Small-scale effect on the vibration of thin nanoplates subjected to a moving nanoparticle via nonlocal continuum theory

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The vibration of elastic thin nanoplates traversed by a moving nanoparticle involving Coulomb friction is investigated using the nonlocal continuum theory of Eringen. The eigen function technique and the Laplace transform method are employed to solve the governing equations of the nanoplate. The explicit expressions of the in-plane and transverse displacements are obtained when the moving nanoparticle traverses the nanoplate on an arbitrary straight line. In a special case, the obtained results are also compared with those of other researchers and a reasonably good agreement is achieved. The effects of small-scale parameters and velocity of the moving nanoparticle on the dynamic response as well as the dynamic amplitude factors (DAFs) of the in-plane and transverse displacements are then explored in some detail. The results indicate that the magnitude of DAF of the transverse displacement of the nanoplate increases with the first small-scale effect parameter, irrespective of the values of the second small-scale effect parameter and the velocity of the moving nanoparticle. As the first small-scale effect parameter grows, the maximum values of DAF as a function of the moving nanoparticle velocity increase and generally occur in the lower levels of the moving nanoparticle velocity.

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1. Introduction

Nanotechnology is currently being considered one of the most promising technologies to be investigated. This novel technology would have remarkable effects on aerospace and defense industries, human health sciences, civil structures, and mechanical devices soon. Many efforts have been devoted to fabricate nanodevices with superior physical, electromechanical and chemical properties. From the point of functionality, the nanodevices could generally be categorized as nanowires, nanorods, nanotubes, nanoplates, and nanoshells. Nanoplates are a new sub-group of nanostructures with a two-dimensional shape [1]. Plate-like nanodevices could be utilized in energy storage and conversion [2], chemical and biological sensors [3], solar cells [4], field emission [5–7], photo-catalytic degradation of organic dye [8], and transporting of nanocars. The later application of the nanoplates has come to the real world since 2005, when a team of scientists [9,10] at Rice university could successfully test the first nanocar with some degrees of control. The designed nanocar was a molecule composed of an H-shaped chassis with fullerene groups attached at the four corners to act as wheels. Upon heating the surface of the nanoplate to 200 °C, the molecules could move forward and back as they roll on their fullerene wheels. The nanocar could roll about because the fullerene wheel is fitted to the alkyne axle through a carbon–carbon single bond. Investigating the dynamic response of the thin nanoplates under a point weight load of the nanocar is of
concern in this work. To this end, we should use an appropriate mechanical model to study the problem since conducting experimental tests in the nanoscale is so difficult and costly. On the other hand, the continuum models are less complex and do not involve more huge computational efforts than atomistic-based models. Moreover, recent studies show that the results of the appropriate continuum models achieve good agreements with those of molecular dynamics [11–13].

A brief survey of the literature reveals that the nonlocal continuum theory of Eringen [14–16] has gained much popularity among researchers [17–20] for analyzing the beam-like nanostructures during recent years. Perhaps the reason of this fact would be the simplicity of this new theory in introducing the small-scale parameters to the classical stress tensors. Therefore, the equations of motion within the context of nonlocal elasticity theory of Eringen could be readily extracted from the classical or local governing equations. In this new theory, the stress at a point of a continuum is affected by the strain field at other points of the continuum for considering the effect of interatomic bonds. As a result, lower the dimensions of the nanostructure, higher the effect of the interatomic bond length on the natural frequencies of the nanostructures [17,18,20].

Regarding nanostructures under moving nanoparticles, Kiani and Mehri [21] investigated vibration of nanotube structures under a moving nanoparticle using nonlocal Rayleigh, Timoshenko, and higher-order beam theories. The potentials of various nonlocal beam models in capturing the dynamic deflection of the nanotube structure were examined. Additionally, the role of the small-scale effect parameter, the slenderness ratio of the nanotube structure, and velocity of the moving nanoparticle on the time history of deflection as well as the dynamic amplitude factors of the moving nonlocal beams were studied in some detail. In other works, Kiani [22,23] explored vibration of double-walled carbon nanotubes (DWCNTs) under excitation of a moving nanoparticle by using nonlocal double body Euler–Bernoulli, Timoshenko, and higher-order beams acted together due to the van der Waals interaction forces. The equations of motion were derived for the double body classical and shear deformable beams which are connected by a flexible layer under a moving nanoparticle within the context of nonlocal continuum theory of Eringen. The analytical solutions of the transverse displacements as well as bending moment were then provided for the DWCNTs with fully simply supported boundary conditions. The critical velocities of the moving nanoparticle as well as the dynamic deflections of the innermost and outermost tubes associated with them were also given for different nonlocal beam theories. The role of the small-scale effect parameter, slenderness of DWCNTs and velocity of the moving nanoparticle on dynamic deflections and nonlocal bending moments of the innermost and outermost tubes as well as their maximum values were then investigated. The results revealed that the critical velocity of the moving nanoparticle increases with the slenderness of DWCNTs and the magnitude of the vdW interaction force. However, the critical velocity of the moving nanoparticle generally decreases with the small-scale effect as well as the ratio of the mean diameter to the thickness of the innermost tube. It was also noticed that the predicted maximum dynamic deflections and nonlocal bending moments of the innermost and outermost tubes by using the nonlocal Euler–Bernoulli and Timoshenko beam theories are generally the lower and upper bounds of those obtained by the nonlocal higher-order beam theory (NHOBT). The application of the NHOBT was strongly recommended for more rational modeling of DWCNTs under a moving nanoparticle, particularly when the slenderness ratio of the innermost nanotube is lower than 20. Recently, Kiani [24] examined the dynamic response of a single-walled carbon nanotubes (SWCNTs) subjected to a moving nanoparticle in the framework of the nonlocal continuum theory of Eringen. The inertial effects of the moving nanoparticle and the existing friction between the nanoparticle surface and the inner surface of the SWCNT were also incorporated into the formulations of the problem. To this end, the equivalent continuum structure associated with the SWCNT was considered and it was modeled using nonlocal Rayleigh beam theory under simply supported boundary conditions. The governing equations were then established both in the strong and weak forms. The set of linear equations were solved in the time domain using generalized Newmark-β method [25]. The effects of mass weight of the moving nanoparticle, its velocity, and small-scale effect parameter on the dynamic amplitude factors of longitudinal and transverse displacements as well as those of nonlocal axial force and bending moment were scrutinized in some detail. Additionally, the possibility of moving nanoparticle separation from the inner surface of the SWCNT was investigated. The role of influential parameters on the possibility of this phenomenon was also addressed and discussed.

Up to now, a large body of research has been devoted to explore dynamic response of macro-scale plates under moving loads using classical continuum theory [26–31]. However, nonlocal continuum theory of Eringen has been only applied to some problems of nanoplates to explore transverse vibration [32,33], in-plane vibration [34] and buckling [35,36] of the plate-like nanostructures. According to the literature, no study has been reported for the effects of the moving nanoparticles on the vibration of nanoplate until now. This issue would be of great importance when plate-like nanostructures are utilized as transporter devices for delivering of atoms or molecules from one place to another one. In such a scale, the wavelengths of propagated sound waves within the nanostructures may be comparable with the interatomic bond length or dimensions of the nanoplate. In such conditions, the classical continuum theory fails to predict the realistic dynamic behavior of the plate-like nanostructures. In the nonlocal continuum theories, the interatomic bond lengths are appropriately incorporated into the formulations of the problem via a parameter called small-scale effect parameter; however, the magnitude of this parameter should be justified by comparison of the predicted dispersion curves with those of atomistic-based models. As a result, nonlocal models enable us to investigate the vibration of the plate-like nanostructure for a wide range of velocity of the moving nanoparticles. The main advantage of using nonlocal continuum theories is that the needed computational efforts with such theories are highly lower than those of atomistic-based models. It would have many practical applications as well, especially when dynamical control of such nanostructures is of concern. It implies that dynamic control of plate-like nanostructures could be performed more efficiently via nonlocal
continuum theories, since in a very short time the dynamic responses of the interested points of the nanoplate are reported and then, the nanoplate could be appropriately controlled via special nanomechanisms.

In this work, the equations of motion of an elastic isotropic thin nanoplate under a moving nanoparticle including the friction effect are extracted within the framework of the nonlocal continuum theory of Eringen. In the case of a moving nanoparticle traverses the nanoplate on an arbitrary straight line, the explicit expressions of the dynamic in-plane and transverse displacements of the nanoplate are obtained during the courses of excitation and free vibration. Furthermore, the effects of velocity of the moving nanoparticle and small-scale parameters on the dynamic response as well as dynamic amplitude factors are investigated in some detail.

2. Definition of the problem

Consider a rectangular isotropic thin nanoplate with uniform thickness $t_p$ as depicted in Fig. 1. The nanoplate is of width $b$, length $a$, density $\rho_p$, Young’s modulus $E_p$, Poisson’s ratio $\nu_p$, and shear modulus $G_p$ where $G_p = E_p/(2(1+\nu_p))$. It is assumed that the values of $t_p/a$ and $t_p/b$ are such that the hypotheses of the Kirchhoff plate theory are satisfied. The origin of the coordinate system is located at a corner of the mid-plane of the nanoplate. The $x$-, $y$- and $z$-axes are taken along the length, width and thickness of the nanoplate, respectively. The nanoplate is subjected to the out-of-plane action of a moving nanoparticle of mass weight $mg$ moves on the upper surface of the nanoplate. The moving nanoparticle would be in contact with the nanoplate during excitation and the inertial effects of the moving nanoparticle would be negligible. Furthermore, the in-plane motion of the nanoplate is caused by the existing friction between the surfaces of the moving nanoparticle and the nanoplate. Friction in the nanoscale is a very complex phenomenon, and it can make up a challenging research subject in its own right. In this paper, the author makes no attempt to use advanced friction laws. Alternatively, the simple Coulomb friction model is adopted, which allow us to investigate the tangential effect of the moving nanoparticle on a nanoplate. Therefore, the force of friction is expressed as $f_s = \mu_k mg$ where $\mu_k$ denotes the coefficient of kinetic friction.

3. Governing equations

The local (or classical) Kirchhoff plate theory is based on the following displacement field:

\[
\begin{align*}
    u_x(x,y,z,t) &= u_0(x,y,t) - z \ W_0(x,y,t), \\
    u_y(x,y,z,t) &= v_0(x,y,t) - zW_0(y,x,t), \\
    u_z(x,y,z,t) &= w_0(x,y,t),
\end{align*}
\]

where $u_0(x,y,t)$, $v_0(x,y,t)$ and $w_0(x,y,t)$ represent the displacement components of the mid-plane of the nanoplate along $x$-, $y$- and $z$-axes, correspondingly. Within the context of linear elasticity, the strain components of the problem are expressed as

\[
e_{xx} = u_{0,x} - ZW_{0,xx},
\]
the constitutive relations between stress and strain components within the framework of local continuum theory read

\[ \sigma_{xx}^l = \frac{E_p}{1-\nu_p} (\epsilon_{xx} + \nu_p \epsilon_{yy}), \]

\[ \sigma_{xy}^l = \frac{E_p}{1-\nu_p} (\nu_p \epsilon_{xx} + \epsilon_{yy}), \]

\[ \sigma_{xy} = G_p \gamma_{xy}, \]

moreover, the local components of the in-plane forces \((N_{x1}^l)\) and moments \((M_{y1}^l)\) are defined as

\[ N_{x1}^l = \int_{-t_s/2}^{t_s/2} \sigma_{x1}^l \, dz; x_1 = x, y, z. \]

\[ M_{y1}^l = \int_{-t_s/2}^{t_s/2} z \sigma_{y1}^l \, dz, \]

by substituting Eq. (3) into Eq. (4) and using Eq. (2), the components of the in-plane forces and moments in terms of displacements are obtained as

\[ N_{x1}^l = C_p (u_{0,x} + v_p v_{0,y}), \]

\[ N_{y1}^l = C_p (v_p u_{0,x} + v_{0,y}), \]

\[ N_{y1}^l = \frac{1}{2} (1-v_p) C_p (u_{0,x} + v_{0,y}), \]

and

\[ M_{x1}^l = -D_p (w_{0,xx} + v_p w_{0,yy}), \]

\[ M_{y1}^l = -D_p (v_p w_{0,xx} + w_{0,yy}), \]

\[ M_{y1}^l = -D_p (1-v_p) w_{0,xy}, \]

where \( C_p = E_p t_p / (1-\nu_p^2) \) and \( D_p = E_p t_p^2 / (12-12(1-v_p^2)) \). Similarly, the nonlocal forces and moments are, respectively, related to the nonlocal stresses by

\[ N_{x2}^n = \int_{-t_s/2}^{t_s/2} \sigma_{x2}^n \, dz \]

and

\[ M_{y2}^n = \int_{-t_s/2}^{t_s/2} z \sigma_{y2}^n \, dz. \]

In contrast to the local continuum theory, the nonlocal continuum theory of Eringen says that the stress at a point depends on the strains at all the points of the continuum. According to the nonlocal continuum theory of Eringen [14,15], the nonlocal constitutive behavior for an elastic solid is represented by the following relation

\[ \sigma_{xy}^n = \frac{1}{2} \left( 1 - \frac{l_1^2}{l_2^2} \right) \sigma_{xy}^l, \]

where \( l_1 \) and \( l_2 \) denote the first and second small-scale parameters, respectively, \( \sigma_{xy}^l \) and \( \sigma_{xy}^n \), in order present the local and nonlocal stress components, and the Laplacian \((\nabla^2)\) and bi-harmonic \((\nabla^4)\) differential operators are defined as \( \nabla^2 \sigma = \nabla \cdot (\nabla \sigma) \) and \( \nabla^4 \sigma = \nabla^2 (\nabla^2 \sigma) \). The small-scale parameters could be adjusted according to the obtained frequencies from atomistic models. In this regard, Duan et al. [37] calibrated nonlocal Timoshenko beam theory for single-walled carbon nanotubes (SWCNTs) using molecular dynamics. It was demonstrated that the values of \( l_1 \) rest on slenderness ratios, mode shapes, and end conditions of the SWCNTs. On the other hand, the values of \( l_2 \) are commonly taken into account in the range of 0–2 nm [34,38,39] for dynamic analyses of CNTs. In another work, Duan and Wang [40] used the value of \( l_1 \) ranging from 0 to 2 nm for axisymmetric bending analysis of circular plates at micro- and nanoscale levels. Further investigations are still needed to propose more realistic values of \( l_1 \) and \( l_2 \) for nanostructures by adjustment of the results of nonlocal continuum theory with those of appropriate atomistic models.

Using the Hamilton’s principle, the governing equations of motion of the nanoplate under a moving nanoparticle by considering friction effect are readily obtained as

\[ N_{xx}^l + N_{yy}^l + \mu_t mg \delta(x-x_m, y-y_m) H(a-x_m) H(b-y_m) = I_0 u_{0,tt}, \]

\[ N_{xy}^l + N_{yx}^l + \mu_t mg \delta(x-x_m, y-y_m) H(a-x_m) H(b-y_m) = I_0 v_{0,tt}, \]

\[ M_{xx}^l + 2M_{xy}^l + M_{yy}^l + mg \delta(x-x_m, y-y_m) H(a-x_m) H(b-y_m) = I_0 w_{0,tt} - I_2 (w_{0,xtt} + w_{0,ytt}). \]
where \( t \) is the time parameter, \( \delta \) and \( H \) are, respectively, the Dirac delta and Heaviside step functions, \( \mu_xmg \) and \( \mu_ymg \) are in turn the components of the kinetic friction force in \( x \) and \( y \) directions, \( (x_m, y_m) \) is the coordinates of the moving nanoparticle on the mid-plane of the nanoplate, and \( n = \int_{-\beta/2}^{\beta/2} \rho_p x^2 \, \mathrm{d}z \); \( n = 0.2 \). Applying Eq. (7) to Eqs. (5) and (6) through using Eqs. (4) and (8) leads to the governing equations of the nanoplate in terms of displacements

\[
C_p \left( u_{0,xx} + \frac{1-v_p}{2} u_{0,xy} + \frac{1+v_p}{2} v_{0,xy} \right) = (1-\mu_p^2) \frac{\nabla^2}{\nabla^2} (I_0 u_{0,tt} - \mu_x m g \delta (x-x_m, y-y_m) H(a-x_m) H(b-y_m)),
\]

\[
C_p \left( v_{0,yy} + \frac{1-v_p}{2} u_{0,xy} + \frac{1+v_p}{2} v_{0,xy} \right) = (1-\mu_p^2) \frac{\nabla^2}{\nabla^2} (I_0 v_{0,tt} - \mu_y m g \delta (x-x_m, y-y_m) H(a-x_m) H(b-y_m)).
\]

\[-D_p \nabla^4 w_0 = (1-\mu_p^2) \frac{\nabla^2}{\nabla^2} (I_0 w_{0,tt} - l_2 (w_{0强有力的tt} + w_{0,xy} - m g \delta (x-x_m, y-y_m) H(a-x_m) H(b-y_m)),
\]

for more convenient in analyzing of the problem, the following dimensionless parameters are defined

\[
\tilde{\Pi}_0 = \frac{u_0}{a}, \quad \tilde{V}_0 = \frac{v_0}{a}, \quad \tilde{W}_0 = \frac{w_0}{a}, \quad \tilde{\tau} = \frac{1}{a^2} \sqrt{\frac{D_p}{I_0}}, \quad \tilde{\zeta} = \frac{x}{a}, \quad \tilde{\eta} = \frac{y}{b}.
\]

where \( \tilde{\tau} \) is the dimensionless time parameter, and \( (\tilde{\zeta}, \tilde{\eta}) \) denotes the dimensionless coordinates. By introducing Eq. (10) into Eq. (9), the dimensionless equations of motion of the problem are obtained as

\[
\tilde{c}_p \left( \tilde{\Pi}_{0,\tilde{\zeta}\tilde{\zeta}} + \frac{1-v_p}{2} k^2 \tilde{\Pi}_{0,\tilde{\zeta}\tilde{\eta}} + \frac{1+v_p}{2} k \tilde{\Pi}_{0,\tilde{\eta}\tilde{\eta}} \right) = \Xi (\tilde{\Pi}_{0,\tilde{\tau}\tilde{\tau}} - \mu_1 f \delta (\tilde{\xi} - \tilde{\xi}_m, \tilde{\eta} - \tilde{\eta}_m) H(1-\tilde{\xi}_m) H(1-\tilde{\eta}_m)),
\]

\[
\tilde{c}_p \left( k^2 \tilde{V}_{0,\tilde{\eta}\tilde{\eta}} + \frac{1-v_p}{2} \tilde{V}_{0,\tilde{\zeta}\tilde{\zeta}} + \frac{1+v_p}{2} k \tilde{V}_{0,\tilde{\eta}\tilde{\eta}} \right) = \Xi (\tilde{V}_{0,\tilde{\tau}\tilde{\tau}} - \mu_2 f \delta (\tilde{\xi} - \tilde{\xi}_m, \tilde{\eta} - \tilde{\eta}_m) H(1-\tilde{\xi}_m) H(1-\tilde{\eta}_m)),
\]

\[
\tilde{w}_{0,\tilde{\xi}\tilde{\xi}\tilde{\xi}} + 2k^2 \tilde{w}_{0,\tilde{\xi}\tilde{\eta}\eta} + k^4 \tilde{w}_{0,\tilde{\eta}\eta\eta} = \Xi (\tilde{f} \delta (\tilde{\xi} - \tilde{\xi}_m, \tilde{\eta} - \tilde{\eta}_m) H(1-\tilde{\xi}_m) H(1-\tilde{\eta}_m) - \tilde{f}_{0,\tilde{\xi}} + \tilde{f}_{0,\tilde{\xi}_0} + 2k^2 \tilde{w}_{0,\tilde{\eta}\eta\eta\eta})).
\]

where the dimensionless parameters and operator \( \Xi \) in Eq. (11) are as follows:

\[
\Xi = 1 - \mu_1^2 \left( \frac{\partial^2}{\partial \xi^2} + k^2 \frac{\partial^2}{\partial \eta^2} \right) + \mu_2^2 \left( \frac{\partial^2}{\partial \xi^2} + 2k^2 \frac{\partial^2}{\partial \xi^2} \frac{\partial^2}{\partial \eta^2} + k^4 \frac{\partial^4}{\partial \eta^4} \right).
\]

4. Solving the governing equations

The above system of three dimensionless second-order partial differential equations should be solved for the unknown displacement components of the mid-plane of the nanoplate (i.e., \( \Pi_0, V_0 \) and \( W_0 \)). In the case of simply supported boundary conditions with no traction on the pyramid surfaces of the nanoplate, the following in-plane and out-of-plane boundary conditions are applied:

the in-plane boundary conditions:

\[
\Pi_0 (\tilde{\zeta}, \tilde{\eta}, \tilde{\tau}) = \Pi_0 (\tilde{\zeta}, 1, \tilde{\tau}) = 0, \quad \Pi_0 (0, \tilde{\eta}, \tilde{\tau}) = \Pi_0 (1, \tilde{\eta}, \tilde{\tau}) = 0,
\]

\[
N_{\tilde{\xi}}^{\Pi}(0, \tilde{\eta}, \tilde{\tau}) = N_{\tilde{\eta}}^{\Pi}(1, \tilde{\eta}, \tilde{\tau}) = 0, \quad N_{\tilde{\xi}}^{\Pi}(\tilde{\xi}, 0, \tilde{\tau}) = N_{\tilde{\eta}}^{\Pi}(\tilde{\xi}, 1, \tilde{\tau}) = 0.
\]

the out-of-plane boundary conditions:

\[
W_0 (0, \tilde{\eta}, \tilde{\tau}) = W_0 (1, \tilde{\eta}, \tilde{\tau}) = W_0 (\tilde{\xi}, 0, \tilde{\tau}) = W_0 (\tilde{\xi}, 1, \tilde{\tau}) = 0,
\]

\[
M_{\tilde{\xi}}^{\Pi}(0, \tilde{\eta}, \tilde{\tau}) = M_{\tilde{\eta}}^{\Pi}(1, \tilde{\eta}, \tilde{\tau}) = M_{\tilde{\xi}}^{\Pi}(\tilde{\xi}, 0, \tilde{\tau}) = M_{\tilde{\eta}}^{\Pi}(\tilde{\xi}, 1, \tilde{\tau}) = 0.
\]

An analytical method based on the eigen function expansion is used to transform the system of partial differential equations in Eq. (11) into the equivalent algebraic forms. To this end, the following admissible displacements are considered in solving of the problem

\[
\Pi_0 (\tilde{\xi}, \tilde{\eta}, \tilde{\tau}) = \sum_{i=1}^{M} \sum_{j=1}^{N} \Pi_{ij} (\tilde{\xi}) \cos (i \pi \tilde{\xi}) \sin (j \pi \tilde{\eta}),
\]

\[
\Pi_0 (\tilde{\xi}, \tilde{\eta}, \tilde{\tau}) = \sum_{i=1}^{M} \sum_{j=1}^{N} \Pi_{ij} (\tilde{\xi}) \cos (i \pi \tilde{\xi}) \cos (j \pi \tilde{\eta}),
\]
\[ \mathbf{W}_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \mathbf{W}_i(\tau) \sin(i\pi \xi) \sin(j\pi \eta), \]  

(15)
much more, the Dirac delta function could be represented in terms of \( \cos(i\pi \xi) \sin(j\pi \eta) \), \( \sin(i\pi \xi) \cos(j\pi \eta) \), and \( \sin(i\pi \xi) \sin(j\pi \eta) \) as follows:

\[ \delta(\xi - \xi_m, \eta - \eta_m) = 4 \sum_{i=1}^{M} \sum_{j=1}^{N} \cos(i\pi \xi_m) \sin(j\pi \eta_m) \cos(i\pi \xi) \sin(j\pi \eta), \]

\[ \delta(\xi - \xi_m, \eta - \eta_m) = 4 \sum_{i=1}^{M} \sum_{j=1}^{N} \sin(i\pi \xi_m) \cos(j\pi \eta_m) \cos(i\pi \xi) \sin(j\pi \eta), \]

\[ \delta(\xi - \xi_m, \eta - \eta_m) = 4 \sum_{i=1}^{M} \sum_{j=1}^{N} \sin(i\pi \xi_m) \sin(j\pi \eta_m) \cos(i\pi \xi) \sin(j\pi \eta). \]  

(16)

where \((\xi_m, \eta_m)\) denotes the dimensionless coordinates of the moving nanoparticle. Substituting Eqs. (15) and (16) into Eq. (11) leads to the following set of ordinary differential equations

\[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{ij,xx} \\ \mathbf{W}_{ij,yy} \\ \mathbf{W}_{ij,tt} \end{bmatrix} + \begin{bmatrix} a_1^2 & a_2^2 & 0 \\ a_2^2 & a_3^2 & 0 \\ 0 & 0 & a_4^2 \end{bmatrix} \begin{bmatrix} \mathbf{W}_{ij} \\ \mathbf{V}_{ij} \\ \mathbf{U}_{ij} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \]

(17)

where

\[ a_1^2 = \mathcal{C}_p \pi^2 \left( \frac{1 - \nu_p}{2} k \right) / \lambda', \quad a_2^2 = (1 + \nu_p) \mathcal{C}_p \pi^2 \lambda / (2 \lambda'), \]

\[ a_3^2 = \mathcal{C}_p \pi^2 \left( \frac{1 - \nu_p}{2} k \right) / \lambda', \quad a_4^2 = \pi^2 (i^2 + k^2) / (\lambda' + 2 \pi^2 (i^2 + k^2)^2), \]

\[ f_1 = 4 \mu_2 \mathcal{J} \cos(i\pi \xi_m) \sin(j\pi \eta_m) \mathcal{H}(1 - \xi_m) \mathcal{H}(1 - \eta_m), \]

\[ f_2 = 4 \mu_2 \mathcal{J} \sin(i\pi \xi_m) \cos(j\pi \eta_m) \mathcal{H}(1 - \xi_m) \mathcal{H}(1 - \eta_m), \]

\[ f_3 = 4 \mathcal{J} \sin(i\pi \xi_m) \sin(j\pi \eta_m) \mathcal{H}(1 - \xi_m) \mathcal{H}(1 - \eta_m), \]

\[ \lambda' = 1 + \pi^2 \mu_2 (i^2 + k^2) + \pi^4 \mu_2 (i^2 + k^2)^2. \]

In order to investigate the dynamic response of the nanoplate, it is assumed that the moving nanoparticle is traveling on an arbitrary straight line with a constant velocity, \( \nu \). Let the equation of the purpose line in the \( xoy \) plane be \( y = px + q \), therefore, the dimensionless location of the moving nanoparticle is described by

\[ \xi_m = \nu \sqrt{12(1 - \nu_p^2) \rho_p} / k \sqrt{\mathcal{E}_p(1 + \rho^2)}, \]

\[ \eta_m = \nu \sqrt{12(1 - \nu_p^2) \rho_p} / k \sqrt{\mathcal{E}_p(1 + \rho^2)} \times q / \rho, \]

(19)

at the time \( \tau = 0 \), the moving nanoparticle enters to the nanoplate with the rest initial conditions (i.e., \( \mathbf{W}(\xi, \eta, 0) = \mathbf{W}(\xi, \eta, 0) = \mathbf{W}(\xi, \eta, 0) = 0 \)). Laplace transform method is utilized to solve Eq. (17) in the time domain. By denoting \( \mathcal{L} \) as the Laplace operator and recalling a property of the Laplace transform as \( \mathcal{L} \left[ \mathbf{f}(\tau) \right] = s \mathcal{L} \left[ f(\tau) \right] (\tau = 0) \), the following expression can be deduced after application of the Laplace transform to Eq. (17)

\[ \begin{bmatrix} s^2 + a_1^2 & a_2^2 & 0 \\ a_2^2 & s^2 + a_3^2 & 0 \\ 0 & 0 & s^2 + a_4^2 \end{bmatrix} \begin{bmatrix} \mathcal{L}[\mathbf{W}_{ij}] \\ \mathcal{L}[\mathbf{V}_{ij}] \\ \mathcal{L}[\mathbf{U}_{ij}] \end{bmatrix} = \begin{bmatrix} 4 \mu_2 \mathcal{J} (s^2 - g_1^2) \mathcal{G}(\cos(h_1) + \omega s^2 + g_2^2 + g_3^2) \sin(h_1) \\ 4 \mu_2 \mathcal{J} (s^2 - g_1^2) \mathcal{G}(\cos(h_1) - \omega s^2 + g_2^2 + g_3^2) \sin(h_1) \\ 4 \mathcal{J} (s^2 - g_1^2) \mathcal{G}(2 s g_2 \cos(h_1)) \end{bmatrix}, \]

(20)
where
\[
g_i = \frac{i\nu \sqrt{12(1-v_r^2)} \rho_p}{k \sqrt{E_p(1+p^2)}}, \quad g_l = \frac{j\nu p \sqrt{12(1-v_r^2)} \rho_p}{k k \sqrt{E_p(1+p^2)}}, \quad \text{and} \quad h_j = j\nu q/b.
\]
According to the values of \( g_i \) and \( g_l \), the dynamic response of the nanoplate in two cases, namely \( g_i \neq g_l \) and \( g_i = g_l \), will be derived in the following parts during excitation.

Case I: \( g_i \neq g_l \). In this case, one can find the expressions of \( \mathcal{L}[\Pi_0] \), \( \mathcal{L}[\mathcal{P}_0] \), and \( \mathcal{L}[\mathcal{W}_0] \) by solving the set of linear equations in Eq. (20)

\[
\mathcal{L}[\Pi_0] = \frac{A^s s^5 + D^s s^4 + B^s s^3 + E^s s^2 + C^s + F^s}{(s^2 + r_1^2)(s^2 + r_2^2)(s^2 + (g_i + g_l)^2)(s^2 + (g_i - g_l)^2)},
\]

\[
\mathcal{L}[\mathcal{P}_0] = \frac{A^r s^5 + D^r s^4 + B^r s^3 + E^r s^2 + C^r + F^r}{(s^2 + r_1^2)(s^2 + r_2^2)(s^2 + (g_i + g_l)^2)(s^2 + (g_i - g_l)^2)},
\]

\[
\mathcal{L}[\mathcal{W}_0] = \frac{A^w s^5 + B^w s + C^w}{(s^2 + a_4^2)(s^2 + (g_i + g_l)^2)(s^2 + (g_i - g_l)^2)},
\]

(21)

where

\[
A^s = 4\mu_j a_j^s \sin(h_j), \quad B^s = 4\mu_j a_j^s (a_3^j + a_4^j + g_j^2) \sin(h_j), \quad C^s = 4\mu_j a_j^s (a_3^j + g_j^2) + 2\mu_j a_j^s g_j \sin(h_j),
\]

\[
D^s = 4\mu_j g_j \cos(h_j), \quad E^s = 4\mu_j g_j (a_3^j + a_4^j + g_j^2) - \mu_j a_j^s g_j \cos(h_j), \quad F^s = -4\mu_j a_j^s (a_3^j + g_j^2) + 2\mu_j a_j^s g_j \cos(h_j),
\]

\[
A^r = 0, \quad B^r = -4\mu_j a_j^r + 2\mu_j g_j \sin(h_j), \quad C^r = -4\mu_j a_j^r (g_j^2 + g_j^2) + 2\mu_j a_j^r g_j \sin(h_j),
\]

\[
D^r = 4\mu_j g_j \cos(h_j), \quad E^r = -4\mu_j g_j (a_3^j + a_4^j + g_j^2) \cos(h_j), \quad F^r = -4\mu_j a_j^r + 2\mu_j a_j^r (g_j^2 + g_j^2) \cos(h_j),
\]

\[
A^w = 2g_j g_k \cos(h_j), \quad B^w = 8g_j g_k \cos(h_j), \quad C^w = 4g_j (g_j^2 - g_k^2) \sin(h_j),
\]

\[
r_1^2 = a_3^j + a_4^j + \sqrt{(a_3^j - a_4^j)^2 + 4a_4^2}, \quad r_2^2 = (a_3^j + a_4^j) + \sqrt{(a_3^j - a_4^j)^2 + 4a_4^2},
\]

(22)
a rational fraction provided in Eq. (21) can be decomposed into simpler functions by application of partial fraction decomposition

\[
\mathcal{L}[\Pi_0] = \frac{sA_1 + B_1}{s^2 + r_1^2} + \frac{sA_2 + B_2}{s^2 + r_2^2} + \frac{sA_3 + B_3}{s^2 + (g_i + g_l)^2} + \frac{sA_4 + B_4}{s^2 + (g_i - g_l)^2},
\]

\[
\mathcal{L}[\mathcal{P}_0] = \frac{sA_1 + B_1}{s^2 + r_1^2} + \frac{sA_2 + B_2}{s^2 + r_2^2} + \frac{sA_3 + B_3}{s^2 + (g_i + g_l)^2} + \frac{sA_4 + B_4}{s^2 + (g_i - g_l)^2},
\]

\[
\mathcal{L}[\mathcal{W}_0] = \frac{sA_1 + B_1}{s^2 + a_4^2} + \frac{sA_2 + B_2}{s^2 + (g_i + g_l)^2} + \frac{sA_3 + B_3}{s^2 + (g_i - g_l)^2},
\]

(23)

where the values of the parameters \( A_i, B_i, A'_i, B'_i, \ i = 1, \ldots, 4 \), \( A''_i \) and \( B''_i, j = 1, 2, 3 \) have been provided in Appendix A. By application of the inverse Laplace transform to Eq. (23), the dynamic displacements of the nanoplate during excitation are readily obtained as

\[
\begin{align*}
\Pi_0(\xi, \eta, \tau) &= \sum_{i=1}^{M} \sum_{j=1}^{N} \left(A_i^s \cos(r_1 \tau) + A_i^c \cos(r_2 \tau) + A_i^s \cos((g_i + g_l) \tau) + A_i^c \cos((g_i - g_l) \tau) + \frac{B_1^i}{r_1} \sin(r_1 \tau) + \frac{B_2^i}{r_2} \sin(r_2 \tau) + \frac{B_3^i}{g_i + g_l} \sin((g_i + g_l) \tau) + \frac{B_4^i}{g_i - g_l} \sin((g_i - g_l) \tau)\right) \cos(i\nu \xi) \sin(j\nu \eta),
\end{align*}
\]

\[
\mathcal{P}_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left(A_i^r \cos(r_1 \tau) + A_i^c \cos(r_2 \tau) + A_i^s \cos((g_i + g_l) \tau) + A_i^c \cos((g_i - g_l) \tau) + \frac{B_1^i}{r_1} \sin(r_1 \tau) + \frac{B_2^i}{r_2} \sin(r_2 \tau) + \frac{B_3^i}{g_i + g_l} \sin((g_i + g_l) \tau) + \frac{B_4^i}{g_i - g_l} \sin((g_i - g_l) \tau)\right) \sin(i\nu \xi) \cos(j\nu \eta),
\]

\[
\mathcal{W}_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left(A_i^w \cos(a_4 \tau) + A_i^w \cos((g_i + g_l) \tau) + A_i^w \cos((g_i - g_l) \tau) + \frac{B_1^i}{a_4} \sin(a_4 \tau) + \frac{B_2^i}{g_i + g_l} \sin((g_i + g_l) \tau) + \frac{B_3^i}{g_i - g_l} \sin((g_i - g_l) \tau)\right) \sin(i\nu \xi) \sin(j\nu \eta),
\]

(24)
Case II: $g_i - g_j$. In this case, it is deduced from Eq. (20)

\[
\begin{bmatrix}
  s^2 + a_1^2 & a_2^2 & 0 \\
  a_1^2 & s^2 + a_3^2 & 0 \\
  0 & 0 & s^2 + a_4^2
\end{bmatrix}
\begin{bmatrix}
  \mathcal{L}(\Pi_{ij}) \\
  \mathcal{L}(\Pi_{ji}) \\
  \mathcal{L}(\mathcal{W}_{ij})
\end{bmatrix}
= \begin{bmatrix}
  \frac{4\mu_j f(sg_j \cos(h_j) + (s^2 + 2g_j^2 \sin(h_j)))}{\sigma(t^2 + 4g_j^2)} \\
  \frac{4\mu_j f(sg_j \cos(h_j) - 2g_j^2 \sin(h_j))}{\sigma(t^2 + 4g_j^2)} \\
  \frac{4\mu_j f(sg_j \sin(h_j) + 2g_j^2 \cos(h_j))}{\sigma(t^2 + 4g_j^2)}
\end{bmatrix},
\]

(25)

by solving the set of linear equations in Eq. (25) for the unknown variables $\mathcal{L}(\Pi_{ij})$, $\mathcal{L}(\Pi_{ji})$, and $\mathcal{L}(\mathcal{W}_{ij})$

\[
\mathcal{L}(\Pi_{ij}) = \frac{A's^4 + D's^3 + B's^2 + E's + C'}{(s^2 + r_1^2)(s^2 + r_2^2)(s^2 + 4g_j^2)},
\]

\[
\mathcal{L}(\Pi_{ji}) = \frac{A''s^4 + D''s^3 + B''s^2 + E''s + C''}{(s^2 + r_1^2)(s^2 + r_2^2)(s^2 + 4g_j^2)},
\]

\[
\mathcal{L}(\mathcal{W}_{ij}) = \frac{A'''s + B'''}{s(s^2 + a_4^2)(s^2 + 4g_j^2)},
\]

(26)

by application of the inverse Laplace transform to Eq. (26), the solution would be obtained as

\[
\Pi_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( A_i r_1 \cos(r_1 \tau) + A_2 r_2 \cos(r_2 \tau) + A_3 (2g_j \tau) \right) \frac{1}{A_4 r_1 \sin(r_1 \tau) + B_1 r_2 \sin(r_2 \tau) + B_3 (2g_j \tau)} \cos(i\pi \xi \sin(j\pi \eta)),
\]

\[
\mathcal{V}_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( A_4 r_1 \sin(r_1 \tau) + A_5 r_2 \sin(r_2 \tau) + A_3 (2g_j \tau) \right) \frac{1}{A_4 r_1 \sin(r_1 \tau) + B_1 r_2 \sin(r_2 \tau) + B_3 (2g_j \tau)} \sin(i\pi \xi \cos(j\pi \eta)),
\]

\[
\mathcal{W}_0(\xi, \eta, \tau) = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( A_6 r_1 \sin(r_1 \tau) + A_7 r_2 \sin(r_2 \tau) + B_1 r_2 \cos(r_2 \tau) + B_3 (2g_j \tau) \right) \frac{1}{A_4 r_1 \sin(r_1 \tau) + B_1 r_2 \sin(r_2 \tau) + B_3 (2g_j \tau)} \sin(i\pi \xi \sin(j\pi \eta)),
\]

(27)

where

\[
A_1 = \frac{-D'r_1^2 + E}{(r_2^2 - r_1^2)(4g_j^2 - r_1^2)}, \quad A_2 = \frac{-D'r_2^2 + E}{(r_2^2 - r_2^2)(4g_j^2 - r_2^2)}, \quad A_3 = \frac{-4D'g_j^2 + E}{(r_2^2 - 4g_j^2)(r_2^2 - 4g_j^2)},
\]

\[
B_1 = \frac{A'r_1^2 - B'r_2^2 + C}{(r_2^2 - r_1^2)(4g_j^2 - r_1^2)}, \quad B_2 = \frac{A'r_2^2 - B'r_2^2 + C}{(r_2^2 - r_2^2)(4g_j^2 - r_2^2)}, \quad B_3 = \frac{16A'g_j^2 - 4B'g_j^2 + C}{(r_2^2 - 4g_j^2)(r_2^2 - 4g_j^2)},
\]

\[
A_1'' = \frac{A''}{a_4^2 - 4g_j^2}, \quad A_2'' = \frac{B''}{a_4^2 - 4g_j^2}, \quad B_1'' = \frac{B''}{a_4^2(a_4^2 - 4g_j^2)}, \quad B_3'' = \frac{B''}{a_4^2(a_4^2 - 4g_j^2)}.
\]

(28)

In order to derive the dynamic response of the nanoplate during free vibration, the following set of linear algebraic equations should be solved in the time domain

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  \Pi_{ij,\tau} \\
  \mathcal{V}_{ij,\tau} \\
  \mathcal{W}_{ij,\tau}
\end{bmatrix} + \begin{bmatrix}
  a_1^2 & a_2^2 & 0 \\
  a_1^2 & a_3^2 & 0 \\
  0 & 0 & a_4^2
\end{bmatrix}
\begin{bmatrix}
  \Pi_{ij} \\
  \mathcal{V}_{ij} \\
  \mathcal{W}_{ij}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}, \quad \tau > \tau_f,
\]

(29)

where $\tau_f = \sqrt{\frac{D}{\mu_j} t_f / a^2}$, and $t_f$ denotes the departure time of the moving nanoparticle from the nanoplate. According to the dynamic response of the nanoplate during the course of excitation, the following initial conditions should be imposed to Eq. (29)

\[
\Pi_{ij}(\tau_f) = \hat{U}_{ij}, \quad \mathcal{V}_{ij}(\tau_f) = \hat{V}_{ij}, \quad \mathcal{W}_{ij}(\tau_f) = \hat{W}_{ij},
\]

\[
\Pi_{ij,\tau}(\tau_f) = \hat{U}_{ij}, \quad \mathcal{V}_{ij,\tau}(\tau_f) = \hat{V}_{ij}, \quad \mathcal{W}_{ij,\tau}(\tau_f) = \hat{W}_{ij},
\]

(30)
where the values of \( \hat{U}_y, \hat{V}_y, \hat{W}_y \), and their corresponding velocities have been given in Appendix B. By application of the Laplace transform to Eq. (29),

\[
\begin{bmatrix}
  s^2 + a_1^2 & a_2^2 & 0 \\
  a_2^2 & s^2 + a_3^2 & 0 \\
  0 & 0 & s^2 + a_4^2
\end{bmatrix}
\begin{bmatrix}
  \mathcal{L}\{\hat{U}_y\} \\
  \mathcal{L}\{\hat{V}_y\} \\
  \mathcal{L}\{\hat{W}_y\}
\end{bmatrix}
= \begin{bmatrix}
  s\hat{U}_y + \hat{\dot{U}}_y \\
  s\hat{V}_y + \hat{\dot{V}}_y \\
  s\hat{W}_y + \hat{\dot{W}}_y
\end{bmatrix},
\]

by solving the set of equations in Eq. (31), the unknowns would be determined as follows:

\[
\mathcal{L}\{\hat{U}_y\} = \frac{\hat{\dot{A}}'}{s^3 + \hat{\dot{C}}' s^2 + \hat{\dot{B}}' s + \hat{\dot{D}}'},
\]

\[
\mathcal{L}\{\hat{V}_y\} = \frac{\hat{\dot{A}}''}{s^3 + \hat{\dot{C}}'' s^2 + \hat{\dot{B}}'' s + \hat{\dot{D}}''},
\]

\[
\mathcal{L}\{\hat{W}_y\} = \frac{s\hat{W}_y + \hat{\dot{W}}_y}{s^3 + \frac{a_4^2}{a_2^2}(s^2 + (g_i + g_l)^2)(s^2 + (g_i - g_l)^2)},
\]

where

\[
\hat{A}' = \hat{\dot{U}}_y, \quad \hat{B}' = a_3^2 \hat{\dot{U}}_y - a_1^2 \hat{\dot{V}}_y, \quad \hat{C}' = \hat{\dot{U}}_y, \quad \hat{D}' = a_3^2 \hat{\dot{U}}_y - a_1^2 \hat{\dot{V}}_y,
\]

\[
\hat{\dot{A}}' = \hat{\ddot{V}}_y, \quad \hat{\dot{B}}' = a_3^2 \hat{\ddot{V}}_y - a_1^2 \hat{\ddot{U}}_y, \quad \hat{\dot{C}}' = \hat{\ddot{V}}_y, \quad \hat{\dot{D}}' = a_3^2 \hat{\ddot{V}}_y - a_1^2 \hat{\ddot{U}}_y,
\]

(33)

(34)

(35)

by taking the inverse Laplace transform of Eq. (34), the dynamic response of the nanoplate during free vibration could be readily obtained as follows:

\[
\mathcal{U}_0(\xi, \eta, \tau') = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( \hat{\tilde{A}}_i \cos(r_1 \tau') + \hat{\tilde{A}}_2 \cos(r_2 \tau') + \frac{\hat{\tilde{B}}_i}{r_1} \sin(r_1 \tau') + \frac{\hat{\tilde{B}}_2}{r_2} \sin(r_2 \tau') \right) \cos(\pi \xi) \sin(\pi \eta),
\]

\[
\mathcal{V}_0(\xi, \eta, \tau') = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( \hat{\tilde{A}}_i \cos(r_1 \tau') + \hat{\tilde{A}}_2 \cos(r_2 \tau') + \frac{\hat{\tilde{B}}_i}{r_1} \sin(r_1 \tau') + \frac{\hat{\tilde{B}}_2}{r_2} \sin(r_2 \tau') \right) \sin(\pi \xi) \cos(\pi \eta),
\]

\[
\mathcal{W}_0(\xi, \eta, \tau') = \sum_{i=1}^{M} \sum_{j=1}^{N} \left( \hat{\tilde{W}}_y \cos(a_4 \tau') + \hat{\tilde{W}}_y \sin(a_4 \tau') \right) \sin(\pi \xi) \cos(\pi \eta),
\]

(36)

where \( \tau' = \tau - \tau_f \).

5. Results and discussions

For more convenient in presenting the results, the following normalized parameters are introduced: \( u_N = \mathcal{U}_0 / \mathcal{U}_0 \), \( v_N = \mathcal{V}_0 / \mathcal{V}_0 \), and \( w_N = \mathcal{W}_0 / \mathcal{W}_0 \) where \( \mathcal{U}_0 \) and \( \mathcal{V}_0 \) are the dimensionless in-plane components of the static displacement field due to the applied friction force at the center of the nanoplate. Moreover, \( \mathcal{W}_0 \) is the dimensionless out-of-plane
displacement (i.e., deflection) of the nanoplate due to the statically applied weight of the nanoparticle at the center of the nanoplate. For a simply supported nanoplate, the explicit expressions of these parameters are readily obtained as follows:

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \sum_{i=1}^{\left[ \frac{M}{2} \right]} \sum_{j=1}^{\left[ \frac{N}{2} \right]} (-1)^{i+j-1} (1-v_p) (1+\mu_1^2 (2j-1)^2 + \mu_2^2 (2j-1)^2 + \pi^2 (2i-1)^2) \frac{1}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \sum_{i=1}^{\left[ \frac{M}{2} \right]} \sum_{j=1}^{\left[ \frac{N}{2} \right]} (-1)^{i+j-1} (1-v_p) (1+\mu_1^2 (2j-1)^2 + \mu_2^2 (2j-1)^2 + \pi^2 (2i-1)^2) \frac{1}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \sum_{i=1}^{\left[ \frac{M}{2} \right]} \sum_{j=1}^{\left[ \frac{N}{2} \right]} (-1)^{i+j-1} (1-v_p) (1+\mu_1^2 (2j-1)^2 + \mu_2^2 (2j-1)^2 + \pi^2 (2i-1)^2) \frac{1}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

Furthermore, the dynamic amplitude factors of the dynamic displacements at the center of the nanoplate are expressed as

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \frac{\pi}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

hence, the dynamic amplitude factors of the dynamic displacements at the center of the nanoplate are expressed as

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \frac{\pi}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

furthermore, the dynamic amplitude factors of the dynamic displacements at the center of the nanoplate are expressed as

\[
\sqrt{\frac{1}{\pi}} \frac{\mu_2}{\rho_0} = \frac{\pi}{\sqrt{\pi}} \sin((2i-1)\pi \eta),
\]

5.1. Verification of the obtained results with those of other works

5.1.1. Development of dispersion relations for thin nanoplates

A harmonic wave propagating through the nanoplate in a direction \( \mathbf{r} \) is represented by

\[
\langle \mathbf{u}_0, \mathbf{v}_0, \mathbf{w}_0 \rangle = \langle \mathbf{U}_0, \mathbf{V}_0, \mathbf{W}_0 \rangle e^{i \mathbf{K} \cdot \mathbf{r}},
\]

where \( i = \sqrt{-1} \), \( \mathbf{U}_0, \mathbf{V}_0, \) and \( \mathbf{W}_0 \) are the amplitudes of the propagated waves in the x, y, and z directions, respectively, \( \mathbf{K} = k_x \mathbf{i} + k_y \mathbf{j} \) is the vector of dimensionless wave number, \( \mathbf{r} = (x+y) \mathbf{i} \) is the dimensionless position vector for the mid-plane, and \( \varpi \) denotes the dimensionless circular frequency of the propagated wave. By substituting Eq. (38) into Eq. (11) and finding the non-trivial solutions for \( \mathbf{U}_0, \mathbf{V}_0, \) and \( \mathbf{W}_0 \), the dimensionless dispersion relations for the in-plane and out-of-plane waves are obtained as

\[
\begin{align*}
\varpi_1 &= \sqrt{\frac{1}{\lambda_1}} \sqrt{\frac{T_p (1-v_p)}{k_1}}, \\
\varpi_2 &= \sqrt{\frac{1}{\lambda_1}} \sqrt{\frac{T_p (1-v_p)}{k_1}}, \\
\varpi_3 &= \sqrt{\frac{1}{\lambda_1}} \sqrt{\frac{T_p (1-v_p)}{k_1}},
\end{align*}
\]

where \( |\mathbf{K}| = \sqrt{k_x^2 + k_y^2} \) and \( \lambda_2 = 1 + \mu_1^2 |k|^2 + \mu_2^2 |k|^4 \). Note that \( \varpi_1 \) and \( \varpi_2 \) are the dimensionless circular frequencies associated with the in-plane waves. Furthermore, \( \varpi_3 \) represents the dimensionless circular frequency corresponding to the flexural wave. In the absence of the second small-scale parameter and neglecting the moment inertia of the thin plate (i.e., \( I_2 = 0 \)), one can arrive at the flexural circular frequency as follows:

\[
\varpi_3 = \sqrt{\frac{1}{\lambda_1} \frac{T_p (1-v_p)}{k_1}},
\]

where \( |\mathbf{K}| = \sqrt{k_x^2 + k_y^2} \) is the length of the wave number vector, \( \mathbf{k} = k_x \mathbf{i} + k_y \mathbf{j} \), in which \( k_x = \bar{k}_x / a \) and \( k_y = \bar{k}_y / b \). The obtained circular frequency of the flexural wave is similar to that predicted by Wang et al. [41], when in their proposed formulation, all the values of initial in-plane stresses, Winkler foundation modulus, and stiffness of the attached shear layer to the nanoplate are set equal to zero.

The ratio of the nonlocal frequency to the local frequency is denoted by the parameter \( \beta \); therefore, according to Eq. (39), \( \beta \) is expressed in the following form

\[
\begin{align*}
\beta_1 &= \left( \frac{\varpi_1}{\varpi_2} \right)_{n=0} = \frac{1}{\sqrt{\lambda_1}^2}, \quad i = 1, 2, \\
\beta_2 &= \left( \frac{\varpi_3}{\varpi_2} \right)_{n=0} = \frac{1}{\sqrt{\lambda_1}^2},
\end{align*}
\]

the obtained values for \( \beta_i \); \( i = 1, 2 \) are identical to those derived by Murmu and Pradhan [34] in the particular case of \( \mu_2 = 0 \).

It is noted herein that the values of \( l_1 \) and \( l_2 \) should be obtained through comparing the predicted dispersion curves from Eq. (39) with those of an appropriate atomistic-based model. However, no comprehensive study has been performed
until now regarding this matter for two-dimensional problems. Wang et al. [41] used \( l_1 = 0, 1.5, \) and 2 nm for studying flexural wave propagation in nanoplate embedded in an elastic matrix with initial stress. Pradhan and Phadikar [32] considered the value of \( l_1 \) in the range of 0–2 nm for assessing the in-plane and out-of-plane dynamic behavior of nanoplates. Murmu and Pradhan [34] also used \( l_1 = 0, 0.5, 1, 1.5, \) and 2 nm in their proposed nonlocal model to examine vibration of nanosingle-layered graphene sheets which are embedded in an elastic medium. Other works show that the values of the first small-scale parameter are commonly taken into account in the range of 0–2 nm [40,42] for the vibration analysis of nanoplates based on the nonlocal continuum theory. On the other hand, no inclusive study has been conducted on the determination of the second small-scale parameter, however, it is expected that the effects of the associated terms with that, on the dynamic behavior of nanoplates would be negligible. Despite all the performed studies on the nonlocal elasticity modeling of nanoplates, the need for determination of the accurate values of the small-scale parameters is still a vital task. In doing so, a higher-order continuum model is established based on inter-particle distances for two-dimensional nanostructures (see Appendix C.1). In a special case, it is shown that the proposed model is reduced to that of Askes et al. [46] for one-dimensional microstructures. Thereafter, a correlation between an assumed higher-order strain-gradient model and the nonlocal elasticity theory of Eringen is presented. To this end, a simple solution is proposed which relates the small-scale parameters to the inter-particle distances (see Appendix C.2). According to this correlation, it is shown that for a one-dimensional case, \( l_2/l_1 = 0.88 \). In the present work, effects of the small-scale parameters on the vibration behavior of a nanoplate acted upon by a moving nanoparticle are also of particular interest.

5.1.2. A macro-scale plate under a moving load: \( \mu_1 = \mu_2 = 0 \)

For the sake of verifying the accuracy of the obtained results from the presented analytical expressions, a comparison study is performed. Consider an isotropic square plate under a moving load traverses the plate on the line \( y = b/2 \) with a constant velocity \( (\mu_1 = \mu_2 = 0) \). The geometry and mechanical properties of the plate are as: \( a = 4 \) in, \( \beta_p = 0.1 \) in, \( \rho_p = 10^{-3} \) lb sec\(^2\)/in\(^4\), \( E_p = 30 \times 10^6 \) psi, and \( v_p = 0.3 \). The predicted values of DAF\(_w\) by the proposed analytical model with those of other researchers [43–45] have been provided in Table 1. As it is seen in Table 1, there is a good agreement between the obtained results and those of other researchers for different levels of the moving load velocity. Further, the obtained results by the proposed analytical model are generally much closer to those of Malekzadeh et al. [45].

5.2. Numerical studies

In order to investigate the effects of the velocity of the moving nanoparticle and small-scale parameters on dynamic response of the thin nanoplate, a fairly comprehensive parametric study is carried out. To this end, consider a square nanoplate under a moving nanoparticle with the following data: \( E_p = 10^{12} \) Pa, \( v_p = 0.2, \) \( \rho_p = 2500 \) kg/m\(^3\), \( a = 40 \) nm, \( \beta_p = 1 \) nm and \( \mu_k = 0.1 \). In all descriptive examples, \( M = N = 51 \), and it is presumed that the moving nanoparticle enters the nanoplate at a corner and moves along the diagonal of the nanoplate associated with that corner (i.e., \( p = 1 \) and \( q = 0 \)); hence, it is expected that the dynamic displacements in the \( x \) and \( y \) directions must be identical due to the existing symmetry of both geometry and loading about the line \( y = x \). In the numerical analyses, \( l_1 \) and \( l_2 \) are considered in the ranges of 0–2 nm and 0–1.2 nm, respectively. For more generality of the discussions, the normalized velocity is also defined as \( V_N = v/V' \) where

\[
V' = \sqrt{\frac{2\pi^2k^2E_p}{3(1-v_p^2)\rho_p(1+2(\pi\mu_1)^2+4(\pi\mu_2)^2)}}
\]

In Figs. 2(a)–(c), the time history plots of the displacements at the center of the nanoplate are presented for different values of the velocity of the moving nanoparticle and first small-scale parameter in the case of \( \mu_2 = 0 \). As it is expected, the in-plane displacements reach to their maximum values when the moving nanoparticle is just passing the center of the nanoplate. The peaks of out-of-plane displacement for different levels of the moving nanoparticle velocity has been occurred in the excitation phase; however, these peaks move toward the second phase as the velocity of the moving nanoparticle increases. Furthermore, the first small-scale parameter has a more effect on \( W_N \) when the moving nanoparticle is in the vicinity of the nanoplate’s center. This fact is more obvious for lower levels of the moving nanoparticle velocity (see Figs. 2(b) and (c)). During the course of excitation, the magnitude of the transverse displacement increases with the first small-scale parameter. Nevertheless, the dynamic response of the in-plane displacements decrease

<table>
<thead>
<tr>
<th>( t_l/T^* )</th>
<th>( v ) (in/s)</th>
<th>DAF(_w) [43]</th>
<th>DAF(_w) [44]</th>
<th>DAF(_w) [45]</th>
<th>DAF(_w) (present study)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
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<tr>
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<tr>
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<td>1.359</td>
<td>1.390</td>
<td>1.392</td>
<td>1.3904</td>
</tr>
</tbody>
</table>

\(^*\) Note: The fundamental period of the plate is denoted by \( T \).
as the first small-scale parameter increases. During the free vibration phase of the nanoplate (i.e., $\tau/\tau_f > 1$), the amplitude of displacements clearly increases with both the moving nanoparticle velocity and the first small-scale parameter.

In another study, the effect of second small-scale parameter on the dynamic displacements of the nanoplate’s center is examined. The time history of the displacements of the nanoplate’s center under the mass weight of the moving nanoparticle and the resulted frictional force have been presented in Figs. 3(a)–(c) for different levels of velocity of the moving nanoparticle and second small-scale parameter. The plots of the out-of-plane displacement show that the second small-scale parameter has a visible effect on $DAF_w$ when the moving nanoparticle is traveling in the vicinity of the nanoplate’s center. Moreover, as is seen in Figs. 3(a)–(c), the second small-scale effect has a trivial effect on the in-plane displacements. A brief comparison between the plotted results in Figs. 2 and 3 reveals that the first small-scale parameter has a greater impact on the time history of both in-plane and out-of-plane displacements of the nanoplate’s center than the second small-scale parameter. A detailed scrutiny of the effects of small-scale parameters on the in-plane and out-of-plane dynamic displacements for a wide range of the moving nanoparticle velocity is also presented in the upcoming parts.

Through Figs. 4(a)–(c), the dynamic amplitude factors of the displacements of the nanoplate’s center in terms of the velocity of the moving nanoparticle are plotted for different values of the first and second small-scale parameters. For a constant $\mu_2$, as the parameter $\mu_1$ grows, the maximum value of $DAF_w$ increases and occurs in lower levels of the velocity of the moving nanoparticle; however, this fact is not generally true for the in-plane displacements. For a constant $\mu_1$, as the parameter $\mu_2$ grows, the maximum value of $DAF_w$ increases slightly. Regarding the in-plane displacements, no distinguished peaks for the plots of $DAF_w-V_N$ could be detected, and it cannot be definitely claimed that $DAF_w$ regularly varies with $V_N$. The overall trend of the plots of $DAF_w-V_N$ indicates that $DAF_w$ increases with $V_N$.

The effect of the first small-scale parameter on the dynamic amplitude factors of the nanoplate’s center has been illustrated in Figs. 5(a)–(c) for different values of the velocity of the moving nanoparticle. As a general result, the values of $DAF_w$ increase with $\mu_1$ irrespective of the velocity of the moving nanoparticle. Concerning the in-plane displacements, the magnitude of $DAF_w$ oscillates with $\mu_1$ (see Figs. 5(a)–(c)). In the plotted results, no distinguished peaks for the plots of $DAF_w-\mu_1$ are observed. Additionally, in the case of $\mu_2 = 0.03$, the fluctuations of $DAF_w$ as a function of $\mu_1$ are more obvious than other cases.

Finally, the effect of the second small-scale parameter on $DAF$s of the in-plane and out-of-plane displacements of the nanoplate’s center is studied for different levels of the moving nanoparticle velocity. The graphs of $DAF_u$ and $DAF_w$ in terms of $\mu_2$ are provided in Figs. 6(a)–(c) for $\mu_1 = 0$, 0.025, and 0.05, respectively. As is seen in Figs. 6(a)–(c), the variation of $\mu_2$ has a slight effect on the variation of $DAF_w$, irrespective of the moving nanoparticle velocity. However, $DAF_w$ varies more obviously with $\mu_2$ for lower levels of the moving nanoparticle velocity. Regarding the effect of $\mu_2$ on $DAF_w$, $DAF_u$ generally increases as the magnitude of $\mu_2$ increases, particularly for higher values of the moving nanoparticle velocity.
In the case of $V_N = 0.7$, the effect of $\mu_2$ on DAF would be lower than 7 percent. A brief comparison between the plotted results in Figs. 5 and 6 shows that the first small-scale parameter has a more influence on the variation of both of DAF$_u$ and DAF$_w$ with respect to the second small-scale parameter.

Fig. 3. Time history of the normalized displacements of the nanoplate's center for different values of moving nanoparticle velocity and second small-scale parameter: (a) $V_N = 0.1$, (b) $V_N = 0.4$, (c) $V_N = 0.7$; (-----) $\mu_2 = 0$, (-.-) $\mu_2 = 0.015$, (---) $\mu_2 = 0.03$; $\mu_1 = 0.05$.

Fig. 4. Effect of the velocity of the moving nanoparticle on the dynamic amplitude factors of displacements at the center of the nanoplate for different values of small-scale parameters: (a) $\mu_2 = 0$, (b) $\mu_2 = 0.015$, (c) $\mu_2 = 0.03$; (-----) $\mu_1 = 0$, (-.-) $\mu_1 = 0.015$, (---) $\mu_1 = 0.03$; $\mu_1 = 0.05$. (i.e., $V_N \geq 0.4$). In the case of $V_N = 0.7$, the effect of $\mu_2$ on DAF$_w$ would be lower than 7 percent. A brief comparison between the plotted results in Figs. 5 and 6 shows that the first small-scale parameter has a more influence on the variation of both of DAF$_u$ and DAF$_w$ with respect to the second small-scale parameter.
6. Conclusions

The in-plane and transverse motions of an elastic isotropic thin nanoplate traversed by a moving nanoparticle are explored by considering small-scale effects. To this end, the equations of motion of Kircchoff plate theory under excitation...
Appendix A. The expressions of $A_i$, $B_i$, $A'_i$, $B'_i$, and $B'_j$; $i = 1, \ldots, 4; j = 1, 2, 3$

\begin{align}
A_i &= \frac{A' r_i^2 - B' r_i^2 + C}{(r_i^2 - r_1^2)(g_i + g_j)^2 - r_i^2}, \\
A'_i &= -\frac{A'(g_i + g_j)^2}{4g_i g_j(r_i^2 - (g_i + g_j)^2)} r_i^2, \\
B_i &= \frac{D' r_i^2 - E' r_i^2 + F'}{(r_i^2 - r_1^2)(g_i + g_j)^2 - r_i^2}, \\
B'_i &= -\frac{D'(g_i + g_j)^2}{4g_i g_j(r_i^2 - (g_i + g_j)^2)} r_i^2, \\
A_i &= \frac{A'^2 - B'^2 + C'^2}{(r_i^2 - r_1^2)(g_i + g_j)^2 - r_i^2}, \\
A'_i &= -\frac{A'^2 g_i + g_j}{4g_i g_j(r_i^2 - (g_i + g_j)^2)} r_i^2, \\
B_i &= \frac{D'^2 - E'^2 + F'^2}{(r_i^2 - r_1^2)(g_i + g_j)^2 - r_i^2}, \\
B'_i &= -\frac{D'^2 g_i + g_j}{4g_i g_j(r_i^2 - (g_i + g_j)^2)} r_i^2, \\
A_i &= \frac{B''}{((g_i + g_j)^2 - g_i^2)^2}, \\
A'_i &= -\frac{A''(g_i + g_j)}{4g_i g_j(a_i^2 - (g_i + g_j)^2)}, \\
B_i &= \frac{B''}{4g_i g_j(a_i^2 - (g_i + g_j)^2)}.
\end{align}

Appendix B. The expressions of $\dot{U}_{ij}$, $\dot{V}_{ij}$, $\ddot{W}_{ij}$, $\dddot{U}_{ij}$, $\dddot{V}_{ij}$, and $\dddot{W}_{ij}$

Case (1): $g_i \neq g_j$

\begin{align}
\dot{U}_{ij} &= A_1 \cos(r_1 \tau_j) + A_2 \cos(r_2 \tau_j) + A_3 \cos((g_i + g_j) \tau_j) + A_4 \cos((g_i - g_j) \tau_j) + B'_1 \sin(r_1 \tau_j) + B'_2 \sin(r_2 \tau_j) + B'_3 \sin((g_i + g_j) \tau_j) + B'_4 \sin((g_i - g_j) \tau_j), \\
\dot{V}_{ij} &= A'_1 \cos(r_1 \tau_j) + A'_2 \cos(r_2 \tau_j) + A'_3 \cos((g_i + g_j) \tau_j) + A'_4 \cos((g_i - g_j) \tau_j) + B''_1 \sin(r_1 \tau_j) + B''_2 \sin(r_2 \tau_j) + B''_3 \sin((g_i + g_j) \tau_j) + B''_4 \sin((g_i - g_j) \tau_j), \\
\ddot{W}_{ij} &= A''_1 \cos((a_4 + g_i + g_j) \tau_j) + A''_2 \cos((g_i + g_j) \tau_j) + A''_3 \cos((g_i - g_j) \tau_j) + \frac{B'''_1}{a_4} \sin(a_4 \tau_j) + \frac{B'''_2}{g_i + g_j} \sin((g_i + g_j) \tau_j) + \frac{B'''_3}{g_i - g_j} \sin((g_i - g_j) \tau_j), \\
\dddot{U}_{ij} &= -(A_1 r_1 \sin(r_1 \tau_j) + A_2 r_2 \sin(r_2 \tau_j) + A_3 (g_i + g_j) \sin((g_i + g_j) \tau_j) + A_4 (g_i - g_j) \sin((g_i - g_j) \tau_j) + B'_1 \cos(r_1 \tau_j) + B'_2 \cos(r_2 \tau_j) + B'_3 \cos((g_i + g_j) \tau_j) + B'_4 \cos((g_i - g_j) \tau_j), \\
\dddot{V}_{ij} &= -(A'_1 r_1 \sin(r_1 \tau_j) + A'_2 r_2 \sin(r_2 \tau_j) + A'_3 (g_i + g_j) \sin((g_i + g_j) \tau_j) + A'_4 (g_i - g_j) \sin((g_i - g_j) \tau_j) + B''_1 \cos(r_1 \tau_j) + B''_2 \cos(r_2 \tau_j) + B''_3 \cos((g_i + g_j) \tau_j) + B''_4 \cos((g_i - g_j) \tau_j), \\
\dddot{W}_{ij} &= -A''_1 r_1 \sin(r_1 \tau_j) - A''_2 r_2 \sin(r_2 \tau_j) - A''_3 (g_i + g_j) \sin((g_i + g_j) \tau_j) - A''_4 (g_i - g_j) \sin((g_i - g_j) \tau_j) - B'''_1 \cos(a_4 \tau_j) - B'''_2 \cos((g_i + g_j) \tau_j) - B'''_3 \cos((g_i - g_j) \tau_j). 
\end{align}
\[ \dot{V}_{ij} = -(A_1 r_1 \sin(r_1 \tau) + A_2 r_2 \sin(r_2 \tau) + A_3 (g_i + g_j) \sin((g_i + g_j) \tau) + A_4 (g_i - g_j) \sin((g_i - g_j) \tau)) + B_1 \cos(r_1 \tau) + B_2 \cos(r_2 \tau) + B_3 \cos((g_i + g_j) \tau) + B_4 \cos((g_i - g_j) \tau), \]

\[ \dot{W}_{ij} = -(A_i a_4 \sin(a_4 \tau) + A_j a_4 \sin(a_4 \tau)) + B_1' \cos(a_4 \tau) + B_2' \cos(a_4 \tau) + B_3' \cos(a_4 \tau), \]

Case (II): \( g_i = g_j \)

\[ \dot{U}_{ij} = A_1' \cos(r_1 \tau) + A_2' \cos(r_2 \tau) + A_3' \cos(2g_j \tau) + \frac{B_1'}{r_1} \sin(r_1 \tau) + \frac{B_2'}{r_2} \sin(r_2 \tau) + \frac{B_3'}{2g_j} \sin(2g_j \tau), \]

\[ \dot{V}_{ij} = A_1' \cos(r_1 \tau) + A_2' \cos(r_2 \tau) + A_3' \cos(2g_j \tau) + \frac{B_1'}{r_1} \sin(r_1 \tau) + \frac{B_2'}{r_2} \sin(r_2 \tau) + \frac{B_3'}{2g_j} \sin(2g_j \tau), \]

\[ \dot{W}_{ij} = \frac{A_1''}{a_4} \sin(a_4 \tau) + \frac{A_2''}{2g_j} \sin(2g_j \tau) + B_1'' + B_2'' \cos(a_4 \tau) + B_3'' \cos(2g_j \tau), \]

\[ \dot{U}_{ij} = -(A_1' r_1 \sin(r_1 \tau) + A_2' r_2 \sin(r_2 \tau) + 2A_3' g_j \sin(2g_j \tau)) + B_1' \cos(r_1 \tau) + B_2' \cos(r_2 \tau) + B_3' \cos(2g_j \tau), \]

\[ \dot{V}_{ij} = -(A_1' r_1 \sin(r_1 \tau) + A_2' r_2 \sin(r_2 \tau) + 2A_3' g_j \sin(2g_j \tau)) + B_1' \cos(r_1 \tau) + B_2' \cos(r_2 \tau) + B_3' \cos(2g_j \tau), \]

\[ \dot{W}_{ij} = A_1'' \cos(a_4 \tau) + A_2'' \cos(2g_j \tau) - (B_2'' a_4 \sin(a_4 \tau) + 2B_3'' g_j \sin(2g_j \tau)). \]

**Appendix C. Determination of \( l_1 \) and \( l_2 \) in terms of inter-particle distances**

**C.1. Development of a higher-order strain-gradient model using a two-dimensional discrete model for the nanostructure**

In this part, a higher-order strain-gradient model is derived based on a discrete model for the two-dimensional nanostructure. To this end, the constitutive particles of the nanostructure are modeled by lumped masses which are connected through inter-particle springs. For the sake of convenient, it is assumed that all particles have the same mass \( M \) and they are uniformly distributed within the spatial domain as shown in Fig. 7. The inter-particle distances along the \( x \) and \( y \) directions are \( d_x \) and \( d_y \), respectively. The inter-particle force between any two particles is modeled via an axial spring. The magnitude of spring stiffness relies significantly on the bond length of particles. The stiffness values of the springs along the \( x \) and \( y \) directions are denoted by \( K_x \) and \( K_y \), respectively, and the stiffness of the diagonal springs is represented by \( K_d \) (see Fig. 7). The equations of motion of a particle located in the \( n \)th column and \( m \)th row of the discrete model are expressed as

\[ MU_{nm,tt} + K_d(2U_{nm} - U_{n+1,m} - U_{n-1,m}) + K_d \sin(x) \cos(x)(V_{n+1,m+1} - V_{n+1,m+1} - V_{n+1,m+1} - V_{n+1,m+1}) + K_d \cos^2(x)(4U_{nm} - U_{n+1,m-1} - U_{n+1,m+1} - U_{n+1,m+1}) = 0, \]

**Fig. 7.** A mass-spring model for discretization of the two-dimensional nanostructure.
where $\alpha$ is the angle between the diagonal spring and the $x$-axis, and $U_x$ and $V_y$ denote the $x$ and $y$ components of displacements of the particle located in the $i$th column-$j$th row of the mass-spring model. For the passage from the discrete model to the continuum model, the displacements of the neighboring particles along the $x$- and $y$-axes are obtained such that their values at the particle specified by $(n,m)$ would be equal to $U_{nm}$ and $V_{nm}$, respectively. For this purpose, the displacements of the neighboring particles are approximated using sixth-order Taylor polynomials as in the following:

$$U(x+d_x,y+d_y) = \sum_{n=1}^{6} \sum_{m=0}^{n-m} \left(n \atop n-m\right) \frac{\partial^n U(x,y)}{\partial x^n \partial y^{n-m}} d_x^n d_y^{n-m},$$

$$V(x+d_x,y+d_y) = \sum_{n=1}^{6} \sum_{m=0}^{n-m} \left(n \atop n-m\right) \frac{\partial^n V(x,y)}{\partial x^n \partial y^{n-m}} d_x^n d_y^{n-m},$$

in which $\partial^n U(x,y)/\partial x^n \partial y^m = U(x,y)$, and $U(x,y)$ and $V(x,y)$ denote the displacement fields along $x$ and $y$ directions, respectively. According to Eq. (C.2), the discrete displacements $U_g$ and $V_y$; $i=n-1,n+1; j=m-1,m+1$ of the mass-spring model could be obtained as a function of continuous displacements. Then, these expressions are substituted into Eq. (C.1). Without lose of generality, let $d_x = d_y = d$ and $K_N = K_N = K$; after some manipulations, one could arrive at the continuous form of the equations of motion of the nanoscale:

$$\rho_p U_{tt} = K \left( \frac{\partial^2 U}{\partial x^2} + \frac{d^2 \partial^4 U}{360 \partial x^6} + \frac{d^6 \partial^8 U}{20160 \partial x^{10}} \right) + \frac{K_d}{t_p} \left[ \frac{\partial^2 U}{\partial x^2} + \frac{d^2 \partial^4 U}{360 \partial x^6} + \frac{d^4 \partial^6 U}{360 \partial x^8} + \frac{d^6 \partial^8 U}{360 \partial x^{10}} \right],$$

$$\rho_p V_{tt} = K \left( \frac{\partial^2 V}{\partial y^2} + \frac{d^2 \partial^4 V}{360 \partial y^6} + \frac{d^6 \partial^8 V}{360 \partial y^{10}} \right) + \frac{K_d}{t_p} \left[ \frac{\partial^2 V}{\partial y^2} + \frac{d^2 \partial^4 V}{360 \partial y^6} + \frac{d^4 \partial^6 V}{360 \partial y^8} + \frac{d^6 \partial^8 V}{360 \partial y^{10}} \right],$$

where $\rho_p = M/(d^2 t_p)$. On the other hand, $\varepsilon_{xx} = \partial U/\partial x$, $\varepsilon_{yy} = \partial V/\partial y$, and $\gamma_{xy} = \partial U/\partial y + \partial V/\partial x$. Therefore, Eq. (C.3) could also be expressed in terms of strains and their derivatives. The resulted equations represent the higher-order strain-gradient equations of motion for the two-dimensional nanosheet.

In a special case of $K_d = 0$, it can be shown that the displacements are decoupled and each component of displacements is only expressed as a function of the coordinate of its direction. In such a case, the equation of motion of a one-dimensional nanosheet in terms of higher-order strain gradients is obtained:

$$\rho_p U_{tt} = E_p \frac{\partial^2 \sigma_{xx}}{\partial x^2},$$

where $E_p = k/t_p$ and the only stress component involves higher-order gradients of strain is

$$\sigma_{xx}^h = E_p \left( \varepsilon_{xx} + \frac{d^2 \varepsilon_{xx}}{12 \partial x^2} + \frac{d^4 \varepsilon_{xx}}{360 \partial x^6} + \frac{d^6 \varepsilon_{xx}}{20160 \partial x^{10}} \right).$$

It is worth mentioning that Eq. (C.5) is identical to that derived by Askes et al. [46] when studying of one-dimensional continuous microstructure under static and dynamic loadings was of concern.

### C.2. A correlation between a higher-order strain-gradient model and the nonlocal continuum theory

It is now assumed that $\sigma_{ij}^h$ could be expressed in terms of higher-order gradients of strains as follows:

$$\sigma_{ij}^h = D_{ijkl} \varepsilon_{ij}^h,$$

where

$$\varepsilon_{ij}^h = \varepsilon_{ij}^h + r_3^2 \nabla^2 \varepsilon_{ij}^h + r_2^2 \nabla^4 \varepsilon_{ij}^h,$$

in which $D_{ijkl}$ denotes the fourth-order material property tensor of the nanosheet within the framework of linear elasticity, $\varepsilon_{ij}^h$ is the classical strain tensor (or local strain), and $\varepsilon_{ij}^h$ represents the higher-order strain tensor. By applying the
operators $r_1^2 \nabla^2$ and $r_2^2 \nabla^4$ to both sides of Eq. (C.7),
\begin{equation}
  r_1^2 \nabla^2 \varepsilon_{ij}^h = r_1^4 \nabla^2 \varepsilon_{ij} + r_1^4 \nabla^4 \varepsilon_{ij} + r_1^4 \nabla^6 \varepsilon_{ij}, \tag{C.8}
\end{equation}
\begin{equation}
  r_2^4 \nabla^4 \varepsilon_{ij}^h = r_2^4 \nabla^2 \varepsilon_{ij} + r_2^4 \nabla^4 \varepsilon_{ij} + r_2^4 \nabla^8 \varepsilon_{ij}, \tag{C.9}
\end{equation}
now a linear combination of Eqs. (C.7)–(C.9) should be considered such that the resulted expression would be equal to $\varepsilon_{ij}^l$.

Therefore,
\begin{equation}
  \varepsilon_{ij}^h + \alpha r_1^2 \nabla^2 \varepsilon_{ij}^h + \beta r_2^4 \nabla^4 \varepsilon_{ij}^h \approx \varepsilon_{ij}^l, \tag{C.10}
\end{equation}
by substituting the equivalent value of $\varepsilon_{ij}^h$ from Eq. (C.7) into Eq. (C.10), and solving the resulted equation for $\alpha$ and $\beta$,
\begin{equation}
  \alpha = -1, \quad \beta = \left(\frac{r_1}{r_2}\right)^4 - 1. \tag{C.11}
\end{equation}

On the other hand, the nonlocal strains are related to the local ones using nonlocal continuum theory of Eringen as
\begin{equation}
  e_{ij}^{nl} - \beta \nabla^2 e_{ij}^{nl} + \alpha \nabla^4 e_{ij}^{nl} = e_{ij}^{l}, \tag{C.12}
\end{equation}
a brief comparison of Eqs. (C.10) and (C.12) reveals that the first and second small-scale parameters are
\begin{equation}
  l_1 = \sqrt{-\alpha |r_1|}, \quad l_2 = \sqrt{|\beta| |r_2|}. \tag{C.13}
\end{equation}

Eq. (C.12) furnishes us regarding the relation between the nonlocal continuum theory and higher-order strain-gradient model. For a one-dimensional nanostructure, $r_1 = d/\sqrt{12}$, $r_2 = d/\sqrt{360}$, and, therefore, $\beta = 1.5$. Using Eqs. (C.13) and (C.11), the small-scale parameters associated with the nonlocal continuum theory are obtained as $l_1 = r_1$ and $l_2 = 0.8801r_1$ (or $l_1 = 0.2887d$ and $l_2 = 0.2541d$).

References


