined by

$$a_i(0) = \int_{\Omega} w(x,0)\phi_i(x)\, d\Omega \qquad \text{and} \qquad \dot{a}_i(0) = \int_{\Omega} w_t(x,0)\phi_i(x)\, d\Omega \tag{13.10}$$

In both cases it is required that $w_R(x, t)$ be as small as possible, i.e., that the higher modes do not contribute much to the solution. In practice this is often so. For instance, $N = 3$ is often adequate to describe the longitudinal vibration of a simple cantilevered beam (recall example 12.3.1). Equations (13.6) and (13.8) are finite-dimensional approximations of Equations (13.1) derived by truncating the higher modes of the response of the structure (i.e., setting $w_R = 0$) and as such are referred to as a *truncated modal model*.

13.3 RAYLEIGH–RITZ–GALERKIN APPROXIMATIONS

The Rayleigh quotient was introduced in Section 11.7 as a means of approximating the natural frequencies of a conservative system. Ritz used this concept to calculate an approximate solution for the eigenfunctions (mode shapes) in terms of an assumed series of trial functions. This approach is similar to modal truncation but, rather than using the exact mode shapes as the expanding basis, any complete set of basis functions that satisfy the boundary conditions is used. In other words, the Rayleigh–Ritz (as it is usually called) approximation does not require any knowledge of the eigenfunctions. Furthermore, the Rayleigh quotient can be written in terms of energy, rather than in terms of the eigenvalue problem, reducing the number of derivatives and boundary conditions that need to be satisfied by the choice of 'trial' functions.

Trial functions are functions that (a) satisfy the boundary conditions or at least some of them, (b) are orthogonal to each other, and (c) have enough derivatives to be fit into the equation of motion. Trial functions are further divided up into those that satisfy all of the boundary conditions (called comparison functions) and those that satisfy only the geometric boundary conditions (called admissible functions). In forming the sum of Equation (13.5) there are three classifications of functions that can be used:

1. *Eigenfunctions*. These satisfy the equation of motion plus all the boundary conditions.
2. *Comparison functions*. These are orthogonal and satisfy all the boundary conditions (but not the equation of motion).
3. *Admissible functions*. These are orthogonal and satisfy only the geometric boundary conditions (i.e. things like displacements and slopes).

Boundary conditions are classified as either (a) natural boundary conditions (those that involve force and moment balances) or (b) geometric boundary conditions (those that satisfy displacement and slope conditions at the boundary).

Using trial functions eliminates the need to know the eigenfunctions of the structure before approximating the system. Let $\{\theta_n(x)\}$ be a linearly independent set of basis functions that are complete in a subspace of $D(L_2)$ and satisfy the appropriate boundary conditions. The Nth approximate solution of Equation (13.1) is then given by the expression

$$w_N(x, t) = \sum_{n=1}^{N} a_n(t)\theta_n(x) \tag{13.11}$$

Likewise, $f(x, t)$ is approximated by

$$f_n(x, t) = \sum_{n=1}^{N} b_n(t)\theta_n(x) \tag{13.12}$$

Substitution of Equations (13.9) and (13.10) for $w(x, t)$ and $f(x, t)$ in (13.1), respectively, yields

$$\sum_{n=1}^{N} [\ddot{a}(t)\theta_n + \dot{a}_n(t)L_1\theta_n + a_n(t)L_2\theta_n] = \sum_{n=1}^{N} b_n(t)\theta_n(x) \tag{13.13}$$

Premultiplying Equation (13.13) by $\theta_m(x)$ and integrating (thus using the boundary conditions) yields the finite-dimensional approximation

$$M\ddot{\mathbf{x}} + D\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}(t) \tag{13.14}$$

where the matrices M, D, and K and the vector $\mathbf{f}$ are defined by

$$m_{ij} = \int \theta_i \theta_j \, d\Omega \tag{13.15}$$

$$d_{ij} = \int \theta_i L_1 \theta_j \, d\Omega \tag{13.16}$$

$$k_{ij} = \int \theta_i L_2 \theta_j \, d\Omega \tag{13.17}$$

$$f_i = \int f(x, t)\theta_i \, d\Omega \tag{13.18}$$

Unlike the coefficient matrices of the modal truncation scheme of Equations (13.7) and (13.8), the matrices M, D, and K in this case are not necessarily diagonal. Note, however, that they are symmetric as long as the operators L_1 and L_2 are self-adjoint. The order, N, of the finite-dimensional approximation [Equation (13.11)] is chosen so that $w_n(x, t)$ is as small as possible for the purpose at hand. Note that the difference between the functions $\phi_i(x)$ in Section 13.2 and the $\theta_n(x)$ in this section is that the $\phi_i(x)$ are eigenfunctions of the stiffness operator. In this section the trial functions $\theta_n(x)$ are chosen in a somewhat arbitrary fashion. Hence, this method is also called the *assumed mode method*. The bottom line with approximation methods is that, the closer the trial function to the exact eigenfunction (mode shape), the better is the estimate. If, in fact, an exact set of mode shapes is used, and the damping is proportional, the approximation will be exact.

Starting with the Rayleigh quotient, the Ritz method minimizes the quotient over the coefficients of expansion for $\phi(x)$ and provides an approximation of the system natural frequencies and mode shapes for undamped systems. Let the approximate spatial dependence have the form

$$\phi(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{13.19}$$

where the $\phi_i(x)$ are the trial functions and the constants c_i are to be determined. Recall the statement of the operator eigenvalue problem resulting from separation of variables, as given in Equation (10.7). Rewriting Equation (10.7) with the mass density placed on the right-hand side yields

$$L\phi(x) = \lambda\rho\phi(x) \tag{13.20}$$

subject to the appropriate boundary conditions. Multiplying Equation (13.20) by $\phi(x)$ and integrating yields the Rayleigh quotient

$$\lambda = \frac{\int_0^\ell \phi(x) L\phi(x)\, dx}{\int_0^\ell \rho\phi(x)\phi(x)\, dx} = \frac{N}{D} \tag{13.21}$$

where

$$N = \int_0^\ell \phi(x) L\phi(x)\, dx \qquad \text{and} \qquad D = \int_0^\ell \rho\phi(x)\phi(x)\, dx$$

The Ritz approximation process is to substitute Equation (13.19) into Equation (13.21) and compute the coefficients c_i that minimize the Rayleigh quotient given by Equation (13.21).

Differentiating Equation (13.21) with respect to the coefficients c_i yields

$$\frac{\partial\lambda}{\partial c_i} = \frac{D\left(\dfrac{\partial N}{\partial c_i}\right) - N\left(\dfrac{\partial D}{\partial c_i}\right)}{D^2} = 0 \Rightarrow \frac{\partial N}{\partial c_i} - \lambda\frac{\partial D}{\partial c_i} = 0 \tag{13.22}$$

since D is never zero. Equation (13.22) computes the values of the expansion coefficients that minimize the Rayleigh quotient and hence allow the approximation of the eigenvalues. Next, consider writing N and D in terms of the constants c_i using Equation (13.19). With a little manipulation, it can be shown (see problem 13.11) that Equation (13.22) is the generalized eigenvalue–eigenvector problem

$$K\mathbf{c} = \lambda M\mathbf{c} \tag{13.23}$$

where the column vector $\mathbf{c}$ consists of the expansion coefficients c_i . The elements of the 'mass' and 'stiffness' matrix are given by

$$k_{ij} = \int_0^\ell \phi_i(x) L\phi_j(x)\, dx \qquad \text{and} \qquad m_{ij} = \int_0^\ell \rho\phi_i(x)\phi_j(x)\, dx \tag{13.24}$$

The solution of the generalized eigenvalue problem (13.23) yields an approximation of the eigenvalues λ_n and hence the natural frequencies. The eigenvectors $\mathbf{c}$ approximate the eigenfunctions of the system and hence the mode shapes. The number of approximated frequencies and mode shapes is N, the number of trial functions used in Equation (13.19).

The power of this approach is in finding approximate solutions when Equation (13.20) cannot be solved analytically, such as for odd boundary conditions and/or for spatially varying coefficients such as $\rho(x)$ and $EI(x)$. Note that, if exact eigenfunctions are used, exact frequencies result. Also note that the boundary conditions come into play when evaluating the integrals in Equation (13.24).

13.4 FINITE ELEMENT METHOD

Probably the most popular method of representing distributed-mass structures is the *finite element method* (FEM). This section presents a very brief introduction to the topic. A classic

reference for FEM is Hughes (2000). The method divides the structure of interest into subsections of finite size, called *finite elements*. These elements are connected to adjacent elements at various points on their boundaries, called *nodes*. Once this procedure is finished, the distributed-mass structure is represented by a finite number of nodes and elements referred to as a finite element *grid*, or *mesh*.

The displacement of each element is approximated by some function of the spatial variables between nodes. The next step in the finite element analysis (often abbreviated FEA) is to calculate the energy in each element as a function of the displacement. The total energy of the structure is then expressed as the sum of the energy in each element. External forces are included by using the principle of virtual work to derive forces per element. Lagrange's equations (see, for instance, Meirovitch, 2001) are then applied to the total energy of the structure, which yields the approximate equations of motion. These equations are finite-dimensional. This procedure is illustrated in the following example.

Example 13.4.1

This example considers the longitudinal vibration of a bar of length ℓ and derives a finite element stiffness matrix of the bar. The bar of Figure 13.1 is configured as one finite element with a node at each end. The axial stiffness is regarded as time independent throughout the element, so that the displacement must satisfy

$$EA\frac{d^2u(x)}{dx^2} = 0, \qquad x \in (0, \ell) \tag{13.25}$$

Integrating this expression yields

$$u(x) = c_1 x + c_2, \qquad x \in (0, \ell) \tag{13.26}$$

where c_1 and c_2 are constants of integration. At each node, the value of u is allowed to be a time-dependent coordinate denoted by $u_1(t)$, as labeled in the figure. Using these as boundary conditions, the constants c_1 and c_2 are evaluated to be

$$c_2 = u(t) \tag{13.27}$$

$$c_1 = \frac{u_2(t) - u_1(t)}{\ell} \tag{13.28}$$

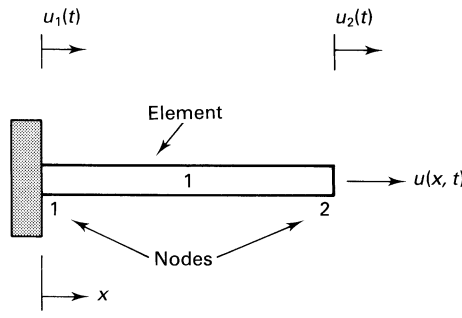

Figure 13.1 Two-node, one-element model of a cantilevered beam.

so that $u(x, t)$ is approximated by

$$u(x, t) = \left(1 - \frac{x}{\ell}\right) u_1(t) + \frac{x}{\ell} u_2(t) \tag{13.29}$$

Next, the nodal forces f_1 and f_2 are related to the displacement $u(x)$ by

$$EAu'(0) = -f_1, \qquad EAu'(\ell) = f_2 \tag{13.30}$$

or

$$EA\frac{u_2 - u_1}{\ell} = -f_1, \qquad EA\frac{u_2 - u_1}{\ell} = f_2 \tag{13.31}$$

where the prime indicates differentiation with respect to x. This last expression can be written in the matrix form

$$K\mathbf{u} = \mathbf{f} \tag{13.32}$$

where $\mathbf{u}(t) = [u_1(t) \quad u_2(t)]^T$, $\mathbf{f} = [f_1(t) \quad f_2(t)]^T$ and

$$K = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{13.33}$$

Here, the vector $\mathbf{u}(t)$ is called the *nodal displacement vector*, the vector $\mathbf{f}(t)$ is called the *nodal force vector*, and the matrix K is the *element stiffness matrix*.

In example 13.4.1, note that the displacement in the element is written in the form

$$u(x, t) = a_1(x)u_1(t) + a_2(x)u_2(t) = \mathbf{a}^T(x)\mathbf{u}(t) \tag{13.34}$$

where $\mathbf{a}(x) = [a_1(x) \;\; a_2(x)]^T$. The functions $u_1(t)$ and $u_2(t)$ are the time-dependent nodal displacements, and in example 13.4.1 they approximate $u(0, t)$ and $u(\ell, t)$ respectively. The functions $a_1(x)$ and $a_2(x)$ are called *shape functions*, or *interpolation functions*. In the example, $a_1(x) = (1 - x/\ell)$ and $a_2(x) = (x/\ell)$. However, the shape functions are not unique in general. They are referred to as interpolation functions because they allow the displacement to be specified, or interpolated, at points along the structure that lie between nodes. As will be illustrated in the following, the solution of the dynamic finite element equations yields only the nodal displacements $u_1(t)$ and $u_2(t)$.

Next, a dynamic model is needed. A mass matrix is required that is consistent with the preceding stiffness matrix for the bar element. The mass matrix can be determined from an expression for the kinetic energy of the element, denoted by $T(t)$ and defined by

$$T(t) = \frac{1}{2}\int_0^\ell \rho(x)u_t(x, t)^2\, dx \tag{13.35}$$

Substitution of $u_t(x, t)$ from Equation (13.32) yields

$$T(t) = \frac{1}{2}\int_0^\ell \rho(x)\dot{\mathbf{u}}^T(t)\mathbf{a}(x)\mathbf{a}^T(x)\dot{\mathbf{u}}^T(t)\, dx \tag{13.36}$$

or

$$T(t) = \frac{1}{2}\dot{\mathbf{u}}^T(t)\left[\int_o^\ell \rho(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx\right]\dot{\mathbf{u}}(t) \tag{13.37}$$

The expression in brackets is clearly a matrix that is defined as the element mass matrix, denoted by M. Examination of Equation (13.37) indicates that the mass matrix is given by

$$M = \int_0^\ell \rho(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx \tag{13.38}$$

Since the mass matrix is calculated by using the same shape functions as the stiffness matrix, the resulting mass matrix is called a *consistent mass matrix*. An alternative means of constructing the mass matrix is just to lump the mass of the structure at the various nodes. If this is done, the result is called an *inconsistent mass matrix*.

Note that the stiffness matrix of Equation (13.36) can also be represented in terms of the *shape* functions $\mathbf{a}(t)$. Examination of the potential energy in the system yields (for the bar of example 13.4.1)

$$K = \int_0^\ell EA(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx \tag{13.39}$$

With K defined by Equation (13.31), the potential energy per element, denoted by $V(t)$, is given by

$$V(t) = \frac{1}{2}\mathbf{u}^T(t)K\mathbf{u}(t) \tag{13.40}$$

Example 13.4.2

Calculate the consistent mass matrix for the bar element of example 13.4.1. Substituting the shape functions of Equation (13.29) into equation (13.38) yields

$$M = \rho\int_0^\ell \left(\begin{bmatrix} \frac{1-x}{\ell} \\ \frac{x}{\ell} \end{bmatrix}\begin{bmatrix} \frac{1-x}{\ell} & \frac{x}{\ell} \end{bmatrix}\right) dx = \frac{\rho\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13.41}$$

These definitions of the finite element mass and stiffness matrix can be assembled by using the potential and kinetic energies along with Lagrange's equations to formulate the approximate equations of a distributed parameter structure.

Recall that Lagrange's equations (see, for instance, Thomson, 1988) simply state that the equations of motion of an n-degree-of-freedom structure with coordinates u_i can be calculated from the energy of the structure by

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_i}\right)\right] - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} = f_i \tag{13.42}$$

where the f_i denote external forces.

With Lagrange's equations, the equations of motion of a structure modeled by one or more finite elements can be derived. This is done by first modeling the structure of interest as several finite elements (like the bar of examples 13.4.1 and 13.4.2). Next, the total energy of each element is added to produce the total energy of the distributed structure. Then Lagrange's equations are applied to produce the dynamic equations for the structure. The procedure is best illustrated by the following example.

Example 13.4.3

Again, consider the bar element of examples 13.4.1 and 13.4.2 and use these to model the vibration of a cantilevered bar. In this example, the clamped free bar will be modeled by three (an arbitrary choice) finite elements – and hence four nodes – as depicted in Figure 13.2. Note that, because of the clamped boundary condition, $u_1(t)=0$. Taking this into consideration, the total potential energy, denoted by $V_T(t)$, is the sum of the potential energy in each element:

$$V_T(t)=\sum_{i=1}^{3} V_i(t) \tag{13.43}$$

With $\ell/3$ substituted for ℓ in Equation (13.33) and the appropriate displacement vector $\mathbf{u}$, $V_T(t)$ becomes

$$\begin{aligned} V_T(t)=&\frac{3EA}{2\ell}\begin{bmatrix}0\\u_2\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}0\\u_3\end{bmatrix}+\frac{3EA}{2\ell}\begin{bmatrix}u_2\\u_3\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_2\\u_3\end{bmatrix}\\ &+\frac{3EA}{2\ell}\begin{bmatrix}u_3\\u_4\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_3\\u_4\end{bmatrix}\end{aligned} \tag{13.44}$$

Calculating the derivatives of V with respect to u_i yields

$$\begin{bmatrix}\dfrac{\partial V_T}{\partial u_2}\\ \dfrac{\partial V_T}{\partial u_3}\\ \dfrac{\partial V_T}{\partial u_4}\end{bmatrix}=\frac{3EA}{\ell}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{bmatrix}u_2\\u_3\\u_4\end{bmatrix} \tag{13.45}$$

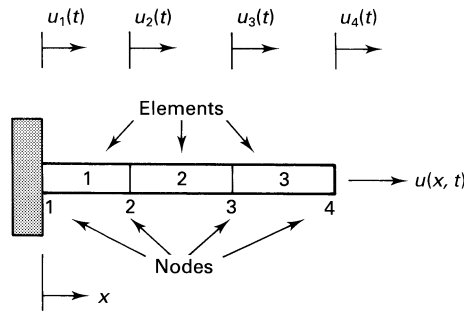

Figure 13.2 Four-node, three-element model of a cantilevered beam.

where the coefficient of the displacement vector $\mathbf{u} = [u_2\ u_3\ u_4]^T$ is the global stiffness matrix, K, for the entire structure based on a three-element finite approximation.

Calculation of the total kinetic energy $T(t)$ yields

$$T = \frac{1}{2}\frac{\rho A \ell}{18}\left\{\begin{bmatrix} 0 \\ \dot{u}_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} \dot{u}_3 \\ \dot{u}_4 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_3 \\ \dot{u}_4 \end{bmatrix}\right\} \tag{13.46}$$

Calculation of the various derivatives of T required for Lagrange's equations yields

$$\begin{bmatrix} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_2}\right) \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_3}\right) \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_4}\right) \end{bmatrix} = \frac{\rho A \ell}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} \tag{13.47}$$

where the coefficient of $\ddot{\mathbf{u}}$ is the consistent mass matrix of the three-element finite element approximation.

Substitution of Equations (13.45) and (13.47) into Lagrange's equation [Equation (13.42)] yields the three-degree-of-freedom model of the undamped bar as

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0} \tag{13.48}$$

This last expression can be solved for the vibration response of the undamped bar at the nodal point. The response between nodes can be interpolated by using the shape functions (or interpolation functions), i.e., $u(x, t) = \mathbf{a}^T\mathbf{u}$.

These procedures can be generalized to any type of distributed-mass structure or combination of structures. The matrices M and K that result are similar to those that result from the Rayleigh–Ritz method. In fact, the finite element method can be thought of as a piecewise version of the Rayleigh–Ritz method. For an accurate representation of a response, 10–20 elements per wavelength of the highest frequency of interest must be used.

13.5 SUBSTRUCTURE ANALYSIS

A distributed-mass structure often yields a large-order finite element model with hundreds or even thousands of nodes. This is especially true of large, complicated, and/or very flexible structures. Substructure analysis is a method of predicting the dynamic behavior of such a complicated large-order system by first dividing the model up into several parts, called substructures, and analyzing these smaller parts first. The dynamic solution of each substructure is then combined to produce the response of the entire structure.

Let the n-dimensional vector $\mathbf{x}$ denote the coordinates of a large finite element model. First, divide the structure up into parts according to the modal coordinates via the following scheme:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{13.49}$$

Here, $\mathbf{x}_1$ represents those nodes associated with the first substructure, and $\mathbf{x}_2$ represents the nodes associated with the second substructure. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be further partitioned into those coordinates that are unique to substructure 1 and those that are common to $\mathbf{x}_1$ and $\mathbf{x}_2$. Divide $\mathbf{x}_1$ into internal coordinates $\mathbf{x}_{1i}$ and common coordinates $\mathbf{x}_c$, i.e., $\mathbf{x}_1^T = \left[\mathbf{x}_{1i}^T \;\; \mathbf{x}_c^T\right]^T$. Likewise, the nodal coordinates for the second substructure, $\mathbf{x}_2$, are partitioned as

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.50}$$

The subset of nodes $\mathbf{x}_c$ is the same in both $\mathbf{x}_1$ and $\mathbf{x}_2$.

Next, partition the mass and stiffness matrices for each of the two (could be $N < n$) parts according to internal ($\mathbf{x}_{2i}$) and external ($\mathbf{x}_c$) coordinates. Let T_1 and V_1 denote the kinetic energy and potential energy, respectively, in substructure 1. These energies are

$$T_1 = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{x}}_{1i} \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} M_{ii}(1) & M_{ic}(1) \\ M_{ci}(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_{1i} \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.51}$$

$$V_1 = \frac{1}{2}\begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} K_{ii}(1) & K_{ic}(1) \\ K_{ci}(1) & K_{cc}(1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.52}$$

Likewise, the energy in substructure 2 is

$$T_2 = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{x}}_{2i} \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} M_{ii}(2) & M_{ic}(2) \\ M_{ci}(2) & M_{cc}(2) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_{2i} \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.53}$$

$$V_2 = \frac{1}{2}\begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} K_{ii}(2) & K_{ic}(2) \\ K_{ci}(2) & K_{cc}(2) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.54}$$

Next, the modes $\mathbf{u}_i$ of each substructure are calculated by assuming that the common coordinates (also called connecting nodes) are free and not really constrained by the rest of the structure, i.e., that the coordinates satisfy the equation of motion

$$\begin{bmatrix} M_{ii}(j) & M_{ic}(j) \\ M_{ci}(j) & M_{cc}(j) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_{2i} \\ \ddot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} K_{ii}(j) & K_{ic}(j) \\ K_{ci}(j) & K_{cc}(j) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} = \mathbf{0} \tag{13.55}$$

for each substructure ($j = 1, 2$). Equation (13.55) is obtained by using the energy expressions of Equations (13.51) through (13.54) substituted into Lagrange's equations. Each of the dynamic substructure equations (13.55) is next solved for the system eigenvalues and eigenvectors. Let $[\phi_{1i}^T(n) \;\; \phi_c^T(n)]^T$ denote the nth eigenvector of substructure 1. The modal matrix of substructure 1, denoted by $\phi(1)$, is the square matrix defined by

$$\phi(1) = \begin{bmatrix} \phi_{1i}(1) & \phi_{1i}(2) \cdots \phi_{1i}(n) \\ \phi_c(1) & \phi_c(2) \cdots \phi_c(n) \end{bmatrix} = \begin{bmatrix} \phi_i(1) \\ \phi_c \end{bmatrix}$$

where $\phi_i(1)$ and ϕ_c are rectangular matrix partitions of $\phi(1)$. These partitions are used to define a new coordinate $\mathbf{q}(1)$ by

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix} = \begin{bmatrix} \phi_i(1) \\ \phi_c \end{bmatrix} \mathbf{q}(1) \tag{13.56}$$

This yields

$$\mathbf{x}_{1i} = \phi_i(1)\mathbf{q}(1) \tag{13.57}$$

where it should be noted that $\phi_i(1)$, a rectangular matrix, relates the internal coordinates of substructure 1, $\mathbf{x}_{1i}$, to the new coordinate $\mathbf{q}(1)$ yet to be determined. This procedure can be repeated using the information from the second substructure to determine $\phi_i(2)$ and to define $\mathbf{q}(2)$. These quantities are related by

$$\mathbf{x}_{2i} = \phi_i(2)\mathbf{q}(2) \tag{13.58}$$

The substitution of Equation (13.57) into the expressions for the energy [Equations (13.51) and (13.52)] yields

$$T(1) = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}(1) \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} \phi_i^T(1)M_{ii}(1)\phi_i(1) & \phi_i^T(1)M_{ic}(1) \\ M_{ci}(1)\phi_i(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}(1) \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.59}$$

$$V(1) = \frac{1}{2}\begin{bmatrix} \mathbf{q}(1) \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} \phi_i^T(1)K_{ii}(1)\phi_i(1) & \phi_i^T(1)K_{ic}(1) \\ K_{ci}(1)\phi_i(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \mathbf{q}(1) \\ \mathbf{x}_c \end{bmatrix} \tag{13.60}$$

Similar expressions are obtained for the energies of the second substructure, $T(2)$ and $V(2)$, by substitution of Equation (13.58) into Equations (13.53) and (13.54). The total energy in the complete structure is now considered to be defined by $[T(1) + T(2)]$ and $[V(1) + V(2)]$.

These energy expressions are substituted into Lagrange's equations to produce the equations of motion in terms of substructure quantities. Lagrange's equations for the system are

$$\begin{bmatrix} \phi_i^T(1)M_{ii}(1)\phi_i(1) & 0 & \phi_i^T(1)M_{ic}(1) \\ 0 & \phi_i(2)M_{ii}(2)\phi_i(2) & \phi_i^T(2)M_{ic}(2) \\ M_{ci}(1)\phi_i(1) & M_{ci}(2)\phi_i(2) & M_{cc}(1)+M_{cc}(2) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}(1) \\ \ddot{\mathbf{q}}(2) \\ \ddot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} \phi_i^T(1)K_{ii}(1)\phi_i(1) & 0 & \phi_i^T(1)K_{ic}(1) \\ 0 & \phi_i(2)K_{ii}(2)\phi_i(2) & \phi_i^T(2)K_{ic}(2) \\ K_{ci}(1)\phi_i(1) & K_{ci}(2)\phi_i(2) & K_{cc}(1)+K_{cc}(2) \end{bmatrix} \begin{bmatrix} \mathbf{q}(1) \\ \mathbf{q}(2) \\ \mathbf{x}_c \end{bmatrix} = \mathbf{0} \tag{13.61}$$

This last expression constitutes a substructure representation of the original structure. The solution of Equation (13.61) is determined by any of the methods discussed in Chapter 3. The matrix coefficients are determined by analyzing each substructure independently. Equations (13.57) and (13.50) are used to recover the solution in physical coordinates from the solution of the substructure equations given by (13.61). Each of the quantities in Equation (13.61) are determined by solving the two substructures separately. Each of these is of an order less than the original system. Equation (13.61) is also of an order less than the original structure. In fact, it is of order n minus the order of $\mathbf{x}_c$. Hence, the response of the entire structure $\mathbf{x}_n$ can be obtained by analyzing several systems of smaller order.

13.6 TRUNCATION IN THE PRESENCE OF CONTROL

The majority of practical control schemes are implemented by actuators and sensors that are fixed at various points throughout the structure and hence behave fundamentally as lumped-mass elements rather than as distributed-mass elements. In addition, most control algorithms are based on finite-dimensional lumped-mass models of small order. Thus, it is quite natural to use a 'truncated' model or other finite-dimensional approximation of distributed-mass structures when designing control systems for them.

This section examines the problem of controlling the vibrations of a distributed-mass structure by using a finite number of lumped-mass actuators and sensors acting at various points on the structure. The approach discussed here is first to cast the structure into an infinite-dimensional matrix equation that is transformed and then truncated. A combination of modal methods and impedance methods is used to solve a simple structural control problem. The goal of the section is to present a simple, representative method of reducing vibration levels in flexible mechanical structures.

Consider a distributed-mass structure described by a partial differential equation of the form

$$Ly(x,t) = f(x,t), \qquad x \in \Omega \tag{13.62}$$

and associated boundary and initial conditions. Here, the functions $y(x,t)$ and $f(x,t)$ are in $\mathcal{L}_2^R(\Omega)$, $y(x,t)$ being the system output, and $f(x,t)$ the system input. This model is an abbreviated formulation of the structures presented in Chapter 9. In terms of the notation of Chapter 9, the operator L is of the form

$$L = \frac{\partial^2}{\partial t^2}(\cdot) + L_1 \frac{\partial}{\partial t}(\cdot) + L_2(\cdot) \tag{13.63}$$

where the output equation is just $y(x,t) = w(x,t)$. If the operator L had an easily calculated inverse, the solution would be given by $y(x,t) = L^{-1}f(x,t)$. To that end, consider taking the Laplace or Fourier transform on the temporal variable of Equation (13.62). This yields

$$Ly(x,s) = f(x,s) \tag{13.64}$$

plus boundary conditions. For ease of notation, no special distinction will be made between y in the time domain and y in the s domain (i.e., between y and its transform), as the remainder of the section deals only with the transformed system.

The control problem of interest here is one that could be implemented by sensors and actuators acting at discrete points along the structure, which may have dynamics of their own. Suppose that the structure is measured at m points along its length, labeled by x_i'. Let $y(s)$ denote an $m \times 1$ column vector with the ith component defined as $y(x_i', s)$, i.e., the time-transformed output (displacement) measured at point x_i'. In addition, r actuators are used to apply time-dependent forces $u_i(t)$, or transformed forces $u_i(s)$, at the r points x_i''. The control action, denoted by $f_c(x,s)$, can be written as

$$f_c(x,s) = \sum_{i=1}^{r} \delta(x - x_i'')u_i(s) = \mathbf{r}^T(x)\mathbf{u}(s) \tag{13.65}$$

Here, $\mathbf{r}(x)$ is an $r \times 1$ vector with ith component $\delta(x - x_i'')$, the Dirac delta function, and $\mathbf{u}(s)$ is an $r \times 1$ vector with ith component $u_i(s)$.

Negative feedback is used, so that the total force applied to the structure is given by

$$f(x, s) = f_{\text{ext}}(x, s) - f_c(x, s) \tag{13.66}$$

where $f_{\text{ext}}(x, s)$ represents an externally applied disturbance force and $f_c(x, s)$ represents the control forces. With the actuator just described, Equation (13.66) becomes

$$f(x, s) = f_{\text{ext}}(x, s) - \mathbf{r}^T \mathbf{u}(s) \tag{13.67}$$

To complete the feedback loop, $\mathbf{u}(s)$ must depend on the output, or measured response, of the structure. Let $H(s)$ be an $r \times m$ transfer matrix defining the dependence of the control action on the output via the expression

$$\mathbf{u}(s) = H(s)\mathbf{y}(s) \tag{13.68}$$

so that Equation (13.67) becomes

$$f(x, s) = f_{\text{ext}} - \mathbf{r}^T(x)H(s)\mathbf{y}(s) \tag{13.69}$$

This last expression represents output feedback control. An alternative here would be to use state feedback control, as discussed in Chapter 7.

Next, consider casting the problem into modal coordinates. Let $\{\Psi_i(x)\}$ be a set of basis functions in $\mathcal{L}_2^{\text{C}}(\Omega)$ and consider $\mathcal{L}\Psi_i(x)$, also in $L_2^{\text{C}}(\Omega)$. Then

$$L\Psi_i = \sum_{j=1}^{\infty} \lambda_{ij}(s)\Psi_j(x) \tag{13.70}$$

where $\lambda_{ij}(s)$ is an expansion coefficient. Note that, if $\lambda_{ij}(s) = 0$ for $i \neq j$, then Equation (13.70) becomes

$$L\Psi_i(x) = \lambda_i(s)\Psi_i(x) \tag{13.71}$$

so that the expansion coefficient, $\lambda_{ii}(s) = \lambda_i(s)$, is an eigenvalue of the operator L with eigenfunction $\Psi_i(x)$. The $\lambda_{ii}(s)$ are also called modal *impedances* (see Section 10.4 for conditions under which this is true).

Example 13.6.1

For a pinned–pinned uniform beam in transverse vibration of length ℓ with no damping

$$\Psi_n(x) = \frac{\sqrt{2}}{\ell}\sin(k_n x) \tag{13.72}$$

$$\lambda_n(s) = \left[k_n^4 + \frac{s^2}{c^2 k^2}\right] EI \tag{13.73}$$

where

$$k_n = \frac{n\pi}{\ell}, \qquad c^2 = \frac{E}{\rho}, \qquad k^2 = \frac{I}{A}$$

Expanding the functions $y(x, s)$ and $f(x, s)$ in terms of this same set of basis functions, $\{\Psi_i(x)\}$, yields

$$y(x, s) = \sum_{i=1}^{\infty} d_i(s)\Psi_i(s) \tag{13.74}$$

and

$$f(x, s) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) \tag{13.75}$$

The expansion coefficients $d_i(s)$ are called the modal response coefficients and the coefficients $c_i(s)$ are called the modal input coefficients.

Next, compute (assuming proper convergence, i.e., that Ψ_i is an eigenfunction of L)

$$Ly = \sum_{i=1}^{\infty} d_i(s)L\Psi_i(x) = f(x, s) \tag{13.76}$$

or

$$\sum_{i=1}^{\infty} \lambda_i(s)d_i(s)\Psi_i(x) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) \tag{13.77}$$

Note that for Equation (13.77) it is assumed that the $\Psi_i(x)$ are, in fact, the eigenfunctions of L.

Using the orthogonality of the $\{\Psi_i(x)\}$, Equation (13.77) implies that

$$\lambda_i(s)d_i(s) = c_i(s) \tag{13.78}$$

for each index i, so that

$$\lambda_i(s) = \frac{c_i(s)}{d_i(s)} \tag{13.79}$$

This gives rise to the interpretation of $\lambda_i(s)$ as a 'modal impedance'. Note also that, as before,

$$d_i(s) = \int_{\Omega} y(x, s)\Psi_i(x)\, d\Omega \tag{13.80}$$

and

$$c_i(s) = \int_{\Omega} f(x, s)\Psi_i(x)\, d\Omega \tag{13.81}$$

If L is not self-adjoint and/or the functions $\Psi_i(x)$ are not the normal modes of the system, then this procedure can be completed using the orthogonality of the complex set of basis functions. In this coupled case, substitution of Equation (13.70) into Equation (13.76) yields

$$\sum_{i=1}^{\infty} d_i(s)\left[\sum_{j=1}^{\infty} \lambda_{ij}(s)\Psi_j(x)\right] = \sum_{j=1}^{\infty} c_j(s)\Psi_j(x) \tag{13.82}$$

Multiplying Equation (13.82) by $\Psi_k^*(x)$, the conjugate of $\Psi_k(x)$, and integrating over Ω yields

$$\sum_{i=1}^{\infty} d_i(s)\lambda_{ik}(s) = c_k(s) \tag{13.83}$$

where the summation over the index j has been eliminated by the assumed orthogonality of the set $\{\Psi_k(x)\}$. Equation (13.83) constitutes the equivalent version of Equation (13.78) for the case in which the system does not possess classical normal modes.

Next, consider applying linear feedback control in a modal coordinate system defined by the set $\{\Psi_k(x)\}$. Equation (13.78) can be written as a single infinite-dimensional matrix equation of the form

$$\Lambda \mathbf{d} = \mathbf{c} \tag{13.84}$$

where Λ is the $\infty \times \infty$ modal impedance matrix with the ijth element defined by $\lambda_{ij}(s)$ and $\mathbf{c}$ and $\mathbf{d}$ are $\infty \times 1$ column matrices defined by $c_i(s)$ and $d_i(s)$ respectively. Defining $\mathbf{\Psi}(x)$ as the $\infty \times 1$ column matrix of eigenfunctions $\mathbf{\Psi}_i(x)$, the other relevant terms can be written as

$$f_{\text{ext}} = \sum_{i=1}^{\infty} e_i(s)\Psi_i(x) = \mathbf{e}^T(s)\mathbf{\Psi}(x) \tag{13.85}$$

$$f(x, s) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) = \mathbf{c}^T(s)\mathbf{\Psi}(x) \tag{13.86}$$

and

$$y(x, s) = \sum_{i=1}^{\infty} d_i(s)\Psi_i(x) = \mathbf{d}^T(s)\mathbf{\Psi}(x) \tag{13.87}$$

where the various column vectors have the obvious definitions. For instance, the vector $\mathbf{e}(s)$ is the vector of expansion coefficients for the external disturbance force f_{ext} with components $e_i(s)$, and so on.

A measurement matrix, denoted by M, can be defined by

$$M_{ij} = \Psi_j(x_i') \tag{13.88}$$

which is an $m \times \infty$ matrix and relates $\mathbf{d}(s)$ directly to $\mathbf{y}(s)$ by

$$\mathbf{y}(s) = M\mathbf{d}(s) \tag{13.89}$$

Likewise, an $r \times \infty$ modal coefficient matrix, denoted by R, can be defined by

$$R_{ij} = \int_{\Omega} r_i(x)\Psi_j(x)\ d\Omega \tag{13.90}$$

which relates $\mathbf{r}(x)$ to $\Psi(x)$ by

$$\mathbf{r}(x) = R\boldsymbol{\Psi}(x) \tag{13.91}$$

Using the orthogonality of $\Psi_i(x)$, the inner product

$$\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega = I_{\infty} \tag{13.92}$$

where I_{∞} denotes the $\infty \times \infty$ identity matrix with elements $\int_{\Omega} \Psi_i(x)\Psi_{\varphi}(x)\ d\Omega = \delta_{ij}$. Note that, if $\boldsymbol{\Psi}(x)$ is complex, then the transpose should be interpreted as the conjugate transpose. Multiplying Equation (13.91) by $\boldsymbol{\Psi}^T$ from the right and integrating over Ω yields

$$R = \int_{\Omega} \mathbf{r}(x)\boldsymbol{\Psi}^T(x)\ d\Omega \tag{13.93}$$

This last expression provides a more useful definition of the modal coefficient matrix R.

A relationship between R, $\mathbf{c}$, and $\mathbf{e}$ can be found by substituting Equations (13.85) and (13.86) into Equation (13.67). This yields

$$\mathbf{c}^T\boldsymbol{\Psi} = \mathbf{e}^T\boldsymbol{\Psi} - \mathbf{r}^T\mathbf{u} \tag{13.94}$$

Since $\mathbf{r}^T\mathbf{u}$ is a scalar, this can also be written as

$$\mathbf{c}^T(s)\boldsymbol{\Psi}(x) = \mathbf{e}^T(s)\boldsymbol{\Psi}(x) - \mathbf{u}^T(s)\mathbf{r}(x) \tag{13.95}$$

Multiplication from the right by $\Psi^T(x)$ and integrating over Ω yields

$$\mathbf{c}^T(s)\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega = \mathbf{e}^T(s)\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega - \mathbf{u}^T(s)\int_{\Omega} \mathbf{r}(x)\boldsymbol{\Psi}^T(x)\ d\Omega \tag{13.96}$$

Using Equations (13.92) and (13.93) then yields

$$\mathbf{c}^T(s) = \mathbf{e}^T(s) - \mathbf{u}^T(s)R \tag{13.97}$$

or

$$\mathbf{c}(s) = \mathbf{e}(s) - R^T\mathbf{u}(s) \tag{13.98}$$

Equation (13.84) now becomes

$$\Lambda\mathbf{d}(s) = \mathbf{e}(s) - R^T\mathbf{u}(s) \tag{13.99}$$

or upon substitution of Equation (13.68) for $\mathbf{u}(s)$

$$\Lambda\mathbf{d}(s) = \mathbf{e}(s) - R^T H(s)\mathbf{y}(s) \tag{13.100}$$

Using Equation (13.89), the last term in Equation (13.100) can be placed in terms of $\mathbf{d}(s)$ to yield

$$\Lambda\mathbf{d}(\mathbf{s}) = \mathbf{e}(\mathbf{s}) - R^T H(\mathbf{s}) M \mathbf{d}(s) \tag{13.101}$$

Assuming that Λ^{-1} exists, this last expression can be manipulated to yield

$$\left[I_\infty + \Lambda^{-\mathbf{1}} Q(s)\right] \mathbf{d}(s) = \Lambda^{-1}\mathbf{e}(s) \tag{13.102}$$

where $Q(s) = R^T H(s) M$. Equation (13.102) represents the closed-loop configuration for the output feedback control of a distributed-parameter structure in terms of infinite-dimensional matrices.

If the infinite matrix inverse $\Lambda^{-\mathbf{1}}$ exists, if the inverse of the impedance matrix $[I_\infty + \Lambda^{-1}Q]$ can be calculated, and if the functions $\{\Psi_i(x)\}$ are known, Equation (13.102) along with Equation (13.74) yields the response $\mathbf{d}(s)$ in terms of the input, $\mathbf{e}(s)$. Several common examples, such as uniform beams and plates of simple geometry, satisfy these assumptions. Unfortunately, in many practical cases these assumptions are not satisfied, and the matrix Λ must be truncated in some fashion. Even in cases where $\Lambda^{-\mathbf{1}}$ can be calculated, the control $Q(s)$ may be such that $[I_\infty + \Lambda^{-\mathbf{1}}Q]$ is difficult to calculate. In cases where truncation of the model is required, Equation (13.102) provides a convenient formula for studying the effects of truncation in the presence of control.

As was true for the procedure of Section 7.8, the truncation method presented here is based on partitioning the various infinite-dimensional matrices of Equation (13.102). Let $\Lambda_{n\infty}^{-\mathbf{1}}$ denote the matrix formed from the matrix $\Lambda^{-\mathbf{1}}$ by partitioning off the first n rows and all the columns. Using this notation, the matrix $\Lambda^{-\mathbf{1}}$ is partitioned as

$$\Lambda^{-1} = \begin{bmatrix} \Lambda_{nm}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & \Lambda_{\infty\infty}^{-1} \end{bmatrix} \tag{13.103}$$

In a similar fashion, the matrices M, R, and Q are partitioned as

$$M = [M_{mn}\, M_{m\infty}] \tag{13.104}$$

$$R^T = \begin{bmatrix} R_{nr}^T \\ R_{\infty r}^T \end{bmatrix} \tag{13.105}$$

and

$$Q = \begin{bmatrix} Q_{nn} & Q_{n\infty} \\ Q_{\infty n} & Q_{\infty\infty} \end{bmatrix} \tag{13.106}$$

The submatrices of Q can all be written in terms of R, M, and H as

$$Q_{nn} = R_{nr}^T H M_{mn} \tag{13.107}$$

$$Q_{n\infty} = R_{nr}^T H M_{m\infty} \tag{13.108}$$

$$Q_{\infty n} = R_{\infty r}^T H M_{mn} \tag{13.109}$$

and

$$Q_{\infty\infty} = R_{\infty r}^T H M_{m\infty} \tag{13.110}$$

Substitution of these partitioned matrices into Equation (13.102) yields the partitioned system

$$\begin{bmatrix} I_n + \Lambda_{mn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n} & \Lambda_{nn}^{-1} Q_{n\infty} + \Lambda_{n\infty}^{-1} Q_{\infty\infty} \\ \Lambda_{\infty n}^{-1} Q_{nn} + \Lambda_{\infty\infty}^{-1} Q_{\infty n} & I_\infty + \Lambda_{\infty n}^{-1} Q_{n\infty} + \Lambda_{\infty\infty}^{-1} Q_{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{d}_n \\ \mathbf{d}_\infty \end{bmatrix} = \begin{bmatrix} \Lambda_{nn}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & \Lambda_{\infty\infty}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{e}_n \\ \mathbf{e}_\infty \end{bmatrix} \tag{13.111}$$

Here, the response vector $\mathbf{d}$ and the input vector $\mathbf{e}$ have also been partitioned, dividing these infinite-dimensional vectors into an $n \times 1$ finite-dimensional part and an $\infty \times 1$ infinite-dimensional part.

The various partitions of Equation (13.103) can be used to interpret the effects of truncating the modal description of a structure at n modes in the presence of a control law. Structures are generally thought of as low-pass filters in the sense that $\lambda_n^{-1} \to 0$ as $n \to \infty$. Thus, for structures it is reasonable to assume that the matrix $\Lambda_{\infty\infty}^{-1}$ is zero for some value of n.

Sensors often behave like low-pass filters as well, so that it is also reasonable to assume that $M_{n\infty}$ is the zero matrix. This, in turn, causes $Q_{n\infty} = Q_{\infty\infty} = 0$. If the actuators are slow enough, it can also be argued that $R_{\infty r}^T = 0$, which causes $Q_{\infty n} = Q_{\infty\infty} = 0$. With these three assumptions, the system of Equation (13.111) is reduced to

$$\begin{bmatrix} I_n + \Lambda_{nn}^{-1} Q_{nn} & 0 \\ \Lambda_{\infty n}^{-1} Q_{nn} & I_\infty \end{bmatrix} \begin{bmatrix} \mathbf{d}_n \\ \mathbf{d}_\infty \end{bmatrix} = \begin{bmatrix} \Lambda_{nn}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_n \\ \mathbf{e}_\infty \end{bmatrix} \tag{13.112}$$

This can be written as the two coupled vector equations

$$(I_n + \Lambda_{nn}^{-1} Q_{nn})\mathbf{d}_n = \Lambda_{nn}^{-1}\mathbf{e}_n + \Lambda_{n\infty}\mathbf{e}_\infty \tag{13.113}$$

and

$$\Lambda_{\infty n}^{-1} Q_{nn}\mathbf{d}_n + \mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n \tag{13.114}$$

These last two equations provide a simple explanation of some of the problems encountered in the control of distributed-mass structures using truncated models.

First, consider the case with $\mathbf{e}_\infty = \mathbf{0}$. This corresponds to a 'band-limited' input. That is, the external disturbance provides energy only to the first n modes. In this case, Equation (13.105) becomes

$$\mathbf{d}_n = (I_n + \Lambda_{nn}^{-1} Q_{nn})^{-1} \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.115}$$

Equation (13.115) can now be used to solve the control problem, i.e., to calculate Q_{nn} such that the response $\mathbf{d}_n$ has a desired form. In fact, Equation (13.115) is equivalent to first approximating a distributed-parameter system by a finite-dimensional system and then designing a finite-dimensional control system for it. However, this is slightly misleading, as can be seen by considering Equation (13.114).

Rearrangement of Equation (13.114) yields

$$\mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n - \Lambda_{\infty n}^{-1} Q_{nn}\mathbf{d}_n \tag{13.116}$$

This states that, unless the dynamics of the structure decouple (i.e., $\Lambda_{\infty n}^{-1} = 0$), or unless it can be argued that $\Lambda_{\infty n}^{-1}$ is small, the higher, uncontrolled modes of the response $\mathbf{d}_\infty$ will be excited by the control action, Q_{nn}. Such unwanted excitation is called *control spillover*.

In the case where $\Lambda_{\infty n}^{-1}$ is close to zero, Equation (13.115) provides a good approximation to the control problem for distributed-mass structures. In fact, the requirement that $\mathbf{e}_\infty = \mathbf{0}$ provides a criterion for determining the proper order, n, to be chosen for the approximation for a given disturbance. The value of n is chosen so that $\mathbf{e}_\infty$ is approximately zero.

Next, consider the case where the sensors are not low-pass filters, i.e., $M_{m\infty} \neq 0$, so that $Q_{n\infty} \neq 0$. In this case the first partition of Equation (13.111) yields

$$(I_n + \Lambda_{nn}^{-1} Q_{nn})\mathbf{d}_n + \Lambda_{nn}^{-1} Q_{n\infty}\mathbf{d}_\infty = \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.117}$$

The equation describing $\mathbf{d}_n$ is recoupled to the truncated dynamics constrained in the vector $\mathbf{d}_\infty$. If the term $Q_{n\infty}$ is erroneously neglected and Equation (13.115) is used to design the control system, then the resulting solution $\mathbf{d}_n$ will be in error, and the resulting calculation of R_{nr}^T and H will be in error. The response will suffer from what is often referred to as *observation spillover*, meaning that the sensors have caused a coupling of the truncated system with the neglected modes, producing error in the closed-loop response.

A similar problem arises if the actuators are 'fast', i.e., if $R_{\infty r}^T \neq 0$. In this case, Equations (13.113) and (13.114) become

$$(I_n + \Lambda_{mn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n})\mathbf{d}_n = \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.118}$$

and

$$(\Lambda_{nn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n})\mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n \tag{13.119}$$

Again, the introduction of the term $Q_{\infty n}$, associated with high-speed actuator excitation, couples the equation for the solution $\mathbf{d}_n$ and the truncated, or residual, solution $\mathbf{d}_\infty$. Thus, if $R_{\infty n}^T$ is not actually zero and Equation (13.115) is used to compute the control law, error will result. The interpretation here is that $R_{\infty n}^T$ excites the neglected modes, $\mathbf{d}_\infty$, and hence causes energy to appear in the neglected part of the model. This again causes control spillover.

13.7 IMPEDANCE METHOD OF TRUNCATION AND CONTROL

The modal description of a structure presented in the previous section lends itself to an interpretation of potential problems encountered when using point actuators and sensors in designing a control system for a distributed-mass structure. In this section an alternative approach is presented that uses the modal equation [Equation (13.102)] but, rather than truncating the response, uses an impedance method to calculate the closed-loop response vector $\mathbf{d}$.

The sensor–actuator admittance (inverse of impedance) matrix $Y(s)$ is defined by

$$Y(s) = M\Lambda^{-1}(s)R^T \tag{13.120}$$

and is related to the dynamic stiffness matrix. Note that $Y(s)$ is a finite-dimensional matrix, the elements of which are infinite sums.

Consider again the infinite matrix description of the structural control problem, as formulated in Equation (13.102). This expression can be written as

$$I\mathbf{d} + \Lambda^{-1}R^T HM\mathbf{d} = \Lambda^{-1}\mathbf{e} \tag{13.121}$$

From Equation (13.89), the vector $M\mathbf{d}$ can be replaced with $\mathbf{y}$ to yield

$$I\mathbf{d} + \Lambda^{-1}R^T H\mathbf{y} = \Lambda^{-1}\mathbf{e} \tag{13.122}$$

Multiplying this expression by M yields

$$\mathbf{y} + M\Lambda^{-1}R^T H\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.123}$$

$$\mathbf{y} + Y(s)H(s)\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.124}$$

This can be written as

$$[I_m + Y(s)H(s)]\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.125}$$

where I_m is the $m \times m$ identity matrix. Thus, the coefficient of $\mathbf{y}$ is a finite-dimensional matrix. Assuming that the coefficient of $\mathbf{y}$ has an inverse, Equation (13.125) can be written as

$$\mathbf{y} = [I_m + Y(s)H(s)]^{-1}M\Lambda^{-1}\mathbf{e} \tag{13.126}$$

This expression can be substituted into Equation (13.122) to yield

$$\mathbf{d} = \{I_\infty - \Lambda^{-1}R^T H[I_m + Y(s)H(s)]^{-1}M\}\Lambda^{-1}\mathbf{e} \tag{13.127}$$

which expresses the system response, $\mathbf{d}$, in terms of the disturbance input, $\mathbf{e}$. Equation (13.127) represents the impedance method of dealing with truncation in the control of distributed-mass structures, as developed by Berkman and Karnopp (1969). The open-loop system, as represented by Λ, still needs to be truncated using the low-pass filter argument of the previous section, i.e., $\Lambda^{-1} \cong \Lambda_n^{-1}$. However, the feedback control portion is now finite-dimensional and of low order (i.e., equal to the number of measurement points or sensors). Hence, truncation or partitioning of an inverse matrix is required in order to compute the control. Instead, the elements of the matrix $[I_m + Y(s)H(s)]$, which are all infinite sums, can be first calculated and/or truncated. Then, the exact inverse can be calculated. Thus, the value of the truncation index for the structure and that for the control can be separately chosen. This approach also allows for the inclusion of actuator dynamics due to the presence of the matrix $H(s)$. The following example serves to clarify this method.

Example 13.7.1

Consider the transverse vibrations of a pined–pinned beam with an actuator providing dynamics (or an impedance), $Z_1(s)$, acting at point x_1. Also, consider a single displacement-measuring sensor,

located at x_1, so that $y(x)$ is the scalar quantity $w(x_1, s)$, i.e., so that the index $m = 1$. Let $\Psi_i(x)$ denote the modes (or eigenfunctions) of the pinned–pinned beam without the actuator attached. From Equation (13.88), the matrix M becomes the $1 \times \infty$ vector

$$M = [\Psi_1(x_1)\ \Psi_2(x_2)\ \cdots]\ \boldsymbol{\Psi}(x_1)$$

Likewise, from Equation (13.82) the matrix R becomes the $1 \times \infty$ vector

$$R = [\Psi_1(x_1)\ \Psi_2(x_1) \cdots]\boldsymbol{\Psi}(x_1)$$

The matrix $H(s)$ in this case is just the scalar element $H(s) = Z_1(s)$.

The sensor actuator admittance matrix becomes the scalar element

$$Y(s) = M\Lambda^{-1}R^T = \sum_{i=1}^{\infty} \lambda_I^{-1}(s)\Psi_i^2(x_I)$$

where the $\lambda_i(s)$ are the open-loop system eigenvalues, since $\Lambda(s)$ is diagonal in this example (i.e., $\Lambda(s)$ consists simply of the eigenvalues of the structure).

The response can now be computed from Equation (13.119) to be

$$\mathbf{d}(s) = \left[\Lambda^{-1}(s) - \frac{[\Lambda^{-1}\boldsymbol{\Psi}^T(x_1)][\Lambda^{-1}\boldsymbol{\Psi}^T(x)]^T}{1 + \sum_{i=1}^{\infty} \lambda_i^{-1}(s)\Psi_i^2(x_1)} \right] \mathbf{e}$$

It is important to note that the response (or, more exactly, the Laplace transform of the response) is calculated here by truncating (approximating) the structural dynamics $\Lambda^{-1}(s)$ and $\Lambda^{-1}\boldsymbol{\Psi}(x_1)$ independently of the control. The actuator representation, $Z_1(s)$, is not truncated at all in this example. This is in contrast to the completely modal approach of the previous section.

As the example illustrates, the modal impedance inversion technique described in this section reduces the problem of truncation in the presence of control from one of approximating an infinite-order matrix with a finite-order matrix to that of approximating infinite sums with partial finite summations.

CHAPTER NOTES

This chapter introduces some methods of approximating distributed-mass models of structures with lumped-mass models more suitable for digital computing. Section 13.2 introduced the obvious and popular method of modal truncation.

Modal methods are quite common and can be found in most texts. See, for instance, Meirovitch (1980, 2001), for a more complete discussion. The Ritz–Galerkin method of Section 13.3 is again very common and is found in most vibration texts at almost every level. The common name of Raleigh–Ritz has always been surrounded with a bit of controversy over who actually first penned the method, and this is nicely settled in Leissa (2005). The finite element method briefly discussed in Section 13.4 is currently the most often used

and written about method. An excellent short introduction to finite element methods can be found in Meirovitch (1986). A more comprehensive approach can be found in the excellent book by Shames and Dym (1985) or the more advanced treatment by Hughes (2000).

Meirovitch's (1980) book contains a complete treatment of the substructure methods discussed in Section 13.5, as does the paper by Hale and Meirovitch (1980). The paper by Craig (1987) reviews the related topic of component mode methods. Sections 13.6 and 13.7, dealing with the topic of truncation and control system design, are taken directly from the paper by Berkman and Karnopp (1969), which was one of the first papers written in this area. Many other approaches to this same problem can be found in the literature. The survey paper by Balas (1982) provides a useful introduction to the topic.

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PROBLEMS

13.1 Estimate the amount of energy neglected in a three-mode approximation of a fixed–fixed beam of length ℓ in the longitudinal vibration.

13.2 Calculate a three-mode approximation of a clamped square plate using modal truncation.

13.3 Use trigonometric functions and perform a Ritz–Galerkin approximation for a transversely vibrating beam (undamped) that is clamped at one end and attached to a spring with constant k and mass m at the other end. Use three terms. Calculate the natural frequencies and compare them with those obtained by the method of Section 12.4.

13.4 Compare the finite element model of example 13.4.1 with a three-mode Ritz–Galerkin model of the same structure. How do the eigenvalues compare with those of the distributed-parameter model?

13.5 Show that the matrices M and K defined by the finite element method of Section 13.4 are both symmetric (in general).

13.6 Derive the finite element matrix for a transversely vibrating beam modeled with three elements.

13.7 Consider a three-degree-of-freedom system with mass matrix $M = I$ and stiffness matrix

$$K = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1.5 & -.5 \\ 0 & -.5 & .5 \end{bmatrix}$$

that corresponds to three masses connected in series by three springs. Define two substructures by letting substructure 1 be the first two masses and substructure 2 be the remaining mass. Calculate the coefficient matrices of Equation (13.61).

13.8 Calculate $\lambda(s)$ for a cantilevered beam in transverse vibration. Use this information to calculate the matrix Λ^{-1}.

13.9 For problem (13.8), suppose that a disturbance force of $\sin 2t$ is applied to the structure at the midpoint and calculate the value of the index n such that $\mathbf{e}_\infty$ is negligible. For simplicity, set each of the physical parameter values to unity.

13.10 Recalculate the equations of example 13.7.1 using two elements as part of the control, i.e. $z_1(s)$ and $z_2(s)$, acting at points x_1 and x_2, respectively.

13.11 Derive Equations (13.23) and (13.24) for the case $N = 2$, by substituting the sum of Equation (3.19) into the Rayleigh quotient and taking the indicated derivatives. (Hint: Use the fact that L is self-adjoint and write the two derivative equations as one equation in matrix form.)

A
Comments on Units

This text omits units on quantities wherever convenient in order to shorten the presentation and to allow the reader to focus on the development of ideas and principles. However, in an application of these principles the proper use of units becomes critical. Both design and measurement results will be erroneous and could lead to disastrous engineering decisions if they are based on improper consideration of units. This appendix supplies some basic information regarding units, which will allow the reader to apply the techniques presented in the text to real problems.

Beginning physics texts define three fundamental units of length, mass, and time and consider all other units to be derived from these. In the *International System* of units, denoted by SI (from the French 'Système International'), the fundamental units are as follows:

Quantity	Unit name	Unit abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s

These units are chosen as fundamental units because they are permanent and reproducible quantities.

The United States has historically used length (in inches), time (in seconds), and force (in pounds) as the fundamental units. The conversion between the US Customary system of units, as it is called, and the SI units is provided as a standard feature of most scientific calculators and in many text and notebook covers. Some simple conversions are

$$1\,\mathrm{kg} = 2.204622622\,\mathrm{lb}$$

$$4.5\,\mathrm{N} \cong 1\,\mathrm{lbf}$$

$$2.54 \times 10^{-2}\,\mathrm{m} = 1\,\mathrm{in}$$

As most equipment and machines used in the United States in vibration design, control, and measurement are manufactured in US Customary units, it is important to be able to convert between the two systems. This is discussed in detail, with examples, by Thomson (1988).

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 ISBN: 0-470-01051-7

In vibration analysis, position is measured in units of length in meters (m), velocity in meters per second (m/s), and acceleration in meters per second squared (m/s^2). Since the basic equation of motion comes from a balance of force, each term in the equation

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \tag{A.1}$$

must be in units of force. Force is mass times acceleration, $kg\,m/s^2$. This unit is given the special name of Newton, abbreviated N (i.e., $1\,N = 1\,kg\,m/s^2$). The coefficients in equation (A.1) then must have the following units:

Quantity	Unit symbol	Unit abbreviation
Mass	m	kg
Damping	c	kg/s
Stiffness	k	kg/s^2

Note that stiffness may also be expressed in terms of N/m, and damping (viscous friction) in terms of N m/s, and these units are normally used. Using other formulae and definitions in the text, the following units and quantities can be derived (some are given special names because of their usefulness in mechanics):

Quantity	Unit name	Abbreviation	Definition
Force	newton	N	$kg\,m/s^2$
Velocity	—	—	m/s
Acceleration	—	—	m/s^2
Frequency	hertz	Hz	1/s
Stress	pascal	Pa	N/m^2
Work	joule	J	N m
Power	watt	W	J/s
Area moment of inertia	—	—	m^4
Mass moment of inertia	—	—	kg/m^2
Density	—	—	kg/m^3
Torque	—	—	N/m
Elastic modulus	—	—	Pa

Because vibration measurement instruments and control actuators are often electromechanical transducers, it is useful to recall some electrical quantities. The fundamental electrical unit is often taken as the ampere, denoted by A. Units often encountered with transducer specification are as follows.

Quantity	Unit name	Abbreviation	Definition
Electrical potential	volt	V	W/A
Electrical resistance	ohm	Ω	V/A
Capacitance	farad	F	A s/V
Magnetic flux	weber	Wb	V s
Inductance	henry	H	V s/N

The gains used in the control formulation for vibration control problems have the units required to satisfy the units of force when multiplied by the appropriate velocity or displacement term. For instance, the elements of the feedback matrix G_f of Equation (7.18) must have units of stiffness, i.e., N/m.

Often these quantities are too large or too small to be convenient for numerical and computer work. For instance, a newton, which is about equal to the force exerted in moving an apple, would be inappropriately small if the vibration problem under study is that of a large building. The meter, on the other hand, is too large to be used when discussing the vibration of a compact disc in a stereo system. Hence, it is common to use units with prefixes, such as the millimeter, which is 10^3 m, or the gigapascal, which is 10^9 Pa. Of course, the fundamental unit kilogram is 10^3 g. The following table lists some commonly used prefixes:

Factor	10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{-2}	10^{3}	10^{6}	10^{9}
Prefix	pico-	nano-	micro-	milli-	centi-	kilo-	mega-	giga-
Abbreviation	P	n	μ	m	c	k	M	G

For example, a common rating for a capacitor is microfarads, abbreviated μF.

Many of the experimental results given in the text are discussed in the frequency domain in terms of magnitude and phase plots. Magnitude plots are often given in logarithmic coordinates. In this situation, the *decibel*, abbreviated db, is used as the unit of measure. The decibel is defined in terms of a power ratio of an electrical signal. Power is proportional to the square of the signal voltage, so that the decibel can be defined as

$$1\,\text{db} = 20\log_{10}\frac{V_1}{V_2}$$

where V_1 and V_2 represent different values of the voltage signal (from an accelerometer, for instance).

The phase is given in terms of either degrees (°) or radians (rad).

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B

Supplementary Mathematics

B.1 VECTOR SPACE

Fundamental to the discipline of matrix theory as well as the operator theory of functional analysis is the definition of a *linear space*, also called a *vector space*. A linear space, denoted by V, is a collection of objects (vectors or functions in the cases of interest here) for which the following statements hold for all elements $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ (this denotes that the vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all constrained in the set V) and for any real-valued scalars α and β:

1. $\mathbf{x}+\mathbf{y} \in V, \qquad \alpha\mathbf{x} \in V.$
2. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}.$
3. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z}).$
4. There exists an element $\mathbf{0} \in V$ such that $0\mathbf{x}=\mathbf{0}$.
5. There exists an element $1 \in V$ such that $1\mathbf{x}=\mathbf{x}$.
6. $\alpha(\beta\mathbf{x})=(\alpha\beta)\mathbf{x}.$
7. $(\alpha+\beta)\mathbf{x}=\alpha\mathbf{x}+\beta\mathbf{x}.$
8. $\alpha(\mathbf{x}+\mathbf{y})=\alpha\mathbf{x}+\alpha\mathbf{y}.$

The examples of linear spaces V used in this text are the set of real vectors of dimension n, the set of complex vectors of dimension n, and the set of functions that are square integrable in the Lebseque sense.

B.2 RANK

An extremely useful concept in matrix analysis is the idea of rank introduced in Section 3.2. Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices with m rows and n columns. Consider the matrix $A \in \mathbb{R}^{m\times n}$. If the columns of matrix A are considered as vectors, the number of linearly independent columns is defined as the *column rank* of matrix A. Likewise, the number of linearly independent rows of matrix A is called the *row rank* of A. The row rank of a matrix and the column rank of the matrix are equal, and this integer is called the *rank* of matrix A.

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 ISBN: 0-470-01051-7

The concept of rank is useful in solving equations as well as checking stability of a system (Chapter 4) or the controllability and observability of a system (Chapter 7). Perhaps the best way to determine the rank of a matrix is to calculate the singular values of the matrix of interest (see Section 7.7 and the following comments). The rank of a matrix can be shown to be equal to the number of nonzero singular values of the matrix. The singular values also provide a very precise way of investigating the numerical difficulties frequently encountered in situations where the rank of the matrix is near the desired value. This shows up as very small but nonzero singular values, as discussed following Equation (8.53).

A simple procedure to calculate the singular values of a matrix A, and hence determine its rank, is provided by calculating the eigenvalues of the symmetric matrix:

$$\tilde{A} = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$$

If $A \in R^{m \times n}$ of rank r, the first r eigenvalues of $\tilde{A}$ are equal to the singular values of A, the next r eigenvalues are equal to the negative of the singular values of A, and the remaining eigenvalues of $\tilde{A}$ are zero. The rank of A is thus the number of positive eigenvalues of the symmetric matrix $\tilde{A}$.

B.3 INVERSES

For $A \in \mathbb{R}^{m \times n}$ the linear equation

$$A\mathbf{x} = \mathbf{b}$$

with $\det A \neq 0$ has the solution $\boldsymbol{x} = A^{-1}\mathbf{b}$, where A^{-1} denotes the unique inverse of matrix A. The matrix A^{-1} is the matrix that satisfies

$$A^{-1}A = AA^{-1} = I_n$$

Next, consider $A \in \mathbb{R}^{m \times n}$. If $m > n$ and if the rank of A is n, then there exists an $n \times m$ matrix A_{L} of rank n such that

$$A_L A = I_n$$

where I_n denotes the $n \times n$ identity matrix. The matrix A_L is called the left inverse of A. If, on the other hand, $n > m$ and the rank of A is m, then there exists an $n \times m$ matrix A_R of rank m, called a right inverse of A, such that

$$AA_R = I_m$$

Where I_m denotes the $m \times m$ identity matrix. If $m = n =$ rank A, then A is nonsingular and $A_R = A_L = A^{-1}$.

Consider the matrix $A^T A$ and note that it is an $n \times n$ symmetric matrix. If A is of rank n (this requires that $m > n$), then $A^T A$ is nonsingular. A solution of

$$A\mathbf{x} = \mathbf{b}$$

for $A \in \mathbb{R}^{m \times n}$ can then be calculated by multiplying both sides of this last expression by $(A^T A)^{-1} A^T$, which yields

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}$$

The quantity $(A^T A)^{-1} A^T$ is called the *generalized inverse* of A, denoted by $A^\dagger$.

The matrix $A^\dagger$ is also called a *pseudoinverse* or *Moore–Penrose inverse* and can be expressed in terms of a singular-value decomposition (Section 7.7) of matrix A. In the notation of Section 7.7, any matrix $A \in \mathbb{R}^{m \times n}$ can be expressed in terms of its singular-value factors as

$$A = U \Sigma V^T$$

where Σ denotes the diagonal matrix of singular values of A and U and V are orthogonal. For the case where $m > n$, if the rank of A is r, then the last $n - r$ (or $m - r$ if $m < n$) singular values are zero, so that Σ has the partitioned form

$$\Sigma = \begin{bmatrix} \Sigma_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where the zeros indicate matrices of zeros of the appropriate size and Σ_r is an $r \times r$ diagonal matrix of the nonzero singular values of A. Define the matrix Σ' by

$$\Sigma' = \begin{bmatrix} \Sigma_r^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The matrix $A^\dagger$ can be shown to be

$$A^\dagger = V \Sigma U^T$$

which is the singular-value decomposition of the generalized inverse. This last expression constitutes a more numerically stable way of calculating the generalized inverse than using the definition $(A^T A^{-1}) A^T$.

The following Moore–Penrose conditions can be stated for the pseudoinverse. If $A \in \mathbb{R}^{m \times n}$ has the singular-value decomposition $A = U \Sigma V^T$, then $A^\dagger = V \Sigma' U^T$ satisfies

$$AA^\dagger A = A$$

$$A^\dagger A A^\dagger = A^\dagger$$

$$(AA^\dagger)^T = AA^\dagger$$

$$(A^\dagger A)^T = A^\dagger A$$

The matrix $A^\dagger$ satisfying all four of these conditions is unique. If A has full rank, then $A^\dagger$ is identical to the left (and right) inverse just discussed.

Finally, note that the least-squares solution of the general equation $A\mathbf{x} = \mathbf{b}$ calculated by using the generalized inverse of A is *not* a solution in the sense that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution in the nonsingular case but is rather a vector $\mathbf{x}$ that minimizes the quantity $||A\mathbf{x} - \mathbf{b}||$.

The preceding is a quick summary of material contained in most modern texts on linear algebra and matrix theory, such as the excellent text by Ortega (1987). Computational issues and algorithms are discussed in the text by Golub and Van Loan (1983), which also mentions several convenient software packages. In most cases, the matrix computations required in the vibration analysis covered in this text can be performed by using standard software packages, most of which are in the public domain.

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ISBN: 0-470-01051-7