Vibrations and instability of double-nanowire-systems as electric current carriers

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Current-carrying nanowires are expected to be building blocks of the upcoming micro-/nano-electromechanical devices, however, little is known on their dynamic interactions in a bundle. As a pivotal step towards realizing such a crucial mechanism, this work is devoted to vibrations and instability of a double-nanowire-system as an electric current carrier. Using Biot–Savart law, the Lorentz interactional forces between doubly parallel current-carrying nanowires are evaluated. Accounting for the surface elastic energy, equations of motion pertinent to the in-plane and out-of-plane vibrations are established. Using analytical techniques, the explicit expressions of both static and purely dynamic parts of the nanowires’ displacements are obtained. For each component of the transverse displacement field, two major vibration modes are observed: in-phase and out-of-phase modes. The frequencies associated with these vibration modes are analytically calculated. Further, the condition corresponds to the dynamic instability of the system is discovered, and the roles of initial tensile force, electric current, and interwire distance on frequencies and stability of the system are addressed.

Keywords: Vibration; instability; double nanowires; direct current; surface effect; Laplace transform.

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1. Introduction

A nanowire is a nanostructure with the width in the range of a fraction of one nanometer to several nanometers. Owing to the superior physical and chemical properties of some special metallic nanowires, they are suggested as ideal building blocks for a diverse range of applications including chemical and physical sensors,1–4 optoelectronics,5–7 energy harvesting,8–10 and micro-/nano-electromechanical systems (MEMS/NEMS).11–14 In the latter application, an ensemble of current-
carrying nanowires may be used.\textsuperscript{15,16} Our knowledge regarding vibrations of magnetically affected nanowires carry electric current is restricted to single nanowires.\textsuperscript{17,18} However, no report on the dynamic behavior of multiple or even double nanowires with the purpose of carrying electric current is now available in the literature. As a key step towards better understanding of the vibration mechanisms of ensembles of current-carrying nanowires, we start the work by studying a system of doubly parallel nanowires as an electrical current carrier.

Generally the length to width ratio of nanowires is more than 1000. It implies that for constrained nanowires, the bending rigidity is adequately negligible such that the flexural strain energy can be rationally excluded from the total strain energy of the nanostructure. This fact becomes more reasonable when the nanowire is acted upon by a considerable tensile force. As a result, a string model would be a suitable alternative for beam models. On the other hand, as the width-to-length ratio decreases, the ratio of the number of surface’s atoms to the number of bulk’s atoms increases. Thereby, the share of the surface energy to total strain energy increases and the mechanical behavior of the nanowire is more influenced by the surface’s atoms. Such an issue is called surface effect and the classical continuum theory (CCT) which cannot display this phenomenon. To overcome this deficiency of the CCT, a sophisticated nonclassical theory (i.e. surface elasticity theory) was proposed by Gurtin–Murdoch\textsuperscript{19–22} in the previous century. In the newly developed model, the material is divided into the bulk zone and surface layer. The volume of the bulk is fairly equal to the volume of the material and the thickness of the surface layer is so tiny that it can be neglected in mechanical analysis of the problem. For the bulk material, the constitutive equations of the CCT or other advanced continuum theories with the given engineering constants of the macro-scale material can be exploited; however, for the surface layer, the mechanical characteristics of the surface including surface density, residual stress, Lame’s constant of the surface layer which are introduced by the surface continuum theory (SCT) should be used. The governing equations of the surface layer in terms of the surface’ stresses are somehow identical to those of the three-dimensional elasticity model of the bulk zone, nevertheless, the constitutive relations of the surface layer are completely different from those of the bulk zone. So far, the SCT of Gurtin-Murdoch has been extensively exploited in predicting various mechanical behaviors of nanowires.\textsuperscript{23–29}

In the present work, we utilize a string model in the context of the SCT to investigate transverse vibrations of doubly parallel current-carrying nanowires in the vicinity of each other.

When an electric current passes through a nanowire, a magnetic field generates around the nanowire which is displayed by the Biot–Savart law. This states that the generated magnetic field at a point is directly proportional with the electric current and is inversely proportional with its distant from the nanowire. In the case of doubly parallel nanowires which carry electric currents, the nanowires exert mutually equal distributed force on each other that can be explained by the Lorentz
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formula. A close scrutiny of the present work shows that these forces inherently rely on the difference of lateral displacements of the nanowires. Therefore, the frequency such a coupled system as well as its free vibration can be affected by the electric current within the nanowires. One of the main objectives of the present work is to explain that under what circumstances the dynamic instability occurs within the nanosystem. Subsequently, it is investigated that how such an extreme condition could be appropriately controlled by changing the physical parameters of the nanosystem.

So far, the influence of magnetic fields on various aspects of vibrations of nanostructures has been addressed. Concerning the dynamic behaviors of nanowires in a magnetic field, Kiani studied magneto-elasto dynamics of elastically rested nanowires acted upon by a longitudinal magnetic shock. A two-dimensional nonlocal elasticity model is developed and the dynamic elastic field within the nanowire is analytically determined. In another work, Kiani explored radial vibrations of nanowires subjected to a magnetic field accounting for eddy-current loss. Using a nonlocal elasticity model, magneto-thermo-elastic fields within the nanowire are explicitly determined, and the roles of influential parameters on vibrations of the magnetically affected nanowire were addressed. Recently, free and forced vibrations of a current-carrying nanowire in the presence of a longitudinal magnetic field has been examined by Kiani. These explorations indicate that how variation of the parameters associated with the transversely applied load, electric current, and magnetic field could affect on the transverse vibrations of nanowires. A close survey of the literature reveals that the effect of generated magnetic field by an electric current in a nanowire on another current-carrying nanowire has not been addressed. In view of the potential applications of current-carrying nanowires in MEMS/NEMS industry and the need for more accurate design of such nanodevices, the primary focus of this work is on the accurate evaluation of the generated magnetic forces in terms of transverse displacement of nanowires.

Herein, transverse vibrations and instability of two parallel nanowires used for carrying direct current are addressed. Using Biot–Savart law and surface elasticity, the equations of motion describe transverse vibration of long double nanowires are obtained in the context of small deformation. Thereafter, an analytical solution is proposed and the explicit expressions of transverse displacements of the nanowires are extracted. Free vibration and dynamic instability of the nanoscale system are noted. Additionally, the roles of the initial tensile force, direct electric current, and interwire distance on the dynamic displacements are examined.

2. Formulations of the Nanomechanical Problem

2.1. Evaluation of the Lorentz’s forces between two deformed current-carrying nanowires

Consider two lengthy parallel nanowires of length $l_b$ and interwire distance $d$ that carry electric currents $I_1$ and $I_2$ as shown in Fig. 1. The geometry and mechanical
properties of the nanowires are identical, and both ends of them are prohibited from any lateral movement. The origin of rectangular coordinate system has been attached to the left support of the first nanowire such that the $x$-axis is coincident with its revolutionary axis and the $z$-axis points downward. The unit base vectors associated with the $x$, $y$ and $z$ axes are represented by $e_x$, $e_y$, and $e_z$, respectively. In all presented formulations, the subscripts 1 and 2 stand for the parameters associated with the first and the second nanowire, respectively. The transverse displacement fields of the $i^{th}$ nanowire along the $y$ and $z$ axes in order are represented by $v_i = v_i(x,t)$ and $w_i = w_i(x,t)$ where $t$ is the time parameter. The deformations $v_i$ and $w_i$ are also called in-plane and out-of-plane transverse displacements, respectively, since the deformations $v_i$ occur in the plane passes through the revolutionary axes of the undeformed nanowires while $w_i$ represent the deformation occurring in the planes are perpendicular to the above-mentioned plane. Herein, transverse vibrations of nanowires and their interactions are of interest. Because of passing the electric current through the nanowires, magnetic fields are generated around the nanowires. According to the Biot–Savart law for lengthy nanowires, the magnetic field vector resulted from the first deformed nanowire at the location of the second deformed nanowire can be evaluated as:

$$B_1 = \frac{\mu_0}{2\pi d'} I_1 \times e_{d'},$$  \hspace{1cm} (1)$$

where $\mu_0$ is the permeability of the free space which is given by $\mu_0 = 4\pi \times 10^{-7} \frac{Tm}{A}$. $d' = ||d'||$ is the distance between deformed nanowires, $d'$ is a vector that specifies the location of the center of the first nanowire with respect to that of the second nanowire, and $e_{d'}$ is its unit base vector.

Using Lorentz’s formula, the resulted magnetic field exerts a force on the adjacent current-carrying nanowire. Such a force per unit length of the second nanowire can be calculated by:

$$f_{21} = I_2 \times B_1 .$$  \hspace{1cm} (2)
According to the geometry of the deformed nanowires, \( I_j = I_j e_x + I_j \frac{\partial w_j}{\partial x} e_y \); \( j = 1, 2 \), and \( d' = (d + \Delta v) e_y + \Delta w e_z \) where \( \Delta v = v_2 - v_1 \) and \( \Delta w = w_2 - w_1 \). By substituting Eq. (1) into Eq. (2),

\[
\mathbf{f}_{21} = \frac{\mu_0 I_1 I_2}{2\pi((d + \Delta v)^2 + (\Delta w)^2)} \left( (d + \Delta v) \frac{\partial v_2}{\partial x} + \Delta w \frac{\partial w_2}{\partial x} \right) e_x \\
+ \left( \frac{\partial w_2}{\partial x} \left( \Delta w \frac{\partial v_1}{\partial x} - (d + \Delta v) \frac{\partial w_1}{\partial x} \right) - (d + \Delta v) \right) e_y \\
+ \left( -\frac{\partial v_2}{\partial x} \left( \Delta w \frac{\partial v_1}{\partial x} - (d + \Delta v) \frac{\partial w_1}{\partial x} \right) - \Delta w \right) e_z \right). \tag{3}
\]

For small dynamic displacements, by neglecting the products of displacement and their first derivatives and excluding the longitudinal component of the magnetic force, Eq. (3) is approximated by:

\[
\mathbf{f}_{21} = -\frac{\mu_0 I_1 I_2}{2\pi d} \left[ \left( 1 - \frac{\Delta v}{d} \right) e_y + \frac{\Delta w}{d} e_z \right]. \tag{4}
\]

Similarly, it can be shown that the exerted force on the first current-carrying nanowire due to the generated magnetic field by the second nanowire is calculated by:

\[
\mathbf{f}_{12} = \frac{\mu_0 I_1 I_2}{2\pi d} \left[ \left( 1 - \frac{\Delta v}{d} \right) e_y + \frac{\Delta w}{d} e_z \right]. \tag{5}
\]

### 2.2. Equations of motion using surface elasticity

Based on the surface elasticity, by neglecting the bending rigidity of the lengthy nanowires, the governing equations describe transverse vibrations of the nanosystem which are as:\(^{18,40}\)

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_i}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_i}{\partial x^2} = q_{v_i}, \tag{6a}
\]

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 w_i}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 w_i}{\partial x^2} = q_{w_i}, \tag{6b}
\]

where \( \rho_b, A_b, \rho_0, S_0, T_0, H_0, q_{v_i}, \) and \( q_{w_i} \) are the bulk density, cross-sectional area, surface density, pyramid of the nanowire’s cross-section, initial tensile force within the nanowire, residual surface stress under unconstrained conditions, and the exerted Lorentz’s forces along the \( y \)- and \( z \)-axes of the \( i \)th nanowire. By introducing Eqs. (4) and (5) to Eqs. (6a) and (6b) and taking into account the weight of the nanowires, the equations of motion of doubly parallel current-carrying nanowires in terms of transverse displacements are expressed by:

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_1}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_1}{\partial x^2} + \frac{\mu_0 I_1 I_2}{2\pi d^2} (v_2 - v_1) = \frac{\mu_0 I_1 I_2}{2\pi d}, \tag{7a}
\]
By introducing Eq. (s) as:

\[
(K \text{Kiani})
\]

Since the deformation fields of the nanowires are resulted from both static and dynamic loads, therefore, we split up displacements into static and dynamic parts as:

\[
v_i(x, t) = v_i^s(x) + v_i^d(x, t), \quad w_i(x, t) = w_i^s(x) + w_i^d(x, t); \quad i = 1, 2, \tag{8}\]

where the superscripts \( s \) and \( d \) are pertinent to the static and dynamic states, respectively. By introducing Eq. (8) to Eqs. (7a)–(7d), the static deformation of the nanowires due to the magneto-static Lorentz’s force as well as nanowires’ weight force can be determined from the following relations:

\[
-(T_0 + H_0) \frac{\partial^2 v_1^s}{\partial x^2} + \frac{\mu_0 I_1 I_2}{2\pi d^2} (v_2^s - v_1^s) = \frac{\mu_0 I_1 I_2}{2\pi d}, \tag{9a}\]

\[
-(T_0 + H_0) \frac{\partial^2 w_1^s}{\partial x^2} = \frac{\mu_0 I_1 I_2}{2\pi d^2} (w_2^s - w_1^s) = \rho_b A_b g, \tag{9b}\]

\[
-(T_0 + H_0) \frac{\partial^2 v_2^s}{\partial x^2} - \frac{\mu_0 I_1 I_2}{2\pi d^2} (v_2^s - v_1^s) = -\frac{\mu_0 I_1 I_2}{2\pi d}, \tag{9c}\]

\[
-(T_0 + H_0) \frac{\partial^2 w_2^s}{\partial x^2} + \frac{\mu_0 I_1 I_2}{2\pi d^2} (w_2^s - w_1^s) = \rho_b A_b g, \tag{9d}\]

with the following boundary conditions:

\[
v_i^s(0) = v_i^s(l_b) = 0; \quad w_i^s(0) = w_i^s(l_b) = 0; \quad i = 1, 2. \tag{10}\]

Further, the purely dynamic displacements of the nanowires are governed by the following relations:

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_i^d}{\partial x^2} + \frac{\mu_0 I_1 I_2}{2\pi d^2} (v_2^d - v_1^d) = 0, \tag{11a}\]

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 w_i^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 w_i^d}{\partial x^2} - \frac{\mu_0 I_1 I_2}{2\pi d^2} (w_2^d - w_1^d) = 0, \tag{11b}\]

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 v_2^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 v_2^d}{\partial x^2} - \frac{\mu_0 I_1 I_2}{2\pi d^2} (v_2^d - v_1^d) = 0, \tag{11c}\]

\[
(\rho_b A_b + \rho_0 S_0) \frac{\partial^2 w_2^d}{\partial t^2} - (T_0 + H_0) \frac{\partial^2 w_2^d}{\partial x^2} + \frac{\mu_0 I_1 I_2}{2\pi d^2} (w_2^d - w_1^d) = 0, \tag{11d}\]

with the following boundary conditions:

\[
v_i^d(0, t) = v_i^d(l_b, t) = 0, \tag{12}\]

\[
w_i^d(0, t) = w_i^d(l_b, t) = 0, \tag{12}\]
and the following general initial conditions:
\[ v_i^d(x,0) = v_{i0}(x), \quad w_i^d(x,0) = w_{i0}(x); \quad i = 1, 2, \]

where \( v_{i0}(x) \) and \( w_{i0}(x) \) are the initial deformations (in addition to those of static displacements) whereas \( \dot{v}_{i0}(x) \) and \( \dot{w}_{i0}(x) \) denote their initial velocities.

3. An Analytical Solution

In this part, it is aimed to determine the static and pure dynamic displacements which are governed by Eqs. 9(a)–9(d) and 11(a)–11(d). To this end, the following dimensionless quantities are considered:

\[
\xi = \frac{x}{l_b}, \quad \tau = \frac{t}{l_b} \sqrt{\frac{E_b}{\rho_b}}, \quad \ddot{v}_i = \frac{v_i}{l_b}, \quad \ddot{w}_i = \frac{w_i}{l_b}, \quad \ddot{d} = \frac{d}{l_b}, \quad \ddot{T}_0 = \frac{T_0}{E_b A_b},
\]

\[
\ddot{H}_0 = \frac{H_0}{E_b A_b}, \quad \ddot{f}_0 = \frac{\mu_0 I_1 I_2 l_0^2}{2\pi d^2 E_b A_b}, \quad \ddot{m}_0 = \frac{\rho_0 S_0}{\rho_b A_b}, \quad \ddot{f}_g = \frac{\rho_b A_b l_b g}{E_b A_b}.
\]

3.1. Static analysis

By introducing Eq. (14) to Eqs. (9a)–(9d), the dimensionless governing equations corresponding to the static deformation of the nanowires are given by:

\[
-(\ddot{T}_0 + \ddot{H}_0) \frac{\partial^2 \ddot{v}_i^s}{\partial \xi^2} + \ddot{f}_0 (\ddot{v}_2^s - \ddot{v}_1^s) = \ddot{f}_0 \ddot{d},
\]

\[
-(\ddot{T}_0 + \ddot{H}_0) \frac{\partial^2 \ddot{w}_1^s}{\partial \xi^2} - \ddot{f}_0 (\ddot{w}_2^s - \ddot{w}_1^s) = \ddot{f}_g,
\]

\[
-(\ddot{T}_0 + \ddot{H}_0) \frac{\partial^2 \ddot{v}_2^s}{\partial \xi^2} - \ddot{f}_0 (\ddot{v}_2^s - \ddot{v}_1^s) = \ddot{f}_0 \ddot{d},
\]

\[
-(\ddot{T}_0 + \ddot{H}_0) \frac{\partial^2 \ddot{w}_2^s}{\partial \xi^2} + \ddot{f}_0 (\ddot{w}_2^s - \ddot{w}_1^s) = \ddot{f}_g,
\]

with the following boundary conditions:

\[ \ddot{v}_i^s(0) = \ddot{v}_i^s(1) = 0; \quad \ddot{w}_i^s(0) = \ddot{w}_i^s(1) = 0. \]

By solving the coupled Eqs. (15a) and (15c) as well as Eqs. (15b) and (15d) for \( \ddot{v}_i^s \) and \( \ddot{w}_i^s \); \( i = 1, 2, \) respectively, the explicit expressions of the dimensionless static displacements of two nanowires are derived as:

\[
\ddot{v}_i^s(\xi) = -\ddot{v}_2^s(\xi) = \ddot{d} \left( 1 - \tan \left( \frac{\sqrt{f_e}}{2} \right) \sin(\sqrt{f_e} \xi) - \cos(\sqrt{f_e} \xi) \right),
\]

\[ \ddot{w}_i^s(\xi) = \ddot{w}_2^s(\xi) = \frac{1}{2} f_w \xi (1 - \xi), \]

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where \( \tilde{f}_e = \frac{f_0}{T_0 + H_0} \) and \( \tilde{f}_w = \frac{f_0}{T_0 + H_0} \). According to Eq. (17a), the magnitudes of the in-plane transverse displacements of double nanowires are the same, however, they have opposite directions. It is interpreted by this fact that the nanowires identically attract each other due to the similar exerted magnetic forces (see Eqs. (4) and (5)).

### 3.2. Dynamic analysis

By introducing Eq. (14) to Eqs. (11a)–(11d), the dimensionless governing equations of the nanoscale system in terms of purely dynamic displacements take the following form:

\[
\frac{\partial^2 \tilde{v}_d^i}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{v}_d^i}{\partial \xi^2} + \tilde{f}_s (\tilde{v}_d^2_i - \tilde{v}_d^1_i) = 0, \quad (18a)
\]

\[
\frac{\partial^2 \tilde{w}_d^i}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{w}_d^i}{\partial \xi^2} - \tilde{f}_s (\tilde{w}_d^2_i - \tilde{w}_d^1_i) = 0, \quad (18b)
\]

\[
\frac{\partial^2 \tilde{v}_d^2_i}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{v}_d^2_i}{\partial \xi^2} - \tilde{f}_s (\tilde{v}_d^2_i - \tilde{v}_d^1_i) = 0, \quad (18c)
\]

\[
\frac{\partial^2 \tilde{w}_d^2_i}{\partial \tau^2} - \tilde{T}_s \frac{\partial^2 \tilde{w}_d^2_i}{\partial \xi^2} + \tilde{f}_s (\tilde{w}_d^2_i - \tilde{w}_d^1_i) = 0, \quad (18d)
\]

where \( \tilde{T}_s = \frac{T_0 + H_0}{1 + m_0} \) and \( \tilde{f}_s = \frac{f_0}{1 + m_0} \). Furthermore, the following boundary conditions (see Eq. (12)):

\[
\tilde{v}_d^i(0, \tau) = \tilde{v}_d^i(1, \tau) = 0, \quad (19)
\]

\[
\tilde{w}_d^i(0, \tau) = \tilde{w}_d^i(1, \tau) = 0,
\]

and the following initial conditions should be satisfied (see Eq. (13)):

\[
\tilde{v}_d^i(\xi, 0) = \tilde{v}_{i0}(\xi), \quad \tilde{w}_d^i(\xi, 0) = \tilde{w}_{i0}(\xi); \quad i = 1, 2,
\]

\[
\frac{\partial \tilde{v}_d^i}{\partial \tau}(\xi, 0) = \hat{v}_{i0}(\xi), \quad \frac{\partial \tilde{w}_d^i}{\partial \tau}(\xi, 0) = \hat{w}_{i0}(\xi), \quad (20)
\]

where \( \hat{v}_{i0}(\xi) = l_b \sqrt{\frac{\rho_b}{E_b}} \tilde{v}_{i0}(\xi) \) and \( \hat{w}_{i0}(\xi) = l_b \sqrt{\frac{\rho_b}{E_b}} \tilde{w}_{i0}(\xi) \).

Now the dimensionless displacements of the nanowires, which are admissible with the boundary conditions, are considered as follows:

\[
\tilde{v}_i(\xi, \tau) = \sum_{m=1}^{\infty} a_{im}(\tau) \sin(m\pi \xi), \quad \tilde{w}_i(\xi, \tau) = \sum_{m=1}^{\infty} b_{im}(\tau) \sin(m\pi \xi); \quad i = 1, 2, \quad (21)
\]

where \( a_{im} \) and \( b_{im} \) are the time-dependent parameters associated with the transverse displacements of the \( m \)th vibration mode of the \( i \)th current-carrying nanowire. By substituting Eq. (21) into Eqs. (18a) and (18c) as well as Eqs. (18b) and (18d), and taking the Laplace transform of both sides of the resulting relations, one can
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arrive at:

\[
\begin{bmatrix}
  s^2 + \bar{T}_s(m\pi)^2 - \bar{f}_s & \bar{f}_s \\
  \bar{f}_s & s^2 + \bar{T}_s(m\pi)^2 - \bar{f}_s
\end{bmatrix}
\mathcal{L}\left\{\begin{array}{c}
a_{1m} \\
a_{2m}
\end{array}\right\} = \left\{\begin{array}{c}
a_{1m}(0)s + \frac{\partial a_{1m}}{\partial \tau}(0) \\
a_{2m}(0)s + \frac{\partial a_{2m}}{\partial \tau}(0)
\end{array}\right\},
\]

and

\[
\begin{bmatrix}
  s^2 + \bar{T}_s(m\pi)^2 + \bar{f}_s & -\bar{f}_s \\
  -\bar{f}_s & s^2 + \bar{T}_s(m\pi)^2 + \bar{f}_s
\end{bmatrix}
\mathcal{L}\left\{\begin{array}{c}
b_{1m} \\
b_{2m}
\end{array}\right\} = \left\{\begin{array}{c}
b_{1m}(0)s + \frac{\partial b_{1m}}{\partial \tau}(0) \\
b_{2m}(0)s + \frac{\partial b_{2m}}{\partial \tau}(0)
\end{array}\right\},
\]

where \(\mathcal{L}\) is the Laplace transform operator and

\[
a_{1m}(0) = 2 \int_0^1 \bar{v}_i \sin(m\pi \xi) d\xi,
\]

\[
\frac{\partial a_{1m}}{\partial \tau}(0) = 2 \int_0^1 \dot{\bar{v}}_0(\xi) \sin(m\pi \xi) d\xi,
\]

\[
b_{1m}(0) = 2 \int_0^1 \bar{w}_i \sin(m\pi \xi) d\xi,
\]

\[
\frac{\partial b_{1m}}{\partial \tau}(0) = 2 \int_0^1 \dot{\bar{w}}_0(\xi) \sin(m\pi \xi) d\xi.
\]

By solving Eqs. (22) and (23) for \(\mathcal{L}\{a_{im}\}\) and \(\mathcal{L}\{b_{im}\}\),

\[
\mathcal{L}\{a_{im}\} = \frac{A_{1im}s^3 + A_{2im}m^2s^2 + A_{3im}s + A_{4im}}{(s^2 + r_{1m}^2)(s^2 + r_{2m}^2)},
\]

\[
\mathcal{L}\{b_{im}\} = \frac{B_{1im}s^3 + B_{2im}m^2s^2 + B_{3im}s + B_{4im}}{(s^2 + r_{2m}^2)(s^2 + r_{3m}^2)}.
\]

where

\[
r_{1m} = \sqrt{(m\pi)^2 \bar{T}_s - 2\bar{f}_s}, \quad r_{2m} = m\pi \sqrt{\bar{T}_s}, \quad r_{3m} = \sqrt{(m\pi)^2 \bar{T}_s + 2\bar{f}_s},
\]

\[
A_{11m} = a_{1m}(0), \quad A_{12m} = \frac{\partial a_{1m}}{\partial \tau}(0), \quad A_{21m} = a_{2m}(0), \quad A_{22m} = \frac{\partial a_{2m}}{\partial \tau}(0),
\]

\[
A_{13m} = a_{1m}(0) (\bar{T}_s(m\pi)^2 - \bar{f}_s) - a_{2m}(0)\bar{f}_s,
\]

\[
A_{14m} = \frac{\partial a_{1m}}{\partial \tau}(0)(\bar{T}_s(m\pi)^2 - \bar{f}_s) - \frac{\partial a_{2m}}{\partial \tau}(0)\bar{f}_s,
\]

\[
A_{23m} = a_{2m}(0)(\bar{T}_s(m\pi)^2 - \bar{f}_s) - a_{1m}(0)\bar{f}_s,
\]

\[
A_{24m} = \frac{\partial a_{2m}}{\partial \tau}(0)(\bar{T}_s(m\pi)^2 - \bar{f}_s) - \frac{\partial a_{1m}}{\partial \tau}(0)\bar{f}_s,
\]
4. Results and Discussion

4.1. Special cases

4.1.1. $r_{1m} = r_{2m} = r_{3m}$: No dynamic interactions

When $\bar{f}_0 = 0$, let us consider $r_{1m} = r_{2m} = r_{3m} = r_m$; hence, $A_{i(j+2)m} = r_m A_{ijm}$ and $B_{i(j+2)m} = r_m B_{ijm}$; $j = 1, 2$. By substituting these values into Eqs. (29a) and (29b), the purely dynamic displacements of the nanowires are expressed by:

$$\bar{w}_i^d(\xi, \tau) = \sum_{m=1}^{\infty} \left( A_{11m} \cos(r_m \tau) + \frac{A_{12m}}{r_m} \sin(r_m \tau) \right) \sin(m \pi \xi), \quad (30a)$$

$$\bar{w}_i^d(\xi, \tau) = \sum_{m=1}^{\infty} \left( B_{11m} \cos(r_m \tau) + \frac{B_{12m}}{r_m} \sin(r_m \tau) \right) \sin(m \pi \xi). \quad (30b)$$
In this case, as it is obvious from Eqs. (30a) and (30b), vibrations of two nanowires become decouple and each nanowire vibrates in accordance with its own initial conditions. In other words, no interaction between nanowires exists since \( \bar{f}_0=0 \).

4.1.2. \( r_{11}=0 \): Dynamic instability

This case corresponds to \( \bar{f}_0=\pi^2T_0 \). By substituting this expression into Eq. (29a) and removing the ambiguity via L’Hopital rule, the displacements of the nanowires along the \( y \)-axis are derived as:

\[
\bar{v}_d^i(\xi, \tau) = \left( \frac{A_{i,31}}{r_{21}^2} + \frac{A_{i,41}}{r_{21}} + \left( \frac{A_{i,31} - r_{21}^2 A_{i,11}}{-r_{21}^2} \right) \cos(r_{21}\tau) + \left( \frac{A_{i,41} - r_{21}^2 A_{i,21}}{-r_{21}^2} \right) \sin(r_{21}\tau) \right) \sin(\pi\xi) + \sum_{m=2}^{\infty} \left( \frac{A_{i,3m} - r_{1m}^2 A_{i,1m}}{r_{2m}^2 - r_{1m}^2} \right) \cos(r_{1m}\tau) + \left( \frac{A_{i,4m} - r_{1m}^2 A_{i,2m}}{r_{2m}^2 (r_{1m}^2 - r_{2m}^2)} \right) \sin(r_{2m}\tau) \right) \sin(m\pi\xi).
\]

(31)

As it is obvious from Eq. (31), the in-plane displacements of the nanowires grow as time goes by and the dynamic instability occurs within the nanosystem. By increasing the displacements of the nanowires, the resulting elastic fields approach to their ultimate values until the nanoscale system collapses.

4.2. Free vibration and dynamic instability

Let the purely dynamic displacements of the nanowires can be expressed in the following harmonic form:

\[
\bar{v}_d^i(\xi, \tau) = \sum_{m=1}^{\infty} a_{0,m} e^{i\varpi_m \tau} \sin(m\pi\xi), \quad (32a)
\]

\[
\bar{w}_d^i(\xi, \tau) = \sum_{m=1}^{\infty} b_{0,m} e^{i\varpi_m \tau} \sin(m\pi\xi); \quad i = 1, 2 \quad (32b)
\]

where \( i = \sqrt{-1} \), \( a_{0,m} \) and \( b_{0,m} \) represent the amplitudes of the dynamic response of the \( i \)th nanowire along the \( y \)- and \( z \)-axes, respectively.

4.2.1. In-plane vibration modes and frequencies

By substituting Eq. (32) into Eqs. (18a) and (18c) and solving the resulting equations for \( \varpi_m \), the natural frequencies as well as amplitude ratios associated with the vibrations of the nanowires along the \( y \)-axis are obtained as follows:

\[
\left( \begin{array}{c} a_{0,1m} \\ a_{0,2m} \end{array} \right) |_I = -1, \quad (\varpi_m)_I = \sqrt{(m\pi)^2 \bar{T}_s - 2\bar{f}_s}.
\]

(33a)
\[
\left( \frac{a_{01m}}{a_{02m}} \right)_{II} = 1, \quad (\varpi_m)_{II} = (m\pi) \sqrt{T_s},
\]

where Eq. (33a) denotes out-of-phase vibration mode of the current-carrying nanowires while Eq. (33b) represents their in-phase vibration mode. As it is explained earlier, the out-of-phase vibration endangers safety of the nanosystem when \(\varpi_{m1} = 0\). The most critical case is \((\varpi_1)_{II} = 0\) in which leads to the following critical values of the initial tensile force, electric current, and interwire distance:

\[
T_{0,cr} = \frac{\mu_0 I_1^2 l_b^2}{\pi^3 d^2} - H_0, \tag{34a}
\]

\[
I_{cr} = \frac{\pi d}{l_b} \sqrt{\frac{\pi (T_0 + H_0)}{\mu_0}}, \tag{34b}
\]

\[
d_{cr} = \frac{I l_b}{\pi} \sqrt{\frac{\mu_0}{\pi (T_0 + H_0)}}. \tag{34c}
\]

Equations (34a)–(34c) state that when the existing initial tensile force in the nanowires is lower than its critical value or when the electric current is greater than its critical level or when the interwire distance is lower than its critical value, dynamic instability would occur within the nanoscale system. In such a condition, any exerted lateral motion along the \(y\)-axis on each nanowire can lead to large displacements in both nanowires. As it is seen in Eq. (33a), increasing of the surface stress effect, initial tensile force, or the interwire distance would provide a more stable system, however, an increase of the electric current would reduce the frequency until its value becomes zero and the nanosystem becomes dynamically unstable.

To show the role of the influential factors on the fundamental frequency, an inclusive parametric study is performed. Consider the silver nanowires with the following mechanical and geometry properties: \(E_b = 76 \times 10^9\) Pa, \(\rho_b = 10,500\) kg/m\(^3\), \(\rho_0 = 10^{-7}\) kg/m\(^2\), \(\tau_0 = 0.89\) N/m, \(l_b = 1500\) nm, and \(r_0 = 15\) nm. In Figs. 2(a)–2(c), the influences of the interwire distance and electric current on the fundamental frequency of the nanosystem are demonstrated. To study the problem in a more general framework, the dimensionless electric current, \(\bar{I}\), and the dimensionless interwire distance, \(\bar{d}\) are defined by:

\[
\bar{I} = I \sqrt{\frac{\mu_0}{2\pi E_b A_b}} \quad \text{and} \quad \bar{d} = d/l_b \quad \text{where} \quad I_1 = I_2 = I \quad \text{and} \quad \bar{f}_s = (\bar{I}/\bar{d})^2.
\]

The plotted results have been provided for three levels of the initial tensile force within the nanowires, namely \(T_0 = 0.5 \times 10^{-3}, 0.9 \times 10^{-3}\) and \(1.3 \times 10^{-3}\). As it is obvious in Figs. 2(a)–2(c), by increasing the initial tensile force within the nanowires, the fundamental frequency of the nanosystem magnifies. The main reason of this fact is that the nanowires become stiffer as the initial tensile force increases; thereby, all the frequencies of the nanosystem increase. The plotted results reveal that the fundamental frequency of the nanosystem increases as the interwire distance magnifies. However, for each level of the interwire distance, the fundamental frequency drastically decreases with the electric current until it approaches zero and the nanosystem becomes dynamically unstable. Such a fact is
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Fig. 2. Variation of the dimensionless fundamental frequency in terms of the dimensionless interwire distance and electric current for different levels of the initial tensile force: (a) $\bar{T}_0 = 0.5 \times 10^{-3}$, (b) $\bar{T}_0 = 0.9 \times 10^{-3}$ and (c) $\bar{T}_0 = 1.3 \times 10^{-3}$.

more obvious for lower levels of the interwire distance. According to the demonstrated results in Figs. 2(a)–2(c), the electric current corresponds to the initiation of dynamic instability increases as the interwire distance increases.

4.2.2. Out-of-plane vibration modes and frequencies

By substituting the equivalent values of $\tilde{\omega}_d^i$ from Eq. (32b) into Eqs. (18b) and (18d) and solving the resulting set of equations for the unknowns $b_{01m}$, the if and only if condition to obtain a nontrivial solution results in:

$$\begin{bmatrix} b_{01m} \\ b_{02m} \end{bmatrix}_{I} = 1, \quad (\bar{\omega}_m)_I = (m\pi)\sqrt{\bar{T}_s},$$

(35a)

$$\begin{bmatrix} b_{01m} \\ b_{02m} \end{bmatrix}_{II} = -1, \quad (\bar{\omega}_m)_II = \sqrt{(m\pi)^2\bar{T}_s + 2\bar{f}_s}.$$  

(35b)

Equations (35a) and (35b) display that there exists two modes of vibrations for doubly parallel nanowires along the z-axis. The first vibration mode (i.e. with the lower frequency), Eq. (35a), shows the in-phase vibration pattern since the modes of two nanowires have identical amplitudes and the amplitudes ratio is independent of the frequency as well as the interactional effects between the nanowires. As an example, consider two nanowires in which similarly deflected only in the $z$ direction.
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at time $t = 0$. If we let the nanosystem to vibrate freely with such an initial condition, two nanowires would vibrate identically (i.e. with similar amplitudes and zero phase delay with respect to each other). Equation 35(a) displays that the vibration in this case is not affected by the electric current within the nanowires (such a fact can be readily explained via Eqs. (4) and (5) by letting $\Delta w = 0$; therefore, the $z$-component of the Biot–Savart force would vanish. This is the main reason of this fact that the vibration of each current-carrying nanowire is not influenced by the vibration of its neighboring one.

The second vibration mode, Eq. (35b), displays the out-of-phase pattern of the vibration. In this case, both nanowires deflect in opposite directions and the interactional effects of two nanowires along the $z$-axis are incorporated into the corresponding frequency. When such interactions are negligible (i.e. $\bar{f}_0 \approx 0$), two nanowires vibrate independently and their dynamic displacements could be readily evaluated from Eqs. (30a) and (30b). Equation (35) also explains that no instability would be generated in the nanoscale system due to any cause of free or forced vibration along the $z$-axis.

5. Conclusions

By implementing a surface theory of elasticity, transverse vibrations of a double-nanowire-system for carrying electric current are investigated. For this purpose, the interactional Lorentz forces between the current-carrying nanowires accounting for their transverse displacements are derived. Thereafter, the equations of motion of the nanosystem are obtained in the context of small deformations. The explicit expressions of transverse displacements of nanowires are derived. The instability of such a system plus to the influential factors on the frequencies of the system are displayed in detail. The major obtained results are summarized as follows:

(i) For each component of transverse displacement, two vibration modes are detected: in-phase and out-of-phase modes. For those transverse displacements in the plane of nanowires, dynamic instability would be possible in the out-of-phase modes; however, for out-of-plane displacements, no dynamic instability would occur. The explicit expressions of the in-plane transverse displacements correspond to the dynamic instability are also evaluated.

(ii) Concerning the in-plane vibration, both in-phase and out-of-phase vibration modes and their corresponding frequencies are analytically obtained. For a specified value of a mode number, the frequencies of the out-of-phase modes are lower than those of the in-phase modes.

(iii) Regarding the out-of-plane vibration, the analytical expressions of both in-phase and out-of-phase vibration modes as well as their pertinent frequencies are derived. For a given mode number, the flexural stiffness of the system in the out-of-plane vibration modes is greater than that of the in-phase vibration modes.
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(iv) The critical values of the interwire distance, initial tensile force, and electric current correspond to the dynamic instability of the nanosystem are obtained. For interwire distance and initial tensile force lower than particular critical values as well as for an electric current greater than a special critical level, the dynamic instability is generated within the nanosystem.

References