Stability and vibrations of doubly parallel current-carrying nanowires immersed in a longitudinal magnetic field

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A R T I C L E   I N F O

Article history:
Received 15 August 2014
Received in revised form 31 October 2014
Accepted 6 November 2014
Available online 13 November 2014
Communicated by R. Wu

Keywords:
Transverse vibration
Doubly parallel nanowires
Surface elasticity theory
Electric current
Longitudinal magnetic field
Biot–Savart law

A B S T R A C T

This paper deals with dynamic interactions of two parallel nanowires carrying electric currents in the presence of a longitudinal magnetic field. Using Biot–Savart law and a surface elasticity model, the equations of motion are obtained. Accounting for both Lorentz and gravity forces, the static and the purely dynamic parts of the total displacements of the nanosystem are explicitly expressed. Two crucial modes of vibration, synchronous and asynchronous patterns, are identified and their characteristics are inclusively explained. It is shown that the nanosystem becomes dynamically unstable under certain conditions in the asynchronous mode. The minimum initial tensile force as well as the maximum values of the electric current and the magnetic field strength corresponding to the dynamic instability are derived. The roles of the crucial factors on the lowest asynchronous frequencies are also addressed and discussed.

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1. Introduction

Because of their superior mechanical behavior [1,2] as well as advanced physical and chemical properties [3–5], nanowires have been regarded as a hot materials in recent years. Nanowires exist in various forms which are made of metals, organic compounds, insulators, and semiconductors. These nanostructures have been widely scrutinized for potential exploitation in optics [6–8], energy conversion and storage [9,10], bioengineering [11,12], medicine [13,14], physical sensing [15,16], chemical sensing [17,18], electronics [19–21], and micro-/nano-electromechanical systems (MEMs/NEMS) [22–25]. In the latter two applications, an ensemble of current-carrying nanowires in the presence of a magnetic field may be used for the considered jobs, however, little is known on their vibrations and dynamic instabilities. Such a fact encouraged the author to explore dynamic behaviors of magnetically affected double-nanowire-systems as a starting point for better realizing vibrations of more complex systems like as networks of current-carrying nanowires.

Due to their high slenderness, nanowires are classified as one-dimensional nanostructures. The length of a nanowire is at least 20 times greater than its lateral sizes which are at the nanometer scale. When a structure becomes so thin such that its thickness reduces to nanoscale, the surface effect becomes important and should be appropriately taken into account. The surface elasticity theory of Gurtin and Murdoch [26,27] is one of the most well-known continuum theories deals with the equations of motion of the surface layers as well as their constitutive relations. So far, such a model has been extensively applied to many physical problems of nanowires including static analysis [28–30], buckling [31–33], vibrations [34–37], and postbuckling [38].

A close scrutiny of the literature shows that there exist a large body of works on the effect of the magnetic fields on vibrations of nanostructures [39–49]. In most of these works, the nanostructures were made of a highly conducting material and no electric current was passing through them. Former works on the influence of the magnetic field on the current-carrying nanowires display that how deformation of the nanowires would lead to the exertion of the Lorentz magnetic force on them [50,51]. In this regard, Kiani [50] investigated forced vibrations of current-carrying nanowires subjected to a longitudinal magnetic field. Using Galerkin and Newmark–β approaches, the equations of motion were discretized in the spatial and time domains, respectively. Additionally, the roles of the frequency of the applied load, surface effect, nonlocality, magnetic field strength, and electric current on the vibrations of the nanostructure were explored. In a complementary work, Kiani [51] addressed dynamic instability of an individual current-carrying nanowire acted upon by a longitudinal...
magnetic field in the context of the surface elasticity theory. By proposing an analytical approach, the influences of the magnetic field strength, direct electric current, surface effect, and initial tensile force on the transverse displacements as well as natural frequencies of the nanostructure were explained. The above-mentioned works were restricted to dynamic analysis of a single nanowire as an electric current carrier in a magnetic field. When one confronts to a double-current-carrying nanowire-system acted upon by a longitudinal magnetic field, each nanowire is not only affected by the longitudinal magnetic field, but also by the generated magnetic field of its adjacent nanowire according to the Biot–Savart law. The resulted magnetic field plus to the longitudinally applied magnetic field yields exertion of the Lorentz forces on the nanowire. Such a fact makes some complexities in solving the problem because of the application of two magnetic fields on each nanowire in which their sources are completely different.

In this paper, an analytical solution is developed to understand the mechanism of dynamic interactions between the magnetically affected double current-carrying nanowires. The origin of the dynamic instability in such a system is also of concern. To this end, using Biot–Savart and Lorentz force laws, the interactional magnetic forces between two long elastic nanowires immersed in a longitudinal magnetic field are evaluated. By adopting an appropriate surface elasticity model, the governing equations associated with the transverse vibrations of the nanowires are derived under gravity and magnetic forces. An analytical solution is then proposed and the corresponding static and dynamic displacements of the nanosystem are obtained. The explicit expressions of the natural frequencies are obtained and their corresponding vibration modes are explained. The conditions lead to the instability of the nanostructure are inclusively displayed. The roles of the interwire distance, strength of the magnetic field, electric current, and initial tensile force on the lowest frequencies pertinent to the asynchronous pattern of vibration are comprehensively explained and discussed. It is hoped that the obtained results in this paper would give useful insights to those researchers who are interested in dynamic behaviors of magnetically affected networks of nanowires used for carrying electric current.

2. Assessment of the mutually exerted magnetic forces

Consider two long, initially straight, parallel nanowires of length $l_0$ immersed in a longitudinal magnetic field, $B_0 = B_0 \mathbf{e}_x$, as illustrated in Fig. 1. These nanowires are located in a free space, separated by a distance $d$, and carry electric current of magnitude $I$ in the $x$ direction. The Cartesian coordinates system has been attached to the nanowire 1 such that the $x$-axis is coincident with its revolutionary axis and the $z$-axis is along the applied gravitational acceleration, $g$. The unit base vectors associated with the $x$, $y$, and $z$ axes are denoted by $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, respectively. The displacement field vectors of the nanowires are represented by: $\mathbf{u}_i = \mathbf{v}_i(x, t) \mathbf{e}_y + \mathbf{w}_i(x, t) \mathbf{e}_z$; $i = 1, 2$ where $r$ is the time parameter. Because of the small deformation, the more accurate electric current vectors in these nanowires would be: $\mathbf{l}_i = l_i \mathbf{e}_x + l_i \frac{\partial \mathbf{v}_i}{\partial x} \mathbf{e}_y + l_i \frac{\partial \mathbf{w}_i}{\partial x} \mathbf{e}_z$; $i = 1, 2$ where $\partial$ represents the partial differential symbol. In this part, we are interested in determining the magnetic force on each nanowire due to the applied magnetic fields.

Based on the Biot–Savart law, the produced magnetic field by the current-carrying nanowire 1 at a point $(x_2, d + v_2, w_2)$ of the nanowire 2 with distance $r$ from the deformed element $ds_1$ of the nanowire 1 is given by:

$$B_1 = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{ds_1 \times \mathbf{e}_r}{r^2},$$

where $\mu_0 = 4\pi \times 10^{-7} \text{Tm/A}$ is the permeability of free space, $ds_1 \approx dx(\mathbf{e}_x + \frac{\partial \mathbf{v}_1}{\partial x} \mathbf{e}_y + \frac{\partial \mathbf{w}_1}{\partial x} \mathbf{e}_z)$ for small deformation, $\mathbf{e}_r$ is the unit base vector pertinent to the position vector of the point with respect to the deformed element of the nanowire 1, and $r = \sqrt{(x_2 - x_1)^2 + (d + \Delta v)^2 + (\Delta w)^2}$ where $\Delta v = v_2 - v_1$ and $\Delta w = w_2 - w_1$. For long nanowires, Eq. (1) is reduced to:

$$B_1 = \frac{\mu_0 I}{2\pi r} \mathbf{e}_r,$$

where $r'$ and $\mathbf{e}_r'$ in order denote the length and unit base vector of $r' = (d + \Delta v) \mathbf{e}_y + \Delta w \mathbf{e}_z$. Therefore, the resulting magnetic field vector at the position of the deformed nanowire 2 is:

$$B_2 = \left( B_0 + \frac{\mu_0 I}{2\pi r'} \left( \frac{\partial \mathbf{v}_1}{\partial x} \sin \alpha - \frac{\partial \mathbf{w}_1}{\partial x} \cos \alpha \right) \right) \mathbf{e}_x - \frac{\mu_0 I}{2\pi r'} \sin \alpha \mathbf{e}_y + \frac{\mu_0 I}{2\pi r'} \cos \alpha \mathbf{e}_z,$$

where $\sin \alpha = \frac{\Delta v}{r'}$ and $\cos \alpha = \frac{d + \Delta v}{r'}$. Based on the Lorentz force law, the exerted magnetic force per unit length of the nanowire 2 is evaluated by:

$$\mathbf{f}_2 = I_2 \times B_2,$$

Fig. 1. Doubly current-carrying nanowires subjected to a longitudinal magnetic field.
by substituting Eq. (3) into Eq. (4), expressing the Taylor series expansion of the resulting relation about the initial equilibrium state up to the first-order, the applied magnetic force on the nanowire 2 in the context of small deformations is obtained as follows:

$$ f_2 = \left( B_0 I \frac{\partial w_2}{\partial x} - \mu_0 I^2 \frac{1 - \Delta w}{d} \right) e_y - \left( B_0 I \frac{\partial v_2}{\partial x} + \mu_0 I^2 \frac{\Delta w}{d} \right) e_z. $$

Similarly, the exerted magnetic force on the unit length of the nanowire 1 because of the generated magnetic field by the nanowire 2 as well as the existing longitudinal magnetic field is given by:

$$ f_1 = \left( B_0 I \frac{\partial w_1}{\partial x} + \mu_0 I^2 \frac{1 - \Delta w}{d} \right) e_y + \left( -B_0 I \frac{\partial v_1}{\partial x} + \mu_0 I^2 \frac{\Delta w}{d} \right) e_z. $$

(6)

3. Equations of motion based on surface elasticity theory

In this part, we are seeking for governing equations to explain how transverse vibrations of each magnetically affected nanowire can affect on the vibrations of its neighboring wire. For this purpose, using surface elasticity theory, the equations of motion describing transverse vibrations of each nanowire are obtained by considering the exerted magnetic and gravity forces. Due to the appearance of both static and dynamic forces in the formulations, the total displacements of the nanowires are decomposed to static and pure dynamic displacements. Therefore, their dimensionless equations of motion are obtained with appropriate initial and boundary conditions.

To describe transverse vibrations of long nanowires, a string continuum-based model is employed. The major difference of a string model with the beam model is that the bending rigidity of the nanostructure is not incorporated into the formulations of the model. For lengthy nanostructures whose major lateral stiffness is provided by the initial tensile loads, application of an appropriate string model results in a fairly rational estimation of their dynamic response. By considering the surface effect, the equations of motion of the nanowires are expressed by [50–52]:

$$ \left( \rho_b A_b + \rho_0 S_0 \right) \frac{\partial^2 u_i}{\partial t^2} - \left( T_0 + H_0 \right) \frac{\partial^2 u_i}{\partial x^2} = f_i + f_{w_i}; \quad i = 1, 2, $$

(7)

where $\rho_b$ is the bulk density, $A_b$ is the cross-sectional area, $\rho_0$ is the surface density, $T_0$ is the initial tensile force within the nanowires, $S_0 = \pi r_0^2$, $H_0 = 4\pi r_0^2$ in which $r_0$ is the residual surface stress under unconstrained conditions and $r_b$ denotes the radius of the solid nanowire, and $f_{w_i} = \rho_b A_b \rho g e_y$ represents the nanowires’ weight force per unit length. In Eq. (7), the effect of the interwire van der Waals (vdW) force has been ignored. It is because of the following reasons: (i) In all given parametric studies, the interwire free space (i.e., $d_0 = d - 2r_b$) is greater than the nanowire’s radius. In such conditions, the generated vdW force due to the transverse displacements can be safely excluded according to the obtained results in Refs. [53,54]. In fact, for $d_0 > r_b$ in the absence of the electric current, each nanowire vibrates individually and independent from its neighboring wire. (ii) For lengthy nanowires which are of concern of the present work, a minimum interwire free space should be forced for getting ride of dynamic instability and reducing the risk of contact between the doubly current-carrying nanowires. As it will be shown, electric current, strength of the longitudinal magnetic field, initial tensile force, and the surface effect are among the major factors that control the maximum static and dynamic deflections.

By substituting Eqs. (5) and (6) into Eq. (7), the transverse equations of motion of the magnetically affected current-carrying nanowires are derived as follows:

$$ \left( \rho_b A_b + \rho_0 S_0 \right) \frac{\partial^2 v_1}{\partial t^2} - \left( T_0 + H_0 \right) \frac{\partial^2 v_1}{\partial x^2} - B_0 I \frac{\partial w_1}{\partial x} + \mu_0 I^2 \frac{1}{2\pi d^2} (v_2 - v_1) = \mu_0 I^2 \frac{1}{2\pi d}, $$

(8a)

$$ \left( \rho_b A_b + \rho_0 S_0 \right) \frac{\partial^2 w_1}{\partial t^2} - \left( T_0 + H_0 \right) \frac{\partial^2 w_1}{\partial x^2} + B_0 I \frac{\partial v_1}{\partial x} - \mu_0 I^2 \frac{1}{2\pi d^2} (w_2 - w_1) = \rho_b A_b g, $$

(8b)

$$ \left( \rho_b A_b + \rho_0 S_0 \right) \frac{\partial^2 v_2}{\partial t^2} - \left( T_0 + H_0 \right) \frac{\partial^2 v_2}{\partial x^2} - B_0 I \frac{\partial w_2}{\partial x} + \mu_0 I^2 \frac{1}{2\pi d^2} (v_2 - v_1) = -\mu_0 I^2 \frac{1}{2\pi d}, $$

(8c)

$$ \left( \rho_b A_b + \rho_0 S_0 \right) \frac{\partial^2 w_2}{\partial t^2} - \left( T_0 + H_0 \right) \frac{\partial^2 w_2}{\partial x^2} + B_0 I \frac{\partial v_2}{\partial x} + \mu_0 I^2 \frac{1}{2\pi d^2} (w_2 - w_1) = \rho_b A_b g. $$

(8d)

Since both ends of the nanowires are prevented from any lateral movement, the following boundary conditions should be satisfied:

$$ v_i(0, t) = w_i(0, t) = 0; \quad v_i(t, 0) = w_i(t, 0) = 0; \quad i = 1, 2. $$

(9)

Additionally, it is assumed that the nanowires start to vibrate from the following general initial conditions:

$$ v_i(x, 0) = v_i^0(x), \quad w_i(x, 0) = w_i^0(x); \quad i = 1, 2, $$

$$ \frac{\partial v_i}{\partial t} (x, 0) = v_i^0(x), \quad \frac{\partial w_i}{\partial t} (x, 0) = w_i^0(x). $$

(10)

Let $v_i(x, t) = v_i^s(x, t)$ and $w_i(x, t) = w_i^s(x, t); i = 1, 2$ where $[\cdot]^s$ and $[\cdot]^d; [\cdot] = v$ or $w$, denote the static and the pure dynamic displacements of the $i$th nanowire, respectively. By introducing these expressions to Eqs. (8)–(10) the governing equations associated with the static deformation of the nanosystem are stated by:

$$ -(T_0 + H_0) \frac{\partial^2 v_1^s}{\partial x^2} - B_0 I \frac{\partial w_1^s}{\partial x} + \mu_0 I^2 \frac{1}{2\pi d^2} (v_2^s - v_1^s) = \mu_0 I^2 \frac{1}{2\pi d}, $$

(11a)

$$ -(T_0 + H_0) \frac{\partial^2 w_1^s}{\partial x^2} + B_0 I \frac{\partial v_1^s}{\partial x} - \mu_0 I^2 \frac{1}{2\pi d^2} (w_2^s - w_1^s) = \rho_b A_b g. $$

(11b)
\[-(T_0 + H_0) \frac{\partial^2 v_i^2}{\partial x^2} - B_0 I \frac{\partial w_i^2}{\partial x} - \frac{\mu_0 I^2}{2\pi d^2} (v_i^2 - v_i^1) = - \frac{\mu_0 I^2}{2\pi d^2}, \tag{11c}\]
\[-(T_0 + H_0) \frac{\partial^2 w_i^2}{\partial x^2} + B_0 I \frac{\partial v_i^2}{\partial x} + \frac{\mu_0 I^2}{2\pi d^2} (w_i^2 - w_i^1) = \rho_b A g, \tag{11d}\]

with the following constraints:

\[v_i^0(0) = v_i^1(0) = 0; \quad w_i^0(0) = w_i^1(0) = 0; \quad i = 1, 2. \tag{12}\]

Furthermore, the equations of motion pertinent to the dynamic state of the magnetically affected current-carrying nanowires would be expressed by:

\[(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_i^d}{\partial x^2} - B_0 I \frac{\partial w_i^d}{\partial x} + \frac{\mu_0 I^2}{2\pi d^2} (v_i^2 - v_i^1) = 0, \tag{13a}\]
\[(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 w_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 w_i^d}{\partial x^2} + B_0 I \frac{\partial v_i^d}{\partial x} - \frac{\mu_0 I^2}{2\pi d^2} (w_i^2 - w_i^1) = 0, \tag{13b}\]
\[(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_i^d}{\partial x^2} - B_0 I \frac{\partial w_i^d}{\partial x} - \frac{\mu_0 I^2}{2\pi d^2} (v_i^2 - v_i^1) = 0, \tag{13c}\]
\[(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 w_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 w_i^d}{\partial x^2} + B_0 I \frac{\partial v_i^d}{\partial x} + \frac{\mu_0 I^2}{2\pi d^2} (w_i^2 - w_i^1) = 0, \tag{13d}\]

with the following boundary conditions:

\[v_i^d(0, t) = v_i^d(t_b, t) = 0, \quad w_i^d(0, t) = w_i^d(t_b, t) = 0, \tag{14}\]

and the initial conditions of the pure dynamic displacements are as:

\[v_i^d(x, 0) = v_i^0(x), \quad w_i^d(x, 0) = w_i^0(x); \quad i = 1, 2, \]
\[\dot{v}_i^d(x, 0) = \dot{v}_i^0(x), \quad \dot{w}_i^d(x, 0) = \dot{w}_i^0(x). \tag{15}\]

where \(v_i^0(x) = v_i^0(x) - v_i^1(x), w_i^0(x) = w_i^0(x) - w_i^1(x), \dot{v}_i^0(x) = \dot{v}_i^0(x), \) and \(\dot{w}_i^0(x) = \dot{w}_i^0(x). \)

In order to analyze the problem at hand more conveniently, the following dimensionless parameters are taken into account:

\[\xi = \frac{x}{l_b}, \quad \tau = \frac{t}{l_b \sqrt{\frac{E_b}{\rho_b}}}, \quad \tilde{v}_i^1 = \frac{v_i^1}{l_b}, \quad \tilde{w}_i^1 = \frac{w_i^1}{l_b}, \quad \tilde{d} = \frac{d}{l_b}, \quad \tilde{T}_0 = \frac{T_0}{E_b A_b}, \quad \tilde{H}_0 = \frac{H_0}{E_b A_b}, \quad \tilde{f}_0 = \frac{B_0 I l_b}{E_b A_b}, \quad \tilde{\mu}_0 = \frac{\mu_0 I}{2\pi d^2 E_b A_b}, \quad \tilde{m}_0 = \frac{\rho_0 S_0}{\rho_b A_b}, \quad \tilde{g} = \frac{\rho_b A_b g}{E_b A_b}, \tag{16}\]

where \([\cdot]\) = s or d, and \(E_b\) is the Young modulus of the nanowires. By introducing Eq. (16) to Eqs. (11) and (12), the dimensionless governing equations corresponding to the static deformation are obtained as:

\[-(\tilde{T}_0 + \tilde{H}_0) \frac{\partial^2 \tilde{v}_i^1}{\partial \xi^2} - \tilde{f}_0 \frac{\partial \tilde{v}_i^1}{\partial \xi} + \tilde{f}_1 (\tilde{v}_2^1 - \tilde{v}_1^1) = \tilde{f}_1 \tilde{d}, \tag{17a}\]
\[-(\tilde{T}_0 + \tilde{H}_0) \frac{\partial^2 \tilde{w}_i^1}{\partial \xi^2} + \tilde{f}_0 \frac{\partial \tilde{w}_i^1}{\partial \xi} - \tilde{f}_1 (\tilde{w}_2^1 - \tilde{w}_1^1) = \tilde{f}_g, \tag{17b}\]
\[-(\tilde{T}_0 + \tilde{H}_0) \frac{\partial^2 \tilde{v}_i^2}{\partial \xi^2} - \tilde{f}_0 \frac{\partial \tilde{v}_i^2}{\partial \xi} - \tilde{f}_1 (\tilde{v}_2^2 - \tilde{v}_1^2) = \tilde{f}_1 \tilde{d}, \tag{17c}\]
\[-(\tilde{T}_0 + \tilde{H}_0) \frac{\partial^2 \tilde{w}_i^2}{\partial \xi^2} + \tilde{f}_0 \frac{\partial \tilde{w}_i^2}{\partial \xi} + \tilde{f}_1 (\tilde{w}_2^2 - \tilde{w}_1^2) = \tilde{f}_g, \tag{17d}\]

with the following boundary conditions:

\[\tilde{v}_i^1(0) = \tilde{v}_i^1(1) = 0; \quad \tilde{w}_i^1(0) = \tilde{w}_i^1(1) = 0. \tag{18}\]

Additionally, the dimensionless equations of motion of the magnetically affected current-carrying nanowires corresponding to their pure dynamic deformation are derived as:

\[\frac{\partial^2 \tilde{v}_i^1}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{v}_i^1}{\partial \xi^2} - \tilde{f}_0 \frac{\partial \tilde{v}_i^1}{\partial \xi} + \tilde{f}_1 s (\tilde{v}_2^1 - \tilde{v}_1^1) = 0, \tag{19a}\]
\[\frac{\partial^2 \tilde{w}_i^1}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{w}_i^1}{\partial \xi^2} + \tilde{f}_0 \frac{\partial \tilde{w}_i^1}{\partial \xi} - \tilde{f}_1 s (\tilde{w}_2^1 - \tilde{w}_1^1) = 0. \tag{19b}\]
\[ \frac{\partial^2 \mathbf{d}}{\partial \tau^2} - T_s \frac{\partial^2 \mathbf{d}}{\partial \xi^2} + F_0 + \frac{\partial \mathbf{d}}{\partial \xi} + F_{1s} \mathbf{d} = \mathbf{0}, \]  

(19c)

\[ \frac{\partial^2 \mathbf{w}_i}{\partial \tau^2} - T_s \frac{\partial^2 \mathbf{w}_i}{\partial \xi^2} + T_{0i} \frac{\partial \mathbf{w}_i}{\partial \xi} + T_{1i} (\mathbf{w}_2 - \mathbf{w}_i) = 0, \]  

(19d)

where \( T_s = \frac{T_{0s} + T_{0s}}{1 + T_{0s}} \), \( T_{0s} = \frac{T_{0s}}{1 + T_{0s}} \) and \( T_{1s} = \frac{T_{1s}}{1 + T_{0s}} \). For these equations, the following boundary conditions should be imposed:

\[ \mathbf{v}_i(0, \tau) = \mathbf{v}_i(1, \tau) = 0, \]

\[ \mathbf{w}_i(0, \tau) = \mathbf{w}_i(1, \tau) = 0, \]

(20)

and the following initial conditions should be satisfied:

\[ \mathbf{v}_i(\xi, 0) = \mathbf{v}_{i0}(\xi), \quad \mathbf{w}_i(\xi, 0) = \mathbf{w}_{i0}(\xi); \quad i = 1, 2, \]

\[ \frac{\partial \mathbf{v}_i}{\partial \tau} (\xi, 0) = \mathbf{v}_{i0}(\xi), \quad \frac{\partial \mathbf{w}_i}{\partial \tau} (\xi, 0) = \mathbf{w}_{i0}(\xi), \]

(21)

where \( \mathbf{v}_{i0} = b_i \sqrt{w_{i0}} \mathbf{v}_{i0} \) and \( \mathbf{w}_{i0} = b_i \sqrt{w_{i0}} \mathbf{w}_{i0} \).

4. Development of an analytical solution

To determine the exact static and dynamic deformations of the nanosystem at hand, useful analytical solutions are proposed. Subsequently, the conditions which lead to the dynamic instability of the nanoscale system will be discussed in some detail.

4.1. A solution to the equations of motion of the pure dynamic state

Eq. (19) can be rewritten in the matrix form:

\[ \frac{\partial^2 \mathbf{d}}{\partial \tau^2} - T_s \frac{\partial^2 \mathbf{d}}{\partial \xi^2} + F_0 + \frac{\partial \mathbf{d}}{\partial \xi} + F_{1s} \mathbf{d} = \mathbf{0}, \]  

(22)

with the following initial and end conditions:

\[ \mathbf{d}(0, \xi) = \mathbf{d}_0, \quad \frac{\partial \mathbf{x}}{\partial \tau}(0, \xi) = \dot{\mathbf{d}}_0, \]  

(23a)

\[ \mathbf{d}(0, \tau) = \mathbf{d}(1, \tau) = \mathbf{0}, \]  

(23b)

where

\[ \mathbf{d}_0 = \begin{bmatrix} \mathbf{v}_{10} \\ \mathbf{w}_{10} \\ \mathbf{v}_{20} \\ \mathbf{w}_{20} \end{bmatrix}, \quad \dot{\mathbf{d}}_0 = \begin{bmatrix} \dot{\mathbf{v}}_{10} \\ \dot{\mathbf{w}}_{10} \\ \dot{\mathbf{v}}_{20} \\ \dot{\mathbf{w}}_{20} \end{bmatrix}, \quad \mathbf{d}(\xi, \tau) = \begin{bmatrix} \mathbf{v}_1(\xi, \tau) \\ \mathbf{w}_1(\xi, \tau) \\ \mathbf{v}_2(\xi, \tau) \\ \mathbf{w}_2(\xi, \tau) \end{bmatrix}, \]

\[ \mathbf{F}_0 = \begin{bmatrix} 0 & -\bar{T}_{0s} & 0 & 0 \\ \bar{T}_{0s} & 0 & 0 & 0 \\ 0 & 0 & -\bar{T}_{0s} & 0 \\ 0 & 0 & 0 & \bar{T}_{0s} \end{bmatrix}, \quad \mathbf{F}_{1s} = \begin{bmatrix} -\bar{T}_{1s} & 0 & \bar{T}_{1s} & 0 \\ 0 & \bar{T}_{1s} & 0 & -\bar{T}_{1s} \\ \bar{T}_{1s} & 0 & -\bar{T}_{1s} & 0 \\ 0 & -\bar{T}_{1s} & 0 & \bar{T}_{1s} \end{bmatrix}. \]  

(24)

Now let \( \mathbf{d} = e^{\mathbf{A}_s \tau} \mathbf{x}(\xi, \tau) \) where \( \mathbf{x} = (x_1, x_2, x_3, x_4)^T \). By substituting this relation into Eq. (22), one can arrive at:

\[ \frac{\partial^2 \mathbf{x}}{\partial \tau^2} - T_s \frac{\partial^2 \mathbf{x}}{\partial \xi^2} + \left( \mathbf{F}_{1s} - \frac{1}{4T_s^2} \mathbf{F}_0^2 \right) \mathbf{x} = \mathbf{0}; \quad \mathbf{A} = \frac{1}{2T_s} \mathbf{F}_0, \]  

(25)

with the following initial and boundary conditions:

\[ \mathbf{x}(0, \xi) = e^{-\mathbf{A}_s \xi} \mathbf{d}_0, \quad \frac{\partial \mathbf{x}}{\partial \tau}(0, \xi) = e^{-\mathbf{A}_s \xi} \dot{\mathbf{d}}_0, \]  

(26a)

\[ \mathbf{x}(0, \tau) = \mathbf{x}(1, \tau) = \mathbf{0}. \]  

(26b)

We consider \( \mathbf{x} = \sum_{m=1}^{\infty} \mathbf{x}_m(\tau) \sin(m\pi \xi) \) where \( \mathbf{x}_m(\tau) \) are the unknown time-dependent vectors. This form of \( \mathbf{x} \) satisfies the end conditions of the magnetically affected nanowires, namely Eq. (26b). By substituting this expression into Eqs. (25) and (26a), the following set of second-order ordinary differential equations is derived:

\[ \frac{\partial^2 \mathbf{x}_m}{\partial \tau^2} + \mathbf{B}_m \mathbf{x}_m = \mathbf{0}. \]  

(27)
with the following initial values:

\[ \mathbf{x}_m(0) = 2e^{-\mathbf{A} \mathbf{r}_m^0} \int_0^1 \mathbf{d}_0 \sin(m \pi \xi) \, d\xi, \]

\[ \frac{\partial \mathbf{x}_m}{\partial \tau}(0) = 2e^{-\mathbf{A} \mathbf{r}_m^0} \int_0^1 \mathbf{d}_0 \sin(m \pi \xi) \, d\xi, \]

where

\[ \mathbf{B} = \begin{bmatrix} (m \pi)^2 T_s - \frac{T_0^2}{4r_1^s} & -\tilde{T}_1s & 0 & 0 \\ 0 & (m \pi)^2 T_s - \frac{T_0^2}{4r_1^s} + \tilde{T}_1s & 0 & -\tilde{T}_1s \\ -\tilde{T}_1s & 0 & (m \pi)^2 T_s - \frac{T_0^2}{4r_1^s} - \tilde{T}_1s & 0 \\ 0 & -\tilde{T}_1s & 0 & (m \pi)^2 T_s - \frac{T_0^2}{4r_1^s} + \tilde{T}_1s \end{bmatrix}. \]

In order to evaluate \( \mathbf{x}_m \) from Eq. (27), Laplace transform method is adopted. After tedious calculations, the explicit expressions of the pure dynamic displacements of the nanowires in the dimensionless form are obtained as:

\[ \psi_1^d(\xi, \tau) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \left( \cos \left( \frac{\tilde{T}_0 \xi}{T_0 + \tilde{T}_0} \right) A_{1jm} \cos(r_{1jm}^* \tau) + B_{1jm} \frac{r_{1jm}^*}{r_{1jm}} \sin(r_{1jm}^* \tau) \right) \sin(m \pi \xi), \]

\[ \psi_2^d(\xi, \tau) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \left( -\sin \left( \frac{\tilde{T}_0 \xi}{T_0 + \tilde{T}_0} \right) A_{1jm} \cos(r_{1jm}^* \tau) + B_{1jm} \frac{r_{1jm}^*}{r_{1jm}} \sin(r_{1jm}^* \tau) \right) \sin(m \pi \xi), \]

\[ \psi_3^d(\xi, \tau) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \left( \cos \left( \frac{\tilde{T}_0 \xi}{T_0 + \tilde{T}_0} \right) A_{3jm} \cos(r_{1jm}^* \tau) + B_{3jm} \frac{r_{1jm}^*}{r_{1jm}} \sin(r_{1jm}^* \tau) \right) \sin(m \pi \xi), \]

\[ \psi_4^d(\xi, \tau) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \left( -\sin \left( \frac{\tilde{T}_0 \xi}{T_0 + \tilde{T}_0} \right) A_{3jm} \cos(r_{1jm}^* \tau) + B_{3jm} \frac{r_{1jm}^*}{r_{1jm}} \sin(r_{1jm}^* \tau) \right) \sin(m \pi \xi), \]

where the values of \( r_{1jm}^* \), \( A_{ijm} \), and \( B_{ijm} \) have been provided in Appendix A.

4.2. A solution to the governing equations of the static state

In this state, the four coupled second-order partial differential equations in Eqs. (17a)-(17d) should be solved such that the boundary conditions in Eq. (18) is appropriately satisfied. These equations can be stated in a more compact form as follows:

\[ -T_s \frac{\partial^2 \mathbf{d}^c}{\partial \xi^2} + \mathbf{F}_0 s \frac{\partial \mathbf{d}^c}{\partial \xi} + \mathbf{F}_{1s} \mathbf{d}^c = \mathbf{f}, \]

where

\[ \mathbf{d}^c = [r_1^c, \psi_1^c, \psi_2^c, \psi_2^c]^T, \quad \mathbf{f} = (\tilde{T} \tilde{d}, \tilde{T}_g, \tilde{d}, \tilde{T}_g). \]

In order to solve Eq. (31) for \( \mathbf{d}^c \), we follow the identical procedure mentioned in the previous part. For this purpose, the sinus Fourier series of the dimensionless static loads, namely \( \tilde{T} \tilde{d} \) and \( \tilde{T}_g \), are expressed by:

\[ (\tilde{T} \tilde{d}, \tilde{T}_g) = (\tilde{T} \tilde{d}, \tilde{T}_g) \sum_{m=1}^{\infty} \frac{2(1 - \cos(m \pi \xi))}{m \pi} \sin(m \pi \xi). \]

Now let \( \mathbf{d}^c = e^{\mathbf{A} \xi} \mathbf{x}^c \) where \( \mathbf{x}^c = \sum_{m=1}^{\infty} \mathbf{x}_m^c \sin(m \pi \xi) \). By substituting these expression into Eq. (31) through using Eq. (33), one can arrive at:
\[ B x_{m}^{i} = f_{m}^{i}, \] 

where

\[ f_{m}^{i}(\xi) = e^{-\lambda_{m}^{i} f} = (f_{1m}^{i}(\xi), f_{2m}^{i}(\xi), f_{3m}^{i}(\xi), f_{4m}^{i}(\xi))^{T}, \]

\[ f_{1m}^{i}(\xi) = \frac{2(1 - \cos(m\pi))}{m\pi} \left( \frac{\bar{T}_{0}}{\bar{T}_{0} + \bar{H}_{0}} \cos \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) - \bar{f}_{g} \sin \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \right). \]

\[ f_{2m}^{i}(\xi) = \frac{2(1 - \cos(m\pi))}{m\pi} \left( \frac{\bar{T}_{0}}{\bar{T}_{0} + \bar{H}_{0}} \sin \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) + \bar{f}_{g} \cos \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \right). \]

\[ f_{3m}^{i}(\xi) = f_{1m}^{i}(\xi), \quad f_{4m}^{i}(\xi) = f_{2m}^{i}(\xi). \] 

Finally, by solving the set of linear equations in Eq. (34) for \( x_{m}^{i} \), the explicit statements of \( x^{i} \) and \( d^{i} \) would be determined. By doing this, the analytical expressions of the dimensionless static displacements are derived as:

\begin{align}
\bar{\nu}_{1}^{i}(\xi) &= \sum_{m=1}^{\infty} \left( \cos \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \left( \frac{1}{\alpha_{m}^{i} + 1} - \frac{1}{\alpha_{m}^{i} - 1} \right) \right) \sin(m\pi \xi), \\
\bar{\nu}_{2}^{i}(\xi) &= \sum_{m=1}^{\infty} \left( \sin \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \left( \frac{1}{\alpha_{m}^{i} + 1} - \frac{1}{\alpha_{m}^{i} - 1} \right) \right) \sin(m\pi \xi), \\
\bar{\nu}_{3}^{i}(\xi) &= \sum_{m=1}^{\infty} \left( \cos \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \left( \frac{1}{\alpha_{m}^{i} + 1} - \frac{1}{\alpha_{m}^{i} - 1} \right) \right) \sin(m\pi \xi), \\
\bar{\nu}_{4}^{i}(\xi) &= \sum_{m=1}^{\infty} \left( \sin \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \left( \frac{1}{\alpha_{m}^{i} + 1} - \frac{1}{\alpha_{m}^{i} - 1} \right) \right) \sin(m\pi \xi),
\end{align}

where the values of \( \alpha_{m} \) and \( \alpha_{1} \) are given in Eq. (A.3).

5. Results and discussion

5.1. Origin of dynamic instability

According to Eqs. (30a)–(30d), the dynamic displacements would be unbounded if at least one root of the characteristic equation, namely Eq. (A.2), becomes zero. Suppose that the minimum root associated with the first vibration mode be zero, namely \( r_{41} = 0 \). In view of Eq. (30a) through using L'Hopital's rule for removing the singularity, the dimensionless dynamic displacements of the magnetically affected nanowires can be determined. In such a case, for example, the transverse displacement of the nanowire 1 along the y-axis is evaluated by:

\[ \nu_{1}^{1}(\xi, \tau) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \left( \cos \left( \frac{\bar{T}_{0}^{\xi}}{\bar{T}_{0} + \bar{H}_{0}} \right) \left( A_{1jm} \cos(r_{1m}^{1} \tau) + B_{1jm} \sin(r_{1m}^{1} \tau) \right) \right) \sin(m\pi \xi) + \left( A_{221} \cos(r_{21}^{2} \tau) + B_{221} \sin(r_{21}^{2} \tau) \right) \sin(\tau \xi). \]

Because of the last term in Eq. (37), transverse displacements of each point of the nanowires drastically increase with time and the nanoscale system becomes unstable. Therefore, instability would occur when at least one of the roots of the characteristic equation becomes zero. For instance, according to Eq. (A.2), the minimum requirement for \( r_{jm} = 0 \) is:

\[ \alpha_{m}^{2} = (\alpha_{0m} - 2\alpha_{1})(\alpha_{0m} + 2\alpha_{1}) = 0. \]
where the parameters \( \alpha_{m0} \) and \( \alpha_1 \) have been introduced in Eq. (A.2). For the first vibration mode, the most crucial term of Eq. (38) is the lowest one, namely \( \alpha_{m0} - 2 \alpha_1 \). By setting this expression equal to zero, the minimum value of the initial tensile force within the nanowires to provide a stable system is derived as:

\[
T_{0, \text{min}} = \frac{\mu_0 I^2 L^2}{2 \pi^3 d^2} + \sqrt{\frac{\mu_0 I^2 L^2}{2 \pi^3 d^2} + \left( \frac{B_0 l l}{2 \pi} \right)^2} - H_0.
\]  

(39)

Eq. (39) displays that the initial tensile force within the nanowires, interwire distance and surface effect are among the major factors that help the dynamic stability of the nanosystem. Nevertheless, an increase of the strength of the longitudinal magnetic field, electric current or length of the nanowire endangers the stability of the system. Such guidelines would be very helpful to the designers of nanostructures to provide more stable nanosystems according to the anticipated jobs and given specifications.

Eq. (39) would be also useful when nanomechanical controlling of the vibrations of the system is of concern. Now consider the case that the initial tensile force has been fixed but the electric current or the magnetic field strength can be variable. For such a case, it can be readily investigated that the maximum levels of the longitudinal magnetic field and the electric current to provide a dynamically stable nanosystem are as:

\[
B_{0, \text{max}} = \sqrt{\frac{4 \pi^2 (T_0 + H_0)^2}{l^4 L^2} - \frac{4 \mu_0 (T_0 + H_0)}{\pi d^2}}.
\]

(40a)

\[
l_{\text{max}} = \frac{H_0 + T_0}{\frac{B_0 l l}{4 \pi^2 (T_0 + H_0)} + \frac{\mu_0 l l}{\pi d^2}}.
\]

(40b)

5.2. Natural frequencies and modes of vibrations

Let introduce \( \mathbf{x} = \sum_{m=1}^{\infty} x_m e^{i \omega_m t} \sin(m \pi \xi) \) to Eq. (27) where \( x_m \) is the amplitude vector associated with the \( m \)th mode of vibration and \( \omega_m \) represents its dimensionless frequency. If the determinant of the resulting coefficients matrix of the amplitude vector is set equal to zero, three group of frequencies are obtained:

\[
\omega_{1m} = \sqrt{\omega_{m0} - 2 \omega_1}, \quad \omega_{2m} = \omega_{3m} = \sqrt{\omega_{m0}}, \quad \omega_{4m} = \sqrt{\omega_{m0} + 2 \omega_1}.
\]

(41)

In the following, the characteristics of the vibration modes pertinent to the above-mentioned natural frequencies are explained in some detail.

Case I: \( \omega_m = \omega_{1m} \): Asynchronous vibrations for the lowest frequencies

In this case, it can be readily shown that the dimensionless transverse displacements are given by:

\[
\begin{align*}
\langle \psi_1 (\xi, \tau) \rangle &= \sum_{m=1}^{\infty} a_{1m} \cos \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi) e^{i \omega_{1m} t}, \\
\langle \psi_2 (\xi, \tau) \rangle &= \sum_{m=1}^{\infty} a_{1m} \sin \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi) e^{i \omega_{1m} t},
\end{align*}
\]

(42)

where \( a_{1m} \) is the amplitude of the \( m \)th mode of vibration. According to Eq. (42), the nanowires exactly vibrate in opposite directions with identical amplitudes. In other words, this case shows asynchronous modes of vibrations such that the \( m \)th vibration modes associated with the transverse displacements \( \psi_1 \), \( \psi_2 \), \( \psi_3 \), and \( \psi_4 \) are given by:

\[
\begin{align*}
\psi_{m1} (\xi) &= \cos \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi), \\
\psi_{m2} (\xi) &= \sin \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi),
\end{align*}
\]

(43)

According to Eq. (43), a combination effect of magnetic field strength, electric current, initial tensile force, and surface effect can influence the modes of vibration of the nanosystem in this case.

Case II: \( \omega_m = \omega_{2m} \): Synchronous vibrations of the medium frequencies

Unlike the previous case, the frequencies of the nanosystem in this mode do not rely on the Lorentz forces applied on the nanowires. In this case, the transverse displacements of doubly parallel current-carrying nanowires in the presence of a longitudinal magnetic field are calculated as:

\[
\begin{align*}
\langle \psi_1 (\xi, \tau) \rangle &= \langle \psi_2 (\xi, \tau) \rangle = \sum_{m=1}^{\infty} (a_{1m} \cos \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) - a_{2m} \sin \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right)) \sin(m \pi \xi) e^{i \omega_{2m} t}, \\
\langle \psi_3 (\xi, \tau) \rangle &= \langle \psi_4 (\xi, \tau) \rangle = \sum_{m=1}^{\infty} (a_{1m} \sin \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right) + a_{2m} \cos \left( \frac{\tau_0 \xi}{2(T_0 + H_0)} \right)) \sin(m \pi \xi) e^{i \omega_{2m} t}.
\end{align*}
\]

(44)

As it is seen in Eq. (44), the counterpart transverse displacements of two nanowires are identical. Therefore, this vibration scenario is called synchronous or in-phase pattern. Such a vibration would be possible if two nanowires start to vibrate from the similar initial boundary
conditions. If this is the case of concern, the calculations show that the interwire magnetic force cannot influence on the vibration pattern of the nanosystem.

It is also worth mentioning that in the absence of the surface effect and the longitudinal magnetic field, we arrive at the classical frequencies and mode shapes of individual nanowires, namely \( \sigma_m = m \pi \sqrt{T_0}, \) and \( \psi_m^v = \psi_m^w = \sin(m \pi \xi) \), respectively.

Case III: \( \sigma_m = \sigma_{sm} \). Asynchronous vibrations of the highest frequencies Based on the computed frequencies in this case, the transverse displacements of the nanowires are calculated as:

\[
\{ \psi_1(\xi, \tau), \bar{\psi}_1(\xi, \tau) \} = \sum_{m=1}^{\infty} a_{qm} \left[ - \sin \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \cos \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \right] \sin(m \pi \xi) e^{i \omega_0 m \tau},
\]

\[
\{ \psi_2(\xi, \tau), \bar{\psi}_2(\xi, \tau) \} = \sum_{m=1}^{\infty} a_{qm} \left[ \sin \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \cos \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \right] \sin(m \pi \xi) e^{i \omega_0 m \tau}.
\]

(45)

The pattern of the vibrational modes at this case is identical to that of the Case I. These patterns show an asynchronous or out-of-phase vibration, however, the corresponding mode shapes of vibration are different from those given in the Case I. These are evaluated as follows:

\[
\psi_m^{v1}(\xi) = - \sin \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi), \quad \psi_m^{v2}(\xi) = \sin \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi),
\]

\[
\psi_m^{w1}(\xi) = \cos \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi), \quad \psi_m^{w2}(\xi) = - \cos \left( \frac{T_0 \xi}{2(T_0 + H_0)} \right) \sin(m \pi \xi).
\]

(46)

Depending on the initial conditions of the out-of-phased nanowires, the patterns mentioned in Case I or III would be possible. For example, if the initially deformed nanowires for the out-of-phase pattern can be presented by an odd function or a linear combination of odd functions, the vibrations of the nanowires occur in accordance with the pattern of Case I.

In regards to the given explanations for the importance of the lower frequencies, those of Case I, we are interested to realize the roles of the major factors affecting such frequencies. In the next part, a comprehensive scrutiny has been provided to investigate this matter.

5.3. Effects of the crucial factors on the lowest asynchronous frequencies

Consider two identical silver nanowires with the following properties: \( \rho_b = 10500 \ \text{kg/m}^3, \) \( \rho_0 = 10^{-7} \ \text{kg/m}^2, \) \( \tau_0 = 0.89 \ \text{N/m}, \) \( l_b = 2000 \ \text{nm}, \) \( r_0 = 25 \ \text{nm}, \) and \( d = 3r_0. \) In Fig. 2(a), the influence of the initial tensile force on the lowest four frequencies associated with the asynchronous pattern of vibration are addressed. The nanowires are carrying electric current with \( I = \frac{T_0}{2 T_0 + H_0} \) where \( T_0 = 10^{-4} \) and are subjected to a longitudinal magnetic field of strength \( B_0 = \frac{B_0}{\sqrt{T_0 \rho_0 \mu_0 l_b}} \) where \( B_0 = 0.1. \) In all plotted results in this part, the graphs pertinent to the cases of with and without consideration of the surface effect are presented by the solid and dashed lines, respectively. The minimum value of the initial tensile force is evaluated by Eq. (39). Fig. 2(a) displays that all asynchronous frequencies of the nanowires magnify as the initial tensile of the nanowire increases. It is mainly related to this fact that the transverse stiffness of the nanowires increases as their initial tensile forces increase. Further, the predicted frequencies by the surface elasticity are generally greater than those of without consideration of the surface effect. The main reason of this fact is related to the residual tensile stress within the surface layer of the nanowire, which is a positive value for silver nanowires. Such a stress leads to an increase of the transverse stiffness of the nanowires; thereby, it helps to provide a more stable nanosystem. According to the plotted results in Fig. 2(a), both theories predict that the effect of the initial tensile force on higher frequencies is more visible. Because of the surface effect, the slopes of the plots pertinent to the model based on the surface elasticity are commonly lower than those of without consideration of the surface effect.

In Fig. 2(b), the influence of the electric current on the first four asynchronous frequencies are investigated. The results are provided for \( T_0 = 10^{-4} \) and \( B_0 = 0.1. \) According to the plotted results, the frequencies would lessen as the electric current increases. For higher levels of the electric current, variation of the electric current would have a more influence on the variation of the frequencies. The corresponding electric current to the zero fundamental frequency (i.e., onset of dynamic instability) is determined via Eq. (40b). Additionally, the existence of the surface effect with positive residual surface stress would lead to a higher value for the electric current corresponds to the instability of the nanosystem. In other words, surface effect would help the nanoscale system to become unstable at higher levels of the electric current.

An interesting parametric study has been conducted to explore the effects of the strength of the longitudinal magnetic field and the interwire distance on the fundamental frequency of the nanoscale system. In Figs. 3(a) and 3(b), the predicted dimensionless fundamental frequencies are demonstrated in terms of dimensionless magnetic field strength and interwire distance in the case of \( T_0 = 0.001 \) and four levels of the electric current (i.e., \( I = 0.0005, 0.001, 0.0015, \) and \( 0.002 \)). According to Fig. 3(a), both classical and surface elasticity models predict that the fundamental frequency of the nanostructure would reduce as the strength of the longitudinal magnetic field magnifies. Further, the influence of the strength of the magnetic field on the variation of the fundamental frequency is much more apparent for higher electric currents. This is chiefly related to the reduction of the transverse stiffness of the nanosystem by a combination effect of magnetic field and electric current. Because of this fact, the nanosystem carries a more electric current becomes dynamically unstable at lower levels of the magnetic field strength. In the absence of the longitudinal magnetic field, electric current has a slight effect on the fundamental frequency. As the strength of the longitudinal magnetic field increases, the influence of an increase of the electric current on the reduction of the fundamental frequency becomes more obvious. In compare to the surface elasticity model, the classical model predicts that the nanosystem becomes unstable at lower levels of the strength of the longitudinal magnetic field.

Fig. 3(b) shows the influence of the interwire distance on the dimensionless fundamental frequencies of the nanosystem for various levels of the electric current in the case of \( B_0 = 0.2. \) As it is seen in Fig. 3(b), by an increase of the interwire distance, the fundamental
frequency of the nanosystem increases. Such a fact is more apparent for those nanosystems carry higher values of the electric current. Furthermore, for higher levels of the interwire distance, variation of the interwire distance has a lower influence on the variation of the fundamental frequencies. Because of the surface effect, the predicted results by the surface elasticity model are greater than those of the classical model.

6. Concluding remarks

Using surface elasticity theory and Biot–Savart law, dynamic interactions and instabilities of doubly parallel nanowires carry direct electric current are investigated when they are affected by a longitudinal magnetic field. By taking into account the exerted magnetic and gravity forces on the nanowires, the equations of motion of the magnetically affected current-carrying nanowires accounting for the surface effect are obtained. For a given general initial conditions, an analytical solution is established to derive static and pure dynamic transverse displacements of the nanosystem. Three sets of natural frequencies and the corresponding vibration modes are explicitly obtained. Two crucial patterns of vibration, synchronous (in-phase) and asynchronous (out-of-phase), are found for such a nanosystem. Two sets of asynchronous modes are pertinent to the lowest and highest sets of frequencies. Concerning the importance of the lower frequencies and their corresponding vibration modes, the roles of interwire distance, electric current, strength of the longitudinal magnetic field, and initial tensile force within the nanowires on them are comprehensively explained. The circumferences which dynamic instability would be generated in the nanosystem are inclusively discussed. The minimum initial tensile force as well as the maximum values of the magnetic field strength and electric current correspond to the initiation of dynamic instability of the nanostructure are analytically expressed. Other major findings of the present work are as:

1. The predicted frequencies by the surface elasticity theory are greater than those of the classical model. It is mainly related to the contribution of the surface effect with positive residual surface stress to the transverse stiffness of the nanosystem.

2. The initial tensile force within the nanowires as well as the surface effect would help the stability of the vibrating nanosystem. Additionally, both strength of the longitudinal magnetic field and the electric current would destabilize the nanosystem. This is mainly attributed to this fact that the first two factors increase the transverse stiffness of the nanowires whereas the two later ones decrease such a stiffness.

3. Fundamental frequency would lessen as the strength of the magnetic field magnifies. Such a fact is more obvious for the nanosystems with higher electric currents. Additionally, for higher levels of the magnetic field, the influence of the electric current on the fundamental frequency of the nanosystem is more apparent.

4. As the interwire distance increases, the fundamental frequency increases. The effect of the interwire distance on the variation of the fundamental frequency of the nanosystem with higher levels of electric current is more obvious. By increasing the interwire distance,
the asynchronous frequencies approach to those of the synchronous modes of vibration since the interactional effects of doubly parallel nanowires weaken.

Acknowledgements

The financial support of the Iran National Science Foundation (INSF) is gratefully acknowledged. The author also wishes to express his gratitude to the anonymous reviewers for their fruitful comments in which lead to the improvement of the present work.

Appendix A. Evaluation of $r_{jm}$, $A_{ijm}$ and $B_{ijm}$

A.1. The values of $r^*_{jm}$; $i, j = 1, 2$

We have:

$$
\begin{align*}
{r^*_{1m}} &= r_{2m}, \quad {r^*_{2m}} = r_{4m}, \quad {r^*_{21m}} = r_{1m}, \quad {r^*_{22m}} = r_{3m}.
\end{align*}
$$

(A.1)

where $r_{jm}$; $j = 1, \ldots, 4$ are the roots of the following fourth-order polynomial in terms of $r$:

$$
\begin{align*}
&{r^4} + 4\alpha_{0m} r^3 + (6\alpha_{0m}^2 - 4\alpha_1^2) r^2 + (4\alpha_0^3 - 8\alpha_{0m}\alpha_1^2) r + \alpha_{0m}^2 (\alpha_{0m} - 2\alpha_1)(\alpha_{0m} + 2\alpha_1) = 0.
\end{align*}
$$

(A.2)

where

$$
\begin{align*}
\alpha_{0m} &= \left(\frac{m\pi}{2T_s}\right)^2 - \frac{I_{0s}}{4T_s}, \quad \alpha_1 = \frac{I_{1s}}{T_s}.
\end{align*}
$$

(A.3)

Eq. (A.2) denotes the characteristic equation of the magnetically affected double current-carrying nanowires. This is a key equation explains that under what circumferences the nanoscale system arrives at the dynamic instability. Further, the dimensionless natural frequencies of the system are its roots whose values are readily calculated as follows:

$$
\begin{align*}
&{r_{1m}} = {r_{2m}} = \sqrt{\alpha_{0m}}, \quad {r_{3m}} = \sqrt{\alpha_{0m} - 2\alpha_1}, \quad {r_{4m}} = \sqrt{\alpha_{0m} + 2\alpha_1}.
\end{align*}
$$

(A.4)
A.2. The values of $A_{1ijm}$ and $B_{1ijm}$: $i, j = 1, \ldots, 4$.

\[
A_{11m} = \frac{1}{2} (x_{1m}(0) + x_{3m}(0)), \quad A_{21m} = \frac{1}{2} (x_{1m}(0) - x_{3m}(0)), \quad A_{31m} = \frac{1}{2} (x_{3m}(0) + x_{1m}(0)), \quad A_{41m} = \frac{1}{2} (x_{2m}(0) + x_{4m}(0)), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu


