Thermo-elasto-dynamic analysis of axially functionally graded non-uniform nanobeams with surface energy

Keivan Kiani*

Department of Civil Engineering, K.N. Toosi University of Technology, P.O. Box 15875–4416, Tehran, Iran

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This paper deals with transverse vibration of axially functionally graded tapered nano-scale beams acted upon by a longitudinal temperature gradient. Using surface elasticity theory of Gurtin–Murdoch, the equations of motion of the nanostructure are displayed based on the hypotheses of the Rayleigh–Timoshenko, and higher-order beam theory. Due to the variation of the material and the cross-section along the nanobeam, seeking an analytical solution to the resulting governing equations is a very cumbersome job. To conquer this difficulty, reproducing kernel particle method is proposed, and the natural frequencies of the thermally affected nanostructure are numerically calculated. Subsequently, the roles of the slenderness ratio, temperature gradient, diameter of the nanobeam, and variation of both the cross section and the material property along the length of the nanobeam on its free dynamic response are investigated. In each parametric study, the effects of both surface energy and shear deformation on the natural frequencies are addressed and explained.

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1. Introduction

Recently, micro-/nano-scaled structures made from functionally graded materials (FGMs) have been proposed as building blocks of micro-/nano-electromechanical systems (MEMS/NEMS) (Fu, Du, Huang, Zhang, & Hu, 2004; Witvrouw & Mehta, 2005) as well as shape memory alloy and thin films (Yoshimura, Suchanek, Watanabe, Sakurai, & Abe 1998; Bogdanski et al., 2002; Fu, Du, & Zhang, 2003; Sioh, 2010). The potential usage of functionally graded nanosensors and nanoactuators is also under investigation. The mechanical behaviors of most of MEMS/NEMS are commonly the same as those of beam-like structures. Therefore, realizing of deformation of functionally graded microbeams/nanobeams leads to better understanding of mechanical response of the above-mentioned systems.

In FGMs, the composition and structure of the constituents would gradually change over volume, resulting in variation of the material’s properties. The basic concept is to provide composite materials by specific alteration the microstructure from one material to another one. Such a fact enables the FGM to have the best of its constitutive materials in the needed directions. These materials are commonly designed for particular functions such as high temperature, fracture, fatigue, stress corrosion cracking, and corrosive resistance. In functionally graded nanobeams, the material properties can be allowed to vary in both longitudinal and transverse directions for the considered jobs. Herein, not only the material properties of the nanobeam of our concern, but also its geometry data would vary across the length. Such a nanostructure is called axially

* Corresponding author. Fax: +98 21 88779476.
E-mail address: k_kiani@kntu.ac.ir, keivankiani@yahoo.com

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functionally graded non-uniform nanobeam (AFGNNB). The main purpose of this study is to examine transverse vibrations of these nanostructures when they are acted upon by a longitudinal thermal field. To this end, appropriate surface energy-based beam models are developed and their capabilities in capturing the natural frequencies are discussed in some detail.

At the nanoscale, the ratio of the surface area to the bulk volume is large enough that the role of elastic strain energy of the surface layer in the mechanical behavior of the nanostructure becomes important. The surface elasticity theory of Gurtin and Murdoch (Gurtin & Murdoch, 1975; 1976; 1978) is one of the successful theories to include such effect in the continuum-based modeling of nanoscaled structures. In brief, this theory explains that the surface is a very thin layer of negligible thickness which has been tightly attached to the underlying bulk. It implies that the displacements of the surface layer are identical to those of the bulk at the vicinity of the surface. However, the mechanical behavior of the surface layer is completely different from that of the bulk. The surface elasticity theory of Gurtin–Murdoch introduces residual surface stress plus to two Lame’s constants to the constitutive relations of the surface layer. The magnitude of these parameters are commonly determined by comparing the obtained results by the surface elasticity-based model and those of appropriate atomistic-based models (Chen, Shi, Zhang, Zhu, & Yan, 2006; Shenoy, 2005) or experiments (Jing et al., 2006; Zheng, Cao, Li, Feng, & Wang, 2010). To investigate the role of surface stress in the elastic properties of nanostructures, Wang, Zhao, and Huang (2010) presented the surface elasticity-based formulations of elastic solids for both Lagrangian and Eulerian contexts. To date, the surface elasticity theory has been widely employed in statics (Fu, Zhang, & Jiang, 2010; Jiang & Yan, 2010; Mahmoud, Eltaher, Alshorbagy, & Meletis, 2012; Yan & Jiang, 2011), free dynamics (Eltaher, Emam, & Mahmoud, 2012b; Ghashlaghi & Hasheminejad, 2011; Kiani, 2014d; 2015c; Wang & Feng, 2010; Wang, 2010), forced vibrations (Ansari, Mohammadi, Shojaei, Gholami, & Sahnani, 2014b; Kiani, 2014a), buckling (Kiani, 2015a; 2015b; Li, Song, Fang, & Zhang, 2011; Wang & Feng, 2009; Wang, 2012), and postbuckling (Ansari, Mohammadi, Faghih Shojaei, Gholami, & Sahnani, 2014a; Sahnani, Bahrami, & Aghdam, 2016) of nanoscale beam-like structures. In all these carried out studies, the cross-section of the nanobeam and the material’s properties are constant across the nanobeam’s length.

Concerning vibrations of nanobeams with non-uniform cross section, Malekzadeh and Shojaee (2013) studied nonlinear flexural vibration of elastically supported nanobeams accounting for both surface and nonlocality effects. To this end, both the Euler–Bernoulli and Timoshenko beam theories were implemented. By using Hamilton’s principle in conjunction with surface elasticity theory of Gurtin–Murdoch and nonlocal continuum theory of Eringen, the governing equations were derived and solved via differential quadrature method. Based on the hypotheses of the Euler–Bernoulli beam by considering the von-Karman strain them and surface energy, On, Altus, and Tadmor (2010) investigated static deformation of nano-scaled non-uniform beams. For special variations of materials along the length, the obtained results by the suggested continuum-based model were checked with those of an atomistic model. Regarding mechanical analysis of functionally graded nanobeams, Sharabiani and Yazdi (2013) investigated nonlinear free vibration of nanobeams with allowance of material variation across the thickness using the surface energy-based Euler–Bernoulli beam theory. In another work, Hosseini-Hashemi and Nazemnezhad (2013) examined nonlinear vibration of thin nanobeams analytically. The nonlinear frequencies of the nanostructure were calculated using multiple scale method, and the influences of the length, volume fraction index, amplitude ratio, and thickness on the natural frequencies were addressed. There exist also several works on transverse vibrations of functionally graded uniform nanobeams using nonlocal elasticity (Eltaher, Alshorbagy, & Mahmoud, 2013; Eltaher, Emam, & Mahmoud, 2012a; Hosseini-Hashemi, Nahas, Fakher, & Nazemnezhad, 2014; Kiani, 2014b; Pedram, 2014). A brief survey of the literature reveals that transverse vibrations of AFGNNBs in the presence of temperature gradient have not been explored thoroughly. Further, the roles of surface effect and shear deformation on their natural frequencies have not been answered convincingly yet.

Due to the variation of both geometry and material properties across the nanobeam’s length, the resulted equations of motion for the developed models are highly spatial dependent. As a result, seeking for an exact or even an analytical solution to them is a very problematic job. To overcome these difficulties, reproducing kernel particle method (RKPM) is adopted. This efficient meshless methodology employs higher-order shape functions and thereby, it enables us to use that for the problems suffer from highly gradient field in partial part or whole part of the domain. Until now, RKPM has been widely used for dynamic analysis of macrobeams (Aluru, 1999; Kiani & Nikkhoo, 2012; Zhou, Zhang, & Zhang, 2005) and nanobeams (Kiani, 2014c). In the present work, the strong form of the equations of motion of each model is obtained in the context of the surface elasticity-based theory of Gurtin–Murdoch. Thereafter, the Galerkin-based RKPM approach is applied to these equations to explore the transverse dynamic behavior of thermally affected AFGNNBs.

Herein, by employing Rayleigh beam theory (RBT), Timoshenko beam theory (TBT), and higher-order beam theory (HOBT) and evaluating the surface energy for each model, the dimensionless equations of motion are constructed using the Hamilton’s principle. Due to variation of the material properties of the bulk and the surface layer along the length of the nanobeam, finding an analytical solution to the problem even for simply supported boundary condition is a cumbersome task. To overcome this dilemma, an efficient meshless methodology is proposed and the essential boundary conditions are enforced by the corrected collocation approach. Subsequently, the natural frequencies associated with the transverse vibration of each model are calculated. The roles of the nanobeam’s diameter, slenderness ratio, temperature gradient, power-law index, and surface energy on the natural frequencies of AFGNNBs are addressed. Specificaly, the influences of the shear deformation and surface energy on the obtained results are explained and discussed. This work can be regarded as an appropriate benchmark for transverse vibration analysis of more complex nanosystems.
consist of multiply interacted nanobeams, for example forests of nanotubes, membranes of nanotubes, and vertically aligned nanostructures.

2. Description of the problem

Consider an elastically embedded tapered functionally graded nanobeam in a thermal environment as shown in Fig. 1. For the sake of surface-energy based modeling of vibration of this nanostructure, it is decomposed into the bulk and the surface layer whose cross-sections and materials properties would generally vary across the length of the nanobeam. The considered nanobeam has circular cross-section; however, the presented formulations (both strong and weak versions of equations of motion) are general and could be applied to any beam-like nanostructure accounting for the surface energy effect. The left and the right ends of the nanobeam are immovable along the x axis. The geometrical and physical properties of the bulk and those of the surface layer of the AFGNNB would vary along the length based on the following power-law relation (Shafiei, Kazemi, & Ghadiri, 2016; Simsek, 2012; 2015):

\[
[\square](x) = [\square]_L + ([\square]_R - [\square]_L) \left( \frac{x}{l_b} \right)^{\gamma}; \gamma = p_m, p_s, \text{ or } p_g,
\]

where \([\square]_L\) and \([\square]_R\) in order represent the effective physical properties of the left-hand-side and the right-hand-side of the nanobeam, \(p_m, p_s, \text{ and } p_g\) denote the power-law indexes pertinent to the variations of physical properties of the bulk, those of the surface layer, and the nanobeam’s radius, respectively, and \(l_b\) is the length of the nanobeam. The vibrating nanobeam is subjected to a longitudinal thermal field such that the temperature of its surrounding environment at the left and the right ends in order are denoted by \(T_{en,L}\) and \(T_{en,R}\). In view of the end conditions, an axial thermal force is generated within both surface layer and bulk. Such a force intends to reduce the lateral stiffness of the nanostructure, and thereby, it would influence on its transverse vibration. One of the purposes of this research work is to explore such an effect by considering both surface effect and shear deformation of the nanobeam.

In the following parts, the resulted thermo-mechanical axial force within the nanobeam is first evaluated. Thereafter, using the RBT, TBT, and HOBT, the strong form of the governing equations pertinent to the transverse vibration of AFGNNBs is displayed. For each model, RKPM is then applied to compute natural frequencies of the nanostructure.

3. Analysis of thermal field and determination of thermo-mechanical axial force

3.1. Thermal field analysis

The governing equation of the thermal field across the length of the AFGNNB reads:

\[
\frac{d}{dx} \left[ k(x) \frac{dT}{dx} \right] = 0,
\]

where \(k(x)\) denotes the coefficient of thermal conductivity, \(d\) represents the differential sign, and \(T = T(x)\) is the thermal field of the nanobeam. The following boundary conditions are considered at the left and the right ends of the nanostructure:

\[
-k_L \frac{dT}{dx} + h_L (T - T_{en,L}) = 0; \quad x = 0, \tag{3a}
\]

\[
k_R \frac{dT}{dx} + h_R (T - T_{en,R}) = 0; \quad x = l_b, \tag{3b}
\]

where \(k_L\) and \(k_R\) in order are the coefficients of thermal conductivity of the left and the right ends, \(h_L\) and \(h_R\) represent the heat transfer coefficients of the left and the right ends of the nanobeam, respectively. Let,

\[
\xi = \frac{x}{l_b}, \quad \beta_L = \frac{h_L l_b}{k_L}, \quad \beta_R = \frac{h_R l_b}{k_R},
\]

therefore, the dimensionless governing equation and the dimensionless boundary conditions are given by:
\[
\frac{d}{d\xi} \left[ k(\xi) \frac{dT}{d\xi} \right] = 0, \quad (5a)
\]
\[
-dT + \beta_L (T - T_{en,L}) = 0; \quad \xi = 0, \quad (5b)
\]
\[
dT + \beta_R (T - T_{en,R}) = 0; \quad \xi = 1. \quad (5c)
\]

Eq. (5a) represents a second-order ordinary differential equation with the Robin boundary conditions in Eq. (5b and c). An analytical solution to Eq. (5) can be sought as follows:

\[
T(\xi) = \left[ \chi_{22} \Gamma_1 - \chi_{12} \Gamma_2 \right] \int_0^\xi \frac{d\xi'}{k(\xi')} + \left[ \chi_{11} \Gamma_2 - \chi_{21} \Gamma_1 \right], \quad (6)
\]
where
\[
\chi_{11} = -\frac{1}{k_L}, \quad \chi_{12} = \beta_L, \quad \Gamma_1 = \beta_L T_{en,L},
\]
\[
\chi_{21} = \beta_R \int_0^1 \frac{d\xi'}{k(\xi')} - \frac{1}{k_R}, \quad \chi_{22} = \beta_R, \quad \Gamma_2 = \beta_R T_{en,R}. \quad (7)
\]

### 3.2. Thermo-mechanical axial force analysis

Due to the nature of the induced thermal field, only longitudinal thermal stress would be generated with the bulk and the surface layer. Since the surface layer has been tightly attached to the bulk, their total strains are identical (i.e., \( \epsilon_{xx}^b = \epsilon_{xx}^s = \epsilon_{xx}^t \)). Therefore, the thermo-mechanical stresses of the bulk and the surface layer would be:

\[
\sigma_{xx}^b = E_b(\chi) (\epsilon_{xx} - \alpha_T(\chi) T(\chi)) \quad \text{and} \quad \sigma_{xx}^s = E_s(\chi) (\epsilon_{xx} - \alpha_T(\chi) T(\chi)),
\]
where \( E_b(\chi) \) and \( E_s(\chi) \) represent the longitudinally varying Young’s modulus of the surface layer and the bulk, respectively, \( \alpha_T(\chi) \) is the coefficient of thermal expansion of the bulk and the surface layer in which vary along the nanobeam. The resulted thermo-mechanical axial force within the nanobeam is evaluated as follows:

\[
N_T = (E_b(\chi) A_b(\chi) + E_s(\chi) S_0(\chi)) \epsilon_{xx} - (E_b(\chi) A_b(\chi) \alpha_T(\chi) + E_s(\chi) S_0(\chi) \alpha_T(\chi)) T(\chi). \quad (8)
\]

where \( A_b(\chi) \) is the cross-sectional area of the bulk, and \( S_0(\chi) \) is the cross-sectional length of the surface layer. If the left and the right ends of the nanobeam would be fixed, their relative longitudinal displacements would be zero (i.e., \( \int_0^L \epsilon_{xx} \, d\chi = 0 \)). Hence, the resulted thermo-mechanical axial force within the AFGNNB due to the exerted thermal field is calculated by:

\[
N_T = -\int_0^1 (E_b(\xi) A_b(\xi) \alpha_T(\xi) + E_s(\xi) S_0(\xi) \alpha_T(\xi)) T(\xi) \, d\xi. \quad (9)
\]

### 4. Basic formulations via surface energy-based RBT

#### 4.1. Strong form of the equations of motion via RBT

According to the RBT, the longitudinal and transverse displacement fields of both bulk and surface layer of the nanobeam \( u^b_n \) and \( u^s_n \) in terms of the transverse deflection of the neutral axis of the nanobeam \( w^R \) are stated by:

\[
u^R = -z \frac{\partial w^R}{\partial x}, \quad u^s = w^R. \]

Thereby, the only strain component of the nanobeam reads: \( \epsilon_{xx}^R = -z \frac{\partial^2 w^R}{\partial x^2} \). Based on the surface elasticity theory \( \text{(Gurtin \\& Murdoch, 1975; 1976; 1978)} \), the nonzero stresses of the non-uniform surface layer are:

\[
\tau_{xx}^R = \tau_0(\chi) - 2(\lambda_0(\chi) + 2\mu_0(\chi)) \frac{\partial^2 w^R}{\partial x^2}, \quad \tau_{xz}^R = n_x \mu_0(\chi) \frac{\partial w^R}{\partial x}, \quad (10)
\]

where \( \tau_0(\chi) \) is the residual surface stress under unconstrained condition, \( \tau_{xx}^R \) and \( \tau_{xz}^R \) are the normal stress and the shear stress of the surface layer, \( \lambda_0(\chi) \) and \( \mu_0(\chi) \) are the varying Lame’s constants of the surface, and \( n_x \) is the z direction component of the unit vector perpendicular to the surface.

Let the bulk normal stress along the x direction to vary between their surface values. Therefore, the longitudinal stress within the bulk takes the following form:

\[
\sigma_{xx}^b = E_b(\chi) \epsilon_{xx}^b + v_b(\chi) \sigma_{22}^b = -z \left( E_b(\chi) - \frac{2\tau_0(\chi) v_b(\chi)}{D_0(\chi)} \frac{\partial^2 w^R}{\partial x^2} + \frac{2\rho_0(\chi) v_b(\chi)}{D_0(\chi)} \frac{\partial^2 w^R}{\partial t^2} \right), \quad (11)
\]

where \( E_b(\chi) \) and \( v_b(\chi) \) in order are Young’s modulus and Poisson’s ratio of the bulk, and \( \rho_0(\chi) \) is the longitudinally varying density of the surface layer.
Using the hypotheses of the RBT and Hamilton’s principle, the transverse vibration of a non-uniform nanobeam with axially varying material accounting for the surface effect is displayed by:

\[
\begin{align*}
\int_{A_b} \rho_b(x) \frac{\partial^2 w_b^R}{\partial t^2} \, dx + \int_{A} \rho_0(x) \frac{\partial^2 w_b^R}{\partial t^2} \, dx + \frac{\partial}{\partial x} \left[ M_{b_b} + \int_{S} \tau_{xx} \, dS \right] + Q_{b_b} - K_{b_b} \frac{\partial w_b^R}{\partial x} &= 0, \\
\int_{A_b} \rho_b(x) \frac{\partial^2 w_b^R}{\partial t^2} \, dA + \int_{A_b} \rho_0(x) \frac{\partial^2 w_b^R}{\partial t^2} \, dA + \frac{\partial}{\partial x} \left[ Q_{b_b} + \int_{S} n_{y} \tau_{xy} \, dS \right] - N_T \frac{\partial^2 w_b^R}{\partial x^2} + K_{b_b} w_b^R &= 0,
\end{align*}
\]

(12a, b)

where \( \rho_b(x) \) is the density of the bulk, \( Q_{b_b}^R \) and \( \int_{S} n_{y} \tau_{xy} \, dS \) are the resultant shear force of the bulk and the surface layer along the z axis, \( M_{b_b}^R \) and \( \int_{S} \tau_{xx} \, dS \) are the bending moment of the bulk and the surface layer about the y axis, and \( dS \) is the length of an infinitesimal element of the surface's cross-section. By combining Eq. (12a and b), one can arrive at:

\[
- \frac{\partial^2 M_{b_b}^R}{\partial x^2} - \left( \int_{S} \left( \frac{\partial^2 \tau_{xx}}{\partial x^2} + \frac{\partial \tau_{xy}}{\partial z} \right) \right) \frac{\partial^2 w_b^R}{\partial x^2} + \int_{A_b} \rho_b \left( \frac{\partial^2 w_b^R}{\partial t^2} + z \frac{\partial^3 w_b^R}{\partial t^2 \partial x} \right) \, dA \\
+ \int_{S} \rho_0 \left( \frac{\partial^2 w_b^R}{\partial t^2} + \frac{\partial^3 w_b^R}{\partial t^2 \partial x} \right) \, dS + K_{b_b} w_b^R - (N_T + K_{b_b}) \frac{\partial^2 w_b^R}{\partial x^2} = 0.
\]

(13)

where

\[
M_{b_b}^R = \int_{A} z \sigma_{xx} \, dA = - \left( E_b(x) I_b(x) - 2 \tau_0(x) v_b(x) I_b(x) \right) \frac{\partial^2 w_b^R}{\partial x^2} - \frac{2 v_b(x) I_b(x) \tau_0(x)}{D_0(x)} \frac{\partial^4 w_b^R}{\partial t^2 \partial x^2},
\]

(14)

by introducing Eq. (14) to Eq. (13), the equation of motion associated with the transverse vibration of thermally affected AFGNBN on the basis of the RBT is obtained as:

\[
(\rho_b(x) A_b + \rho_0(x) S_0^b(x)) \frac{\partial^2 w_b^R}{\partial t^2} - \left( \rho_b(x) I_b(x) + \rho_0(x) I_0^b(x) - \frac{2 v_b(x) I_b(x) \rho_0(x)}{D_0(x)} \right) \frac{\partial^4 w_b^R}{\partial t^2 \partial x^2} \\
+ \frac{\partial^2}{\partial x^2} \left[ E_b(x) I_b(x) + (\lambda_0(x) + 2 \mu_0(x)) I_0^b(x) - \frac{2 v_b(x) I_b(x) \tau_0(x)}{D_0(x)} \right] \frac{\partial^2 w_b^R}{\partial x^2} \\
+ K_{b_b} w_b^R - (\tau_0(x) S_0(x) + N_T + K_{b_b}) \frac{\partial^2 w_b^R}{\partial x^2} = 0.
\]

(15)

In order to study the problem more conveniently, the following dimensionless quantities are considered:

\[
\tau = \frac{t}{l_b^2} \sqrt{\frac{E_b I_b}{\rho_b b L_b}}, \quad \bar{w}_b^R = \frac{w_b^R}{l_b}, \quad \lambda = \frac{l_b}{\sqrt{I_b/L_b}}, \quad \bar{\rho}_b(\xi) = \frac{\rho_b(x)}{\rho_b b},
\]

\[
\bar{A}_b(\xi) = \frac{A_b(x)}{A_b L_b}, \quad I_b(\xi) = \frac{I_b(x)}{I_b}, \quad \bar{N}_T^R = \frac{N_T I_b^2}{E_b I_b}, \quad \bar{K}_T^R = \frac{K_T l_b^4}{E_b I_b}, \quad \bar{K}_r^R = \frac{K_r l_b^4}{E_b I_b},
\]

\[
\chi_1^R(\xi) = \frac{\rho_b(x) S_0^b(x)}{\rho_b A_b L_b}, \quad \chi_2^R(\xi) = \frac{\rho_b(x) I_0^b(x)}{\rho_b A_b L_b} - \frac{2 v_b(x) I_b(x) \rho_0(x)}{D_0(x) \rho_b b L_b},
\]

\[
\chi_3^R(\xi) = \frac{(\lambda_0(x) + 2 \mu_0(x)) I_0^b(x)}{E_b I_b} - \frac{2 v_b(x) I_b(x) \tau_0(x)}{D_0(x) E_b I_b}, \quad \chi_4^R(\xi) = \frac{\tau_0(x) S_0^b(x) l_b^2}{E_b I_b}.
\]

(16)

By introducing Eq. (16) to Eq. (15), the dimensionless governing equation associated with the transverse vibration of the nanostructure is concluded:

\[
\left( \bar{\rho}_b(\xi) \bar{A}_b(\xi) + \chi_1^R(\xi) \right) \frac{\partial^2 \bar{w}_b^R}{\partial \tau^2} - \lambda^2 \left( \bar{\rho}_b(\xi) I_b(\xi) + \chi_2^R(\xi) \right) \frac{\partial^4 \bar{w}_b^R}{\partial \tau^2 \partial \xi^2} \\
+ \frac{\partial^2}{\partial \xi^2} \left[ \bar{E}_b(\xi) I_b(\xi) + \chi_3^R(\xi) \right] \frac{\partial^2 \bar{w}_b^R}{\partial \xi^2} - \left( \chi_4^R(\xi) + \bar{N}_T^R + \bar{K}_T^R \right) \frac{\partial^2 \bar{w}_b^R}{\partial \xi^2} + \bar{K}_r^R \bar{w}_b^R = 0.
\]

(17)

If the surface parameters are set equal to (i.e., \( \chi_1^R = \chi_2^R = \chi_3^R = \chi_4^R = 0 \)), Eq. (17) is reduced to the governing equation of a macro-scaled axially functionally graded beam subjected to temperature gradient at its ends.

4.2. Weak form of the equations of motion based on the RBT via RKPM

Let discretize the transverse displacement in terms of RKPM’s shape functions as follows:

\[
\bar{w}_b^R(\xi, \tau) = \sum_{l=1}^{N_P} \phi_l^R(\xi) \bar{w}_b^R(\tau),
\]

(18)
where $NP$ is the number of RKPM’s particles, $\phi^w_i(\xi)$ is the shape function associated with the $i$th particle, and $\mathbf{W}^R(\tau)$ denotes the time-dependent parameter associated with the $i$th particle. By premultiplying both sides of Eq. (17) by $\delta \mathbf{W}^R$, $\delta$ is the variational sign, taking the integral on the dimensionless spatial domain of the nanobeam, and using integration by parts, the following set of equations are obtainable:

$$\begin{align*}
\left[\mathbf{M}_b^R\right]^{ww} \frac{d^2 \mathbf{W}^R}{d \tau^2} + \left[\mathbf{K}_b^R\right]^{ww} \mathbf{W}^R &= \mathbf{0},
\end{align*}$$

where the dimensionless mass and stiffness matrices are defined by:

$$\begin{align*}
\left[\mathbf{M}_b^R\right]^{ww}_{ij} &= \int_0^1 \left( (\bar{\rho}_b(\xi) \bar{A}_b(\xi) + \chi^R_x(\xi)) \phi^w_i(\xi) \phi^w_j(\xi) + \lambda^{-2} (\bar{D}_b(\xi) \bar{I}_b(\xi) + \chi^R_x(\xi)) \frac{d \phi^w_i}{d \xi} \frac{d \phi^w_j}{d \xi} \right) d\xi, \\
\left[\mathbf{K}_b^R\right]^{ww}_{ij} &= \int_0^1 \left( (\bar{D}_b(\xi) \bar{I}_b(\xi) + \chi^R_x(\xi)) \frac{d^2 \phi^w_i}{d \xi^2} \frac{d^2 \phi^w_j}{d \xi^2} + (\chi^R_x(\xi) + \bar{R}_T + \bar{K}^R) \frac{d \phi^w_i}{d \xi} \frac{d \phi^w_j}{d \xi} + \bar{K}^R \phi^w_i \phi^w_j \right) d\xi,
\end{align*}$$

$\mathbf{W}^R(\tau) = < \mathbf{W}^R_1(\tau),\mathbf{W}^R_2(\tau),\ldots,\mathbf{W}^R_N(\tau)>^T$.

In the case of AFGNNB with simply supported ends, the following conditions should be satisfied:

$$\mathbf{W}^R(0, \tau) = \mathbf{W}^R(1, \tau) = \mathbf{0}; \quad \bar{M}_b^R(0, \tau) = \bar{M}_b^R(1, \tau) = \mathbf{0},$$

where $\bar{M}_b^R = \frac{M_b^R}{t \bar{b}_1^4}$. In order to enforce the essential boundary conditions from all conditions in Eq. (21), the corrected collocation method (Wagner & Liu, 2000) is implemented. By considering a harmonic form for the time-dependent vectors of nodal parameter values, and by solving the resulting set of eigenvalue equations for the dimensionless natural frequencies, $\omega^R_1$, the frequencies of the AFGNNB based on the RBT are evaluated by: $\omega^R_1 = \frac{m_b^R}{\bar{b}_1} \sqrt{\frac{T_0 t \bar{b}_1}{\rho_b t \bar{b}_1}}$.

5. Basic formulations via surface energy-based TBT

5.1. Strong form of the equations of motion via TBT

Using the TBT, the longitudinal and transverse displacement fields of both bulk and surface layer of the nanobeam ($u^T_x$ and $u^T_y$) as a function of the angle of deflection and deflection of the neutral axis ($\theta^T_x$ and $\theta^T_y$) are expressed by: $u^T = -2\theta^T_x$ and $u^T = -2\theta^T_y$. Thereby, the nonvanishing small strains would be: $\varepsilon^T_x = -z^T \frac{d \theta^T_x}{d x}$ and $\gamma^T_x = \frac{d \theta^T_x}{d y} - \theta^T_x$. According to the surface elasticity theory by Gurtin and Murdoch (1975; 1976; 1978), the only stresses within the surface layer of the AFGNNB are stated by:

$$\tau^T_{xx} = \tau_0(x) - z \left( \lambda_0(x) + 2\mu_0(x) \right) \frac{d \theta^T_x}{d x}, \quad \tau^T_{yy} = n_z \tau_0(x) \frac{d \theta^T_x}{d x}.$$

It is assumed that $\sigma^T_{zz}$ would linearly vary across the thickness of the nanobeam between those values of the surface layer at the bottom and the top axes. By employing the generalized Hook’s law, the longitudinal stress ($\sigma^T_x$) and the shear stress ($\sigma^T_{xz}$) of the non-uniform bulk are given by:

$$\sigma^T_{xx} = \bar{E}_b(x) \left( -\frac{D_0(\chi)}{E_0(\chi)} + \frac{2b_0(\chi)}{D_0(\chi)} \right) \left( \frac{\partial \tau^T_x}{\partial x} - \rho_0(x) \frac{\partial w^T}{\partial t^2} \right),$$

$$\sigma^T_{xz} = \bar{G}_b(x) \left( \frac{\partial w^T}{\partial x} - \theta^T_y \right).$$

By taking an infinitesimal longitudinal element of the nanobeam, plotting the free-body-diagram of the surface layer and the bulk, and expressing the dynamic equilibrium equations using Newton’s second law,

$$\begin{align*}
\int_{S_b} \rho_b \frac{\partial^2 u^T_x}{\partial t^2} \, dA + \int_{S} \rho_0 \frac{\partial^2 u^T_x}{\partial t^2} \, dS - \frac{\partial M_b^T}{\partial x} - \int_{S} z \frac{\partial \tau^T_x}{\partial x} \, dS + Q^T_b + K_t \theta^T_y &= 0, \\
\int_{S_b} \rho_b \frac{\partial^2 u^T_y}{\partial t^2} \, dA + \int_{S} \rho_0 \frac{\partial^2 u^T_y}{\partial t^2} \, dS - \frac{\partial Q^T_b}{\partial x} - N_t \frac{\partial w^T}{\partial x} + \int_{S} \frac{\partial \tau^T_x}{\partial x} n_z \, dS + K_t w^T &= 0.
\end{align*}$$
where $Q_{by}^T$ is the resultant shear force within the bulk along the z axis, and $M_{by}^T$ is the bending moment about the y axis. These internal forces are defined by:

$$M_{by}^T = \int_{A_0} z \sigma_{xz}^T \, dA = -E_b(x)I_b(x) \frac{\partial \theta_y^T}{\partial x} + \frac{2}{D_0(x)} \left( \frac{\partial^2 w^T}{\partial x^2} - \rho_0(x) \frac{\partial^2 \theta_y^T}{\partial t^2} \right). \quad (25a)$$

$$Q_{by}^T = k_s \int_{A_0} \sigma_{yz}^T \, dA = k_s G_b(x)A_b(x) \left( \frac{\partial w^T}{\partial x} - \theta_y^T \right). \quad (25b)$$

By substituting Eq. (25a and b) into Eq. (24a and b), the equations of motion of a thermally affected AFGNNB as a function of deformation fields of the nanobeam based on the TBT are obtained as follows:

$$(\rho_b(x)A_b(x) + \rho_0(x)S_b^0(x)) \frac{\partial^2 \theta_y^T}{\partial t^2} + \frac{\partial}{\partial x} \left[ - \frac{2}{D_0(x)} \frac{\partial^2 w^T}{\partial x^2} + \frac{2}{D_0(x)} \frac{\partial^2 w^T}{\partial t^2} \right] - k_s(x)G_b(x)A_b(x) \left( \frac{\partial w^T}{\partial x} - \theta_y^T \right) - \frac{\partial}{\partial x} \left[ E_b(x)I_b(x) + (\lambda_0(x) + 2 \mu_0(x)) \frac{\partial \theta_y^T}{\partial x} \right] + K_t \theta_y^T = 0, \quad (26a)$$

$$(\rho_b(x)A_b(x) + \rho_0(x)S_b^0(x)) \frac{\partial^2 w^T}{\partial t^2} - \frac{\partial}{\partial x} \left[ k_s(x)G_b(x)A_b(x) \left( \frac{\partial w^T}{\partial x} - \theta_y^T \right) \right] - \tau_0(x)S_b^0(x) \frac{\partial^2 w^T}{\partial x^2} - N_T \frac{\partial^2 w^T}{\partial x^2} + K_t w^T = 0. \quad (26b)$$

To analyze the problem in a more general framework, the following dimensionless parameters are taken into account:

$$\bar{W}_b^T = \frac{w^T}{I_b}, \quad \bar{\theta}_b^T = \bar{\theta}_y^T, \quad \bar{r} = \frac{t}{I_b} \sqrt{\frac{G_{b,l}}{\rho_{b,l}}}, \quad \bar{K}_b^T = \frac{k_s}{k_{b,d}}, \quad \bar{G}_b^T = \frac{G_{b,l}}{k_{b,d}}, \quad \bar{R}_I^T = \frac{K_I^T}{k_{b,d}A_b},$$

$$\bar{K}_b^T = \frac{k_I^T}{k_{b,d}A_b}, \quad \bar{\eta}(\xi) = \frac{E_b(x)I_b(x)}{k_sG_b(x)A_b}, \quad \bar{R}_T^T = \frac{N_T}{k_sG_b(x)A_b}, \quad \bar{\lambda}_T^T = \frac{\lambda_0(x) + 2 \mu_0(x)}{\rho_b(x)D_0(x)}, \quad \bar{\lambda}_T^T = \frac{\lambda_0(x) + 2 \mu_0(x)}{\rho_b(x)D_0(x)}, \quad \bar{\lambda}_T^T = \frac{\lambda_0(x) + 2 \mu_0(x)}{\rho_b(x)D_0(x)}.$$

By introducing Eq. (27) to Eq. (26a and b), the dimensionless equations of motion of the nanostructure in the presence of the longitudinal temperature gradient are obtained as:

$$\lambda^{-2} \left( \bar{\mu}_b(\xi) I_b(\xi) + \chi_2(\xi) \right) \frac{\partial^2 \bar{\theta}_y^T}{\partial \xi^2} + \frac{\partial}{\partial \xi} \left[ -\lambda^{-2} \chi_1(\xi) \frac{\partial^2 \bar{W}_b^T}{\partial \xi^2} + \chi_4(\xi) \frac{\partial^2 \bar{W}_b^T}{\partial \xi^2} \right]$$

$$- \frac{\partial}{\partial \xi} \left[ \left( \bar{\eta}(\xi) + \chi_5(\xi) \right) \frac{\partial \bar{\theta}_y^T}{\partial \xi} - \bar{K}_b(\xi) \bar{G}_b(\xi) \bar{A}_b(\xi) \left( \frac{\partial \bar{W}_b^T}{\partial \xi} - \bar{\theta}_y^T \right) \right] + \bar{K}_b^T \bar{w}_b^T = 0. \quad (28a)$$

$$\left( \bar{\mu}_b(\xi) \bar{A}_b(\xi) + \chi_1(\xi) \right) \frac{\partial^2 \bar{\theta}_b^T}{\partial \xi^2} - \frac{\partial}{\partial \xi} \left[ \left( \bar{\eta}(\xi) \bar{G}_b(\xi) \bar{A}_b(\xi) \left( \frac{\partial \bar{W}_b^T}{\partial \xi} - \bar{\theta}_y^T \right) \right] \right]$$

$$- \left( \chi_5(\xi) + \bar{K}_T \right) \frac{\partial^2 \bar{W}_b^T}{\partial \xi^2} + \bar{R}_I^T \bar{W}_b^T = 0. \quad (28b)$$

In Eq. (28a and 28b), by vanishing the spatially varying surface parameters (i.e., $\chi_1(\xi) = \chi_2(\xi) = \chi_3(\xi) = \chi_4(\xi) = \chi_5(\xi) = \bar{\lambda}_T(\xi) = 0$), the equations of motion of an axially functionally graded macrobeam with varying cross-section based on the classical TBT are retrieved. Finding an exact or even an analytical solution to the coupled Eq. (28a and 28b) is not an easy task, mostly due to the variation of both geometry and material properties of both bulk and surface layer along the nanobeam’s length. In the following part, a numerical methodology based on the RKPM’s interpolants are presented.

5.2. Weak form of the equations of motion based on the TBT via RKPM

The deformation fields of the TBT-based nanostructure by employing the RKPM are discretized as:

$$\bar{W}_b^T(\xi, \tau) = \sum_{i=1}^{NP} \phi_{b,i}^T(\xi) \bar{W}_b^T(\tau), \quad \bar{\theta}_b^T(\xi, \tau) = \sum_{i=1}^{NP} \phi_{\theta,i}^T(\xi) \bar{\theta}_b^T(\tau).$$

(29)
where $\phi_i^w(\xi)$ and $\phi_j^w(\xi)$ in order are the $i$th shape functions pertinent to the deflection and angle of deflection fields, and, $\mathbf{W}_T^T(\tau)$ and $\partial_T^y(\tau)$ are their corresponding nodal parameter values. To weaken Eq. (28a and b), both their sides are premultiplied by $\partial_T^y$ and $\delta \mathbf{W}_T^T$, respectively. Then, the resulting relations are integrated over the dimensionless spatial domain of the nanostructure. After taking successful integration by parts,

$$
\begin{bmatrix}
\mathbf{M}_b^T & \mathbf{M}_b^T \\
\mathbf{M}_b^T & \mathbf{M}_b^T
\end{bmatrix}
\begin{bmatrix}
\phi_i^w(\xi) \\
\phi_j^w(\xi)
\end{bmatrix}
\begin{bmatrix}
\partial_T^y \\
\partial_T^y
\end{bmatrix}
\begin{bmatrix}
\mathbf{K}_b^T & \mathbf{K}_b^T \\
\mathbf{K}_b^T & \mathbf{K}_b^T
\end{bmatrix}
\begin{bmatrix}
\phi_i^w(\xi) \\
\phi_j^w(\xi)
\end{bmatrix}
\begin{bmatrix}
\mathbf{\Theta}_y^T \\
\mathbf{\Theta}_y^T
\end{bmatrix}
\begin{bmatrix}
\mathbf{W}_T^T \\
\mathbf{W}_T^T
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix},
$$

(30)

where the nonzero dimensionless mass and stiffness submatrices as well as the dimensionless nodal parameter vectors are as:

$$
\begin{align}
\mathbf{M}_b^T &= \lambda^{-2} \int_0^1 \left( \mathbf{P}_b(\xi) \mathbf{I}_b(\xi) + \mathbf{X}^T(\xi) \right) \phi_i^w(\xi) \phi_j^w(\xi) d\xi, \\
\mathbf{K}_b^T &= \lambda^{-2} \int_0^1 \left( \mathbf{K}_b(\xi) \mathbf{I}_b(\xi) + \mathbf{X}^T(\xi) \right) \phi_i^w(\xi) \phi_j^w(\xi) d\xi, \\
\mathbf{\Theta}_y^T &= \begin{bmatrix}
\Theta_{y1}^T \\
\Theta_{y2}^T \\
\ldots \\
\Theta_{ynp}^T
\end{bmatrix},
\end{align}

(31a, 31b, 31c, 31d, 31e, 31f)

For simply supported AFGNNBs modeled on the basis of the TBT, the following boundary conditions should be imposed:

$$
\mathbf{W}_T^T(0, \tau) = \mathbf{W}_T^T(1, \tau) = 0; \quad \mathbf{M}_{b_T}(0, \tau) = \mathbf{M}_{b_T}(1, \tau) = 0,
$$

(32)

where $\mathbf{M}_{b_T} = \frac{M_{b_T}}{\kappa_{\text{mort}} L_{b_T}^2}$. To enforce the essential conditions in Eq. (32), the corrected collocation method by Wagner and Liu (2000) is exploited. Through taking a harmonic form for the time-dependent vectors, and by solving the resulting set of eigenvalue equations for the dimensionless natural frequencies, $\sigma^T$, the frequencies of the nanostructure are computed by:

$$
\omega^T = \frac{\sigma^T}{\eta} \sqrt{\frac{\kappa_{\text{mort}}}{\rho_{\text{mort}}}}.
$$

6. Basic formulations via surface energy-based HOBT

6.1. Strong form of the equations of motion via HOBT

According to the HOBT (Bickford, 1982; Reddy, 2014; Reddy & Phan, 1985), the longitudinal and transverse displacement fields of the continuum-based nanobeam ($u^x_b$ and $u^y_b$) are stated in terms of angle of deflection ($\psi^H$) and deflection ($w^H$) of the neutral axis of the nanobeam as: $u^x_b = -[(z - \alpha(x)z^2)\psi^H + \alpha(x)z^3 \frac{\partial \psi^H}{\partial x}]$ and $u^y_b = w^H(x, t)$ where $\alpha(x) = -\frac{4}{\lambda_{\text{mort}}(x)}$. In the context of small deformation, the only strain fields of the bulk are expressed by: $\epsilon^H_{xx} = -(z - \alpha(x)z^2)\frac{\partial \psi^H}{\partial x} + \alpha(x)z^3 \frac{\partial^2 \psi^H}{\partial x^2}$ and $\gamma^H_{zz} = (1 - 3\alpha(x)z^2)(\frac{\partial \psi^H}{\partial x} - \psi^H)$.

By employing the surface elasticity theory of Gurtin and Murdoch (1975; 1976; 1978), the nonzero stresses of the surface layer of the higher-order beam take the following form:
\[ \tau_{xx}^H = \tau_0(x) - (\lambda_0(x) + 2\mu_0(x)) \left[ (z - \alpha z^2) \frac{\partial \psi_y^H}{\partial x} + \alpha^2 z^3 \frac{\partial^2 W_i^H}{\partial x^2} \right], \]
\[ \tau_{xz}^H = n_z \tau_0(x) \frac{\partial W_i^H}{\partial x}. \]

By assuming linear variation of the bulk’s normal stress along the z direction between those of the bottom and top axes of the surface layer, the longitudinal and shear stresses within the bulk are stated by:
\[ \sigma_{xx}^H = -E_b(x) \left[ (z - \alpha(x)z^2) \frac{\partial \psi_y^H}{\partial x} + \alpha(x)z^3 \frac{\partial^2 W_i^H}{\partial x^2} \right] + \frac{2v_b(x)}{D_0(x)} \left[ z \left( \tau_0(x) \frac{\partial^2 W_i^H}{\partial x^2} - \rho_0(x) \frac{\partial^2 W_i^H}{\partial t^2} \right) \right], \]
\[ \sigma_{xz}^H = (1 - 3\alpha(x)z^2)G_b(x) \left( \frac{\partial W_i^H}{\partial x} - \psi_y^H \right). \]

By considering the surface energy, the kinetic energy, \( T^H \), and the elastic strain energy of the AFGNNB, \( U^H \), based on the HOBT are given by:
\[ T^H = \frac{1}{2} \int_0^l \left( (l_0 + l_0') \left( \frac{\partial^2 \psi_y^H}{\partial t \partial x} \right)^2 + (l_2 + l_2') \left( \frac{\partial \psi_y^H}{\partial t} \right)^2 + \alpha^2 (l_0 + l_0') \right) \left( \frac{\partial \psi_y^H}{\partial t} + \frac{\partial^2 W_i^H}{\partial t \partial x} \right) \right) \] 
\[ + \frac{1}{2} \int_0^l \left( K_1 \left( \psi_y^H \right)^2 + K_0 \left( \psi_y^H \right)^2 \right) \right) \right) \] 
\[ + \frac{1}{2} \int_0^l \int_{S(x)} \left( \tau_{xx}^H \psi_y^H + \tau_{xy}^H \gamma_y^H \right) dS dx, \]
\[ U^H = \frac{1}{2} \int_0^l \left( \frac{\partial \psi_y^H}{\partial x} \right)^2 + \left( \frac{\partial \psi_y^H}{\partial x} \right)^2 \right) + \frac{1}{2} \int_0^l \int_{S(x)} \left( \tau_{xx}^H \psi_y^H + \tau_{xy}^H \gamma_y^H \right) dS dx, \]

where
\[ M_{b_0}^H = \int_{A_b(x)} z^3 \sigma_{xx}^H dA = - \left[ (l_2(x) - \alpha(x)l_4(x)) \frac{\partial \psi_y^H}{\partial x} + \alpha(x)l_4(x) \frac{\partial^2 W_i^H}{\partial x^2} \right] \]
\[ + \frac{2v_b(x)(l_0' + l_0)}{D_0(x)} \left( \tau_0(x) \frac{\partial^2 W_i^H}{\partial x^2} - \rho_0(x) \frac{\partial^2 W_i^H}{\partial t^2} \right), \]
\[ P_{b_0}^H = \int_{A_b(x)} z^3 \sigma_{xz}^H dA = - \left[ (l_4(x) - \alpha(x)l_6(x)) \frac{\partial \psi_y^H}{\partial x} + \alpha(x)l_6(x) \frac{\partial^2 W_i^H}{\partial x^2} \right] \]
\[ + \frac{2v_b(x)(l_0' + l_0)}{D_0(x)} \left( \tau_0(x) \frac{\partial^2 W_i^H}{\partial x^2} - \rho_0(x) \frac{\partial^2 W_i^H}{\partial t^2} \right), \]
\[ Q_{b_0}^H = \kappa(x) \left( \frac{\partial \psi_y^H}{\partial x} - \psi_y^H \right). \]

and
\[ \kappa(x) = \int_{A_b(x)} E_b(x) \left( 1 - 3\alpha(x)z^2 \right) dA, \]
\[ l_m^{m'}(x) = \int_{A_b(x)} z^m dA, l_0^m(x) = \int_{S(x)} z^m dS; \]
\[ J_n(x) = \int_{A_b(x)} E_b(x) z^n dA, \]
\[ l_m^0(x) = \int_{A_b(x)} \rho_0(x) z^n dA, l_0^0(x) = \int_{S(x)} \rho_0(x) z^n dS; \]

Using Hamilton’s principle, the equations of motion of the thermally affected AFGNNB on the basis of the HOBT are obtained as:
\[ \left( l_2 + l_2' \right) - 2\alpha (l_4 + l_4') + \alpha^2 (l_6 + l_6') \frac{\partial^2 \psi_y^H}{\partial t^2} + \alpha^2 (l_0 + l_0') \frac{\partial^2 W_i^H}{\partial t^2} \frac{\partial^2 \psi_y^H}{\partial t \partial x} \]
\[- \frac{\partial M_{b_0}^H}{\partial x} + \alpha \frac{\partial P_{b_0}^H}{\partial x} + Q_{b_0}^H - \int_S z \frac{\tau_{xx}^H}{\partial x} dS + K_1 \psi_y^H = 0, \]
\[ (l_0 + l_0') \frac{\partial^2 W_i^H}{\partial t^2} + \alpha (l_4 - \alpha^2 l_6) \frac{\partial^2 W_i^H}{\partial t^2} \frac{\partial^2 \psi_y^H}{\partial t \partial x} - \frac{\partial Q_{b_0}^H}{\partial x} - \alpha \frac{\partial^2 P_{b_0}^H}{\partial x^2} - \int_S z \frac{\tau_{xx}^H}{\partial x} n_z dS - \alpha (l_4 - \alpha^2 l_6) \frac{\partial^2 W_i^H}{\partial t^2} = 0. \]
by substituting Eqs. (33), (36 a–c), into Eq. (38a and b), the governing equations of the AFGNNB subjected to the longitudinal temperature gradient in terms of deformation fields of the higher-order beam are derived as follows:

\[
\left( I_2(x) + I_2^*(x) - 2\alpha(x)(I_4(x) + I_4^*(x)) + \alpha^2(x)(I_6(x) + I_6^*(x)) \right) \frac{\partial^2 \psi^H_y}{\partial t^2} - \frac{\partial}{\partial x} \left[ \left( \alpha^2(x)(I_6(x) + I_6^*(x)) - \alpha(x)(I_4(x) + I_4^*(x)) + \frac{2\nu_b(x)\rho_0(x)}{D_0(x)}(I_2(x) - \alpha(x)I_4(x)) \right) \frac{\partial^2 W^H_t}{\partial x^2} \right] \\
- \frac{\partial}{\partial x} \left[ \left( \alpha I_4(x) - \alpha^2(x)I_6(x) + (\lambda_0(x) + 2\mu_0(x))\alpha(x)I_4^*(x) \right) - \frac{2\nu_b(x)\tau_0(x)}{D_0(x)}(I_2(x) - \alpha(x)I_4(x) + \alpha^2(x)I_6(x)) \right] \frac{\partial^2 W^H_t}{\partial x^2} - \frac{\partial}{\partial x} \left[ \frac{\kappa(x)}{K_r} \left( \frac{\partial W^H_t}{\partial x} - \psi^H_y \right) \right] + K_r \psi^H_y = 0, \tag{39a}
\]

\[
(I_0(x) + I_0^*(x)) \frac{\partial^2 W^H_t}{\partial t^2} + \frac{\partial}{\partial x} \left[ \frac{\left( 2\alpha(x)\nu_b(x)\rho_0(x)I_4(x) \right)}{D_0(x)} \frac{\partial^2 W^H_t}{\partial x^2} \right] \\
+ \frac{\partial}{\partial x} \left[ \left( \alpha I_4(x) - \alpha^2(x)I_6(x) \right) \frac{\partial^2 \psi^H_y}{\partial x^2} \right] - \frac{\partial}{\partial x} \left[ \frac{\kappa(x)}{K_r} \left( \frac{\partial W^H_t}{\partial x} - \psi^H_y \right) \right] \\
- \frac{\partial}{\partial x} \left[ \left( \tau_0(x)\gamma_0^2(x) + N_f \right) \frac{\partial W^H_t}{\partial x} \right] + \frac{\partial}{\partial x} \left[ \frac{\left( \alpha(x)I_4(x) - \alpha^2(x)I_6(x) \right)}{D_0(x)} \frac{\partial \psi^H_y}{\partial x} \right] + (\lambda_0(x) + 2\mu_0(x))(I_2(x) - \alpha(x)I_4^*(x)) \frac{\partial^2 \psi^H_y}{\partial x^2} \right] + K_r \psi^H_y = 0. \tag{39b}
\]

To investigate the problem in a more general context, we consider the following dimensionless quantities:

\[
\begin{align*}
\psi^H_t &= \frac{W^H_t}{I_b} \psi^H_y, \quad T = \frac{\alpha t}{I_b} \sqrt{\frac{I_0^*}{I_b}}, \quad \gamma^2_1(\xi) = \frac{\alpha(x)I_4(x) - \alpha^2(x)I_6(x)}{I_0.I_0^*}, \\
\gamma^2_2(\xi) &= \frac{\alpha^2(x)I_6(x)}{I_0.I_0^*}, \quad \gamma^2_3(\xi) = \frac{\kappa(x)I_0^*}{\alpha I_6^*}, \quad \gamma^2_4(\xi) = \frac{\alpha(x)I_4(x) - \alpha^2(x)I_6(x)}{\alpha I_6^*}, \\
\gamma^2_5(\xi) &= \frac{\alpha(x)I_4(x) - \alpha^2(x)I_6(x)}{I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*}, \quad \gamma^2_6(\xi) = \frac{\kappa(x)I_0^*}{\alpha I_6^*}, \\
K^H_r &= \frac{K_r I_6^*}{\alpha^2 I_6.I_6^*}, \quad \gamma^2_7(\xi) = \frac{\alpha(x)I_4(x) - \alpha^2(x)I_6(x)}{I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*}, \\
\chi^H_1(\xi) &= \frac{I_2^*}{I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*}, \quad \chi^H_2(\xi) = \frac{\alpha(x)I_4^*(x) - \alpha^2(x)I_6^*(x)}{I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*}, \\
\chi^H_3(\xi) &= \frac{\left( \lambda_0(x) + 2\mu_0(x) \right) - \frac{2\nu_b(x)\rho_0(x)}{D_0(x)}(I_2(x) - \alpha(x)I_4(x))}{(I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*)}, \\
\chi^H_4(\xi) &= \frac{\left( \lambda_0(x) + 2\mu_0(x) \right)(I_2(x) - \alpha(x)I_4(x))}{(I_2.I_2^* - 2\alpha I_4.I_4^* + \alpha^2 I_6.I_6^*)}, \\
\chi^H_5(\xi) &= \frac{\rho_0(x)\gamma_0^2(x) + \frac{2\alpha(x)\nu_b(x)\rho_0(x)I_4(x)}{D_0(x)}}{I_0.I_0^*}, \quad \chi^H_6(\xi) = \frac{\tau_0(x)\gamma_0^2(x)}{\alpha I_6.I_6^*}, \quad \chi^H_7(\xi) = \frac{2
u_b(x)I_4(x)\tau_0(x)}{\alpha I_6.I_6^* D_0(x)}, \\
\chi^H_8(\xi) &= \frac{\rho_0(x)\gamma_0^2(x)}{I_0.I_0^*}, \quad \chi^H_9(\xi) = \frac{\alpha^2(x)I_6(x)}{\alpha I_6.I_6^*},
\end{align*}
\]

by introducing Eq. (40) to Eq. (39a and b), the dimensionless governing equations that display transverse vibration of the thermally affected AFGNNB based on the HOBT take the following form:
\begin{align*}
\left( \varphi^H_1(\xi) + \chi^H_1(\xi) \right) & \frac{\partial^2 \psi^H_1}{\partial \tau^2} + \frac{\partial}{\partial \xi} \left[ \left( \gamma_6^2(\xi) + \chi_2^H(\xi) \right) \frac{\partial^2 \psi^H}{\partial \xi \partial \tau} \right] - \frac{\partial}{\partial \xi} \left[ \left( \gamma_5^2(\xi) + \chi_3^H(\xi) \right) \frac{\partial^2 \psi^H}{\partial \xi^2} \right] = 0.
\intertext{\hspace{0.5cm}}
- \frac{\partial}{\partial \xi} \left[ \left( \gamma_5^2(\xi) + \chi_3^H(\xi) \right) \frac{\partial^2 \psi^H}{\partial \xi \partial \tau} \right] - \frac{\partial}{\partial \xi} \left[ \left( \chi_6^H(\xi) + \chi_4^H(\xi) \right) \frac{\partial^2 \psi^H}{\partial \xi^2} \right] + \mathcal{K}_c \psi^H = 0. \tag{41a}
\end{align*}

\begin{align*}
\left( \varphi^H_2(\xi) + \chi^H_2(\xi) \right) & \frac{\partial^2 \psi^H}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} \left[ \gamma_2^2(\xi) \frac{\partial^2 \psi^H}{\partial \xi \partial \tau} \right] + \frac{\partial}{\partial \xi} \left[ \gamma_3^2(\xi) \frac{\partial^2 \psi^H}{\partial \xi^2} \right] = 0.
\intertext{\hspace{0.5cm}}
- \frac{\partial}{\partial \xi} \left[ \gamma_2^2(\xi) \frac{\partial^2 \psi^H}{\partial \xi \partial \tau} \right] & - \frac{\partial}{\partial \xi} \left[ \left( \chi_6^H(\xi) + \mathcal{N}_1 \right) \frac{\partial \psi^H}{\partial \xi} \right] + \mathcal{K}_c \frac{\partial \psi^H}{\partial \xi} = 0. \tag{41b}
\end{align*}

When the dimensions of the nanostructure increase, the surface parameters would decrease until they become vanishingly small (i.e., $\gamma_1^2(\xi) = \gamma_2^H(\xi) = \chi_1^H(\xi) = \chi_2^H(\xi) = \chi_3^H(\xi) = \chi_4^H(\xi) = \chi_5^H(\xi) \approx 0$). In such a case, Eq. (41a and b) would be reduced to those of the axially-functionally graded beam under a longitudinal thermal field which is modeled based on the HOBT.

6.2. Weak form of the equations of motion based on the HOBT via RKPM

The deflection and angle of deflection of the neutral axis of the nanobeam modeled based on the HOBT are discretized as:

$$\mathbf{\bar{w}}^H(\xi, \tau) = \sum_{i=1}^{NP} \phi^H_i(\xi) \mathbf{\bar{w}}^H(\tau), \quad \mathbf{\bar{\psi}}^H(\xi, \tau) = \sum_{i=1}^{NP} \phi^H_i(\xi) \mathbf{\bar{\psi}}^H(\tau). \tag{42}$$

For weakening the governing equations, Eq. (41a and b) are premultiplied by $\delta \mathbf{\bar{\psi}}^H$ and $\delta \mathbf{\bar{w}}^H$, respectively, and then, the resulted expressions are integrated over the dimensionless special domain. By taking the required integration by parts in view of Eq. (42), the following set of second-order ordinary differential equations is derived:

$$\begin{bmatrix}
\mathbf{M}_{\psi\psi}^H & \mathbf{M}_{\psi\omega}^H & \mathbf{M}_{\psi\omega}^H \\
\mathbf{M}_{\psi\omega}^H & \mathbf{M}_{\omega\omega}^H & \mathbf{M}_{\omega\omega}^H \\
\mathbf{M}_{\psi\omega}^H & \mathbf{M}_{\omega\omega}^H & \mathbf{M}_{\omega\omega}^H
\end{bmatrix}
\begin{bmatrix}
\psi \\
\omega \\
\bar{w}
\end{bmatrix}
\begin{bmatrix}
\frac{d^2 \psi^H}{d \tau^2} \\
\frac{d^2 \omega^H}{d \tau^2} \\
\frac{d^2 \bar{w}^H}{d \tau^2}
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{K}_{\psi\psi}^H & \mathbf{K}_{\psi\omega}^H & \mathbf{K}_{\psi\omega}^H \\
\mathbf{K}_{\psi\omega}^H & \mathbf{K}_{\omega\omega}^H & \mathbf{K}_{\omega\omega}^H \\
\mathbf{K}_{\psi\omega}^H & \mathbf{K}_{\omega\omega}^H & \mathbf{K}_{\omega\omega}^H
\end{bmatrix}
\begin{bmatrix}
\psi \\
\omega \\
\bar{w}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}. \tag{43}$$

where the nonzero dimensionless mass and stiffness submatrices as well as the dimensionless time-dependent nodal parameter vectors are provided by:

$$\begin{align*}
\mathbf{M}_{\psi\psi}^H & = \int_0^1 \left( \varphi^H_1(\xi) + \chi^H_1(\xi) \right) \phi^H_i \phi^H_j d\xi, \tag{44a} \\
\mathbf{M}_{\psi\omega}^H & = \int_0^1 \phi_i d\xi \left[ \left( \gamma_6^2(\xi) + \chi_2^H(\xi) \right) \phi^H_j \right] d\xi, \tag{44b} \\
\mathbf{M}_{\omega\omega}^H & = - \int_0^1 \gamma_2^2(\xi) \frac{d \phi^H_i}{d \xi} \phi^H_j d\xi, \tag{44c} \\
\mathbf{K}_{\psi\psi}^H & = \int_0^1 \left( \varphi^H_2(\xi) + \chi^H_2(\xi) \right) \phi^H_i \phi^H_j + \gamma_2^2(\xi) \frac{d \phi^H_i}{d \xi} \frac{d \phi^H_j}{d \xi} d\xi, \tag{44d} \\
\mathbf{K}_{\psi\omega}^H & = \int_0^1 \left( \gamma_2^2(\xi) \phi^H_i \phi^H_j + \left( \gamma_6^2(\xi) + \chi_2^H(\xi) \right) \frac{d \phi^H_i}{d \xi} \frac{d \phi^H_j}{d \xi} + \mathcal{K}_c \phi^H_i \phi^H_j \right) d\xi, \tag{44e} \\
\mathbf{K}_{\omega\omega}^H & = - \int_0^1 \left( \gamma_2^2(\xi) \phi^H_i \phi^H_j \frac{d \phi^H_j}{d \xi} + \left( \gamma_6^2(\xi) + \chi_2^H(\xi) \right) \frac{d \phi^H_i}{d \xi} d\xi \right) d\xi, \tag{44f} \\
\mathbf{K}_{\psi\omega}^H & = - \int_0^1 \left( \gamma_2^2(\xi) \phi^H_i \phi^H_j + \gamma_2^2(\xi) \phi^H_j \frac{d \phi^H_j}{d \xi} - \gamma_2^2(\xi) \frac{d \phi^H_i}{d \xi} \frac{d \phi^H_j}{d \xi} \right) d\xi. \tag{44g}
\end{align*}
\[ \left[ K^H \right]_{ij}^{vw} = \int_0^1 \left( \left( \gamma^H_i(\xi) + \chi^H_i(\xi) + \overline{N^H_i} \right) \left( \frac{d\phi^w_j}{dx} \frac{d\phi^w_j}{dx} + \frac{d\phi^w_i}{dx} \frac{d\phi^w_i}{dx} \right) + \left( \frac{d^2\phi^w_i}{dx^2} \frac{d^2\phi^w_j}{dx^2} + \overline{K^H_i} \phi^w_i \phi^w_j \right) \right) d\xi, \]  
\[ (44h) \]
\[ \Psi_y^H = \left< \overline{\Psi}_y^H, \overline{\Psi}_y^H, \ldots, \overline{\Psi}_{y_{2p}}^H \right>^T. \]  
\[ (44i) \]
\[ \mathbf{W}^H = \left< \overline{\mathbf{W}}_0^H, \overline{\mathbf{W}}_1^H, \ldots, \overline{\mathbf{W}}_{N_{NP}}^H \right>^T. \]  
\[ (44j) \]

In the case of simply supported AFGNNBs based on the HOBT, the boundary conditions read:
\[ \mathbf{W}^H(0, \tau) = \mathbf{W}^H(1, \tau) = 0; \quad \mathbf{W}_{b_i}^H(0, \tau) = \mathbf{M}_{b_i}^H(1, \tau) = 0, \]
where \( \mathbf{M}_{b_i}^H = \frac{M_{b_i}}{a_{R_i}^L}. \) To impose the essential boundary conditions in Eq. (45), the corrected collocation method (Wagner & Liu, 2000) is employed. By taking a harmonic form for the time-dependent vectors, and by solving the resulting set of linear eigenvalue equations for the dimensionless natural frequencies, \( \omega_i^H \), the frequencies of the thermally affected nanostructure based on the HOBT are provided by:
\[ \omega_i^H = \frac{\alpha \omega_i^H}{\xi} \sqrt{\frac{K_i}{K_{i1}}}. \]

7. Results and discussion

Consider an AFGNNB whose material properties would generally vary across its length between those of Al[111] at the left end and those of the Si[100] at the right end. The material properties of these two constituents are as follows: \( E_b, \tau = 70 \) GPa, \( v_b, \tau = 0.3, \rho_b, \tau = 2700 \) kg/m\(^3\), \( \mu_b, \tau = -5.4251 \) N/m, \( \gamma_0, \tau = 3.4939 \) N/m, \( \tau_0, \tau = 0.5689 \) N/m, \( \rho_0, \tau = 5.4610 \) kg/m\(^2\), \( \alpha_{R, \tau} = 23.1 \times 10^{-8} \) C\(^-1\)K\(^1\), \( k_{L, \tau} = 209 \) W/(m\(^3\)K), \( E_b, R = 210 \) GPa, \( v_b, R = 0.24, \rho_b, R = 2370 \) kg/m\(^3\), \( \mu_0, R = -0.6543 \) N/m, \( \gamma_0, R = -10 \) N/m, \( \tau_0, R = 0.6048 \) N/m, \( \rho_0, R = 3.1710^{-7} \) kg/m\(^3\), \( \alpha_{R, R} = 2.6 \times 10^{-8} \) C\(^-1\)K\(^1\), and \( k_{L, R} = 280 \) W/(m\(^3\)K). The radius of the left and right hand sides cross-sections of the nanobeam in order are 4 and 8 nm, except other values have been specified for these parameters. Additionally, in the lack of data on the thermal expansion coefficient of the surface layer, its value is assumed to be identical to that of its nearby bulk. In constructing the shape functions of particles of RKPM, cubic spline window function and linear base function are used and the dilation parameter is set equal to 3.2 times of the interparticle distance. The RKPM analysis is performed based on 11 particles with uniform distribution across the nanobeam’s length and 6 Gauss points are used in each integration cell for calculations of RKPM’s shape functions. The details of constructing the RKPM’s shape functions and their derivatives are given in Kiani (2014c).

In the following parts, a comparison study is given and thereafter, the roles of important characteristics of the nanostructure on the natural frequencies associated with the transverse vibration are presented.

7.1. A validation study

In order to ensure about the calculations of the RKPM, a comparison study is conducted. To this end, we are interested in verifying the predicted results by the RKPM with those of the assumed mode method (AMM). In AMM analysis of the problem based on the Galerkin approach, the mode shapes that satisfy the geometry boundary conditions are substituted into the RKPM’s shape functions in the formulations of the dimensionless stiffness submatrices. For example, in the case of simply supported AFGNNB, the following mode shapes are taken into account:
\[ \phi^w_i(\xi) = \sin(i\pi \xi), \text{ for RBT}; \]
\[ \phi^w_i(\xi) = \sin(i\pi \xi), \phi^b_i(\xi) = \cos(i\pi \xi), \text{ for TBT}; \]
\[ \phi^w_i(\xi) = \sin(i\pi \xi), \phi^b_i(\xi) = \cos(i\pi \xi), \text{ for HOBT}. \]  
\[ (46) \]

After evaluating the dimensionless submatrices analytically, the natural frequencies of the thermally affected nanostructure are determined by solving the resulted eigenvalue problems for the proposed surface energy-based models. For AMM analysis, the first eleven mode shapes of the nanostructure have been considered. The predicted first five natural frequencies of the thermally affected nanostructure by the AMM and those of the RKPM according to the RBT, TBT, and HOBT are provided in Table 1. The results are given for five levels of the slenderness ratio (i.e., \( \lambda = 10, 20, 30, 40, \) and \( 50 \)) and two levels of the temperature gradient (i.e., \( \Delta T = 50 \) and \( 150 \) °C) in the case of \( \rho_b = 3, \rho_0 = 1, \) and \( K_0 = K_0 = 0. \) As it is obvious from Table 1, there is a reasonably good agreement between the RKPM’s results and those predicted by the AMM. Generally, the predicted results by the TBT are close to those of the HOBT since both these models take into account shear deformation effect in their formulations. By an increase of the slenderness ratio, the discrepancies between the results of various models would decrease because the share of shear strain energy in the total energy of the nanostructure would reduce. The predicted results by various suggested models in Table 1 also display that the temperature gradient is more influential on the natural frequencies of lower modes of more slender nanobeams.
### Table 1

Verification of the predicted first five natural frequencies of the nanobeam by the RKPM and those of the AMM based on the surface energy-based beam models.

<table>
<thead>
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<th>i</th>
<th>$\Delta T=50^\circ C$</th>
<th></th>
<th>$\Delta T=150^\circ C$</th>
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<td>TBT</td>
<td>HOBT</td>
<td>RBT</td>
</tr>
<tr>
<td></td>
<td>RKPM</td>
<td>AMM</td>
<td>RKPM</td>
<td>AMM</td>
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<td>15.828</td>
<td>15.828</td>
</tr>
</tbody>
</table>

### 7.2. Parametric explorations

In this part, the influences of the slenderness ratio, temperature gradient, diameter of the nanobeam, and variations of both material and geometry properties on the natural frequencies of simply supported nanobeams are investigated. For each numerical study, the results are provided based on the suggested models, namely the RBT, TBT, and HOBT. The plotted results on the basis of the classical elasticity theory (CET) are presented by the dashed lines whereas those of the surface elasticity theory (SET) are demonstrated by the solid lines. By comparing the obtained results, the effects of the surface energy as well as the shear deformation on the predicted frequencies are displayed and discussed.

### 7.2.1. Natural frequencies vs. slenderness ratio

An important parametric study is conducted to investigate the role of the slenderness ratio on the free dynamic analysis of AFGNBs. For three levels of the temperature gradient, the plots of the fundamental frequency of the nanobeam are given in Fig. 2(a–c). As it is seen, all suggested models predict that the fundamental frequency would decrease as the slenderness ratio increases. Additionally, the rate of reduction is more obvious for higher levels of the temperature gradients. Generally, the predicted fundamental frequencies by the RBT are greater than those obtained by the TBT and the HOBT, and the results of the HOBT are commonly overestimated by those of the RBT and the TBT. Since shear deformation effect is incorporated into the equations of both RBT and HOBT, the predicted results by the TBT are usually closer to those of the HOBT. In the case of $\Delta T=100^\circ C$, by an increase of the slenderness ratio, the relative discrepancies between the obtained results by the proposed models would reduce. In such a case, the TBT can capture the results of the HOBT with relative error lower than 2.5% for the general range of the slenderness ratio. However, for the AFGNBs with slenderness ratios 10, 20, and 30, the relative discrepancies between the predicted fundamental frequencies by the RBT and those of the HOBT are about 28, 10, and 5.3%, respectively. In the cases of $\Delta T=700$ and 1300 $^\circ C$, the relative discrepancies between the results of the RBT or TBT and those of the HOBT would lessen by increasing of the slenderness ratio up to a certain level. For the slenderness ratios greater than this value, the above-mentioned discrepancies would commonly magnify with the slenderness ratio. For instance, in the case of the $\Delta T=1300^\circ C$, the maximum discrepancy between the results of the RBT or the TBT and those of the HOBT is observed at the slenderness ratio corresponds to the zero frequency, which is called critical slenderness ratio. In fact, this critical value denotes the minimum slenderness ratio that the nanobeam would be dynamically unstable in the context of elastic deformation. Therefore, identification of the critical slenderness ratio is a crucial stage in analysis and design of thermally affected nanobeams. A brief comparison of the plotted results in Fig. 2(b)
and those of Fig. 2(c) reveals that the critical slenderness ratio would reduce by increasing of the temperature gradient at the nanobeam’s ends.

Concerning the role of the surface energy effect on the fundamental frequencies, the demonstrated results in Fig. 2(a–c) show that the predicted results by the suggested models based on the CET are somewhat greater than those obtained by the models accounting for the surface effect. The main reason of this fact is the negative incorporation of the surface energy effect into the lateral stiffness of the nanostructure. This issue is mainly affected by the distribution of Young’s modulus of the surface layer (i.e., $E_0(x) = \lambda_0(x) + 2\mu_0(x)$) across the nanobeam’s length. In the case of $\Delta T = 100^\circ C$, the effect of the surface energy on the obtained fundamental frequencies would reduce as the slenderness ratio increases. However, in the cases of $\Delta T = 700$ and $1300^\circ C$, a close survey of the obtained results displays that the relative discrepancies between the predicted fundamental frequencies based on the CET and those of the SET would magnify as the slenderness ratio grows. In the case of $\Delta T = 1300^\circ C$, the maximum surface energy effect is observed at the critical slenderness ratio for each model.

Fig. 3 (a–c) displays variations of the first natural frequencies of the AFGNNB based on the HOBT as a function of the slenderness ratio for three levels of the temperature gradients (i.e., $\Delta T = 100$, 700, and 1300 $^\circ C$). Generally, variation of the slenderness ratio is more influential on the variation of higher frequencies. In the case of $\Delta T = 100^\circ C$ and low levels of the slenderness ratios (i.e., $10 < \lambda < 15$), the maximum relative discrepancies between the CET and the SET are observed for the frequency of the fifth mode of vibration. However, for $\lambda > 20$, the influence of the surface energy on the lower frequencies are more apparent. In the cases of $\Delta T = 700$ and $1300^\circ C$, the relative discrepancies between the fundamental frequencies based on the SET and those obtained based on the CET are more obvious with respect to other natural frequencies. This fact is more apparent for higher slenderness ratios, and as the temperature gradient increases, the influence of the surface energy on the fundamental frequency of the nanostructure increases.

7.2.2. Natural frequencies vs. temperature

The influence of the temperature gradient on the free transverse vibration of the functionally graded beam-like nanostructure is the subject of another interesting study. In Fig. 4(a–c), the predicted fundamental frequencies by the RBT, TBT, and HOBT based on the CET and the SET in terms of the temperature gradient are plotted for three levels of the slenderness ratio (i.e., $\lambda = 10$, 20, and 40). All the suggested models predict that the fundamental frequency of the longitudinally constrained nanobeam would decrease as the temperature gradient increases. Such a fact is more obvious for more slender nanostructures. In the case of $\lambda = 10$ (see Fig. 4(a)), the RBT(TBT) overestimates the results of the HOBT with relative error lower than 35(3.5)% for the considered range of the temperature gradient. A detailed scrutiny of the plotted results in Fig. 4(a–c) indicates that the relative discrepancies between various models would generally increase as the temperature gradient grows. This issue is also more apparent for nanobeams with higher slenderness ratios. Additionally, in more slender
Fig. 3. Variation of the first five natural frequencies based on the HOBT in terms of the slenderness ratio for different levels of the temperature gradients: (a) $\Delta T=100$, (b) $\Delta T=700$, (c) $\Delta T=1300$ °C; $(p_m=p_s=2$, $p_g=1$, $K_t=K_r=0$; (--) without surface effect, (—) with surface effect; ([□] $\omega_1$, (◦) $\omega_2$, (△) $\omega_3$, (∇) $\omega_4$, (♦) $\omega_5$).

Fig. 4. Variation of the fundamental frequency in terms of the temperature gradient for different levels of the slenderness ratio: (a) $\lambda=10$, (b) $\lambda=20$, (c) $\lambda=40$; $(p_m=p_s=2$, $p_g=1$, $K_t=K_r=0$; (--) without surface effect, (—) with surface effect; ([□] RBT, (◦) TBT, (△) HOBT).
nanobeams, variation of the temperature gradient has a more influence on the variation of the above-mentioned discrepancies. In the case of $\lambda=40$ (see Fig. 4(c)), for special values of the temperature gradient, the fundamental frequency of the nanobeams becomes zero and the functionally graded nanobeam becomes unstable from the dynamics point of view. The temperature gradient associated with zero fundamental frequency is called critical temperature gradient. For the latter case, the relative discrepancies between various models reach to their maximum values at the critical temperature gradients.

In the cases of $\lambda=10$ and 20, the predicted results based on the SET are commonly underestimated by those of the CET. Furthermore, by increasing of the temperature gradient, the role of the surface effect on the obtained fundamental frequencies becomes more obvious. In the case of $\lambda=40$, the predicted fundamental frequencies based on the SET are generally underestimated by those of the CET. At the critical temperature gradients, the discrepancies between the obtained results based on the SET and those of the CET reach to their maximum levels.

In Fig. 5(a–c), the first five fundamental frequencies based on the HOBT as a function of the temperature gradient are demonstrated for $\lambda=10$, 20, and 40. By increasing the temperature gradient, all the predicted natural frequencies would reduce. The rate of reduction of the fundamental frequencies is more obvious with respect to other frequencies. In the case of $\lambda=10$, the relative discrepancies between the predicted fifth frequency based on the SET and that of the CET is more apparent with respect to other frequencies, irrespective of the considered temperature gradient. In the case of $\lambda=20$ and $\Delta T<320^\circ C$, the maximum above-mentioned discrepancies are related to the fifth frequency; however, for $\Delta T>320^\circ C$, surface energy would have the greatest impact on the fundamental frequency of the nanostructure. In the case of $\lambda=40$, the maximum relative discrepancies between the results of the SET and those of the CET are observed for the fundamental frequencies.

7.2.3. Natural frequencies vs. nanobeam’s diameter

The variation of the fundamental frequency of the nanobeam as a function of its diameter is the subject of another investigation. In Fig. 6(a–c), the predicted fundamental frequencies based on the RBT, TBT, and HOBT in terms of the diameter of the nanobeam are plotted for three levels of the temperature gradient (i.e., $\Delta T=100, 700$, and 1300 $^\circ C$). The plotted results are shown for the case of $p_m=p_s=2$, $p_g=1$, $K_{i}=K_{f}=0$. All the suggested models predict that the fundamental frequency would reduce as the diameter of the left end decreases. Such a reduction is more obvious for the nanobeams subjected to higher temperature gradients. Commonly, both RBT and TBT overestimate the results of the HOBT. In the case of $\Delta T=100$ $^\circ C$ (see Fig. 6(a)), the relative discrepancies between the predicted fundamental frequencies by the RBT and those of the HOBT would generally magnify as the diameter of the nanobeam increases. However, the relative discrepancies between the results of the TBT and those of the HOBT would slightly alter with the diameter of the nanobeam. In the case of $\Delta T=700$ $^\circ C$ (see
2.5
2.5
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37x53
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TBT
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TBT
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have
diameter.
In the case of \( \Delta T=1300 \) °C (see Fig. 6(c)), the relative discrepancies between the results of the NRBT or the TBT and those of the HOBT would generally grow by increasing of the nanobeam’s diameter. The maximum discrepancies are observed at the diameter pertinent to the zero fundamental frequency, which is also called critical diameter.

Concerning the role of the surface energy on the free vibration of the nanostructure in the case of \( \Delta T=100 \) °C, all proposed models display that the predicted fundamental frequencies based on the SET are usually overestimated by those of the CET. Commonly, the influence of the surface energy on the fundamental frequency of the nanostructure becomes important by increasing of the nanobeam’s diameter. In the case of \( \Delta T=700 \) °C, the relative discrepancies between the fundamental frequencies based on the SET and those of the CET would decrease by increasing of the diameter up to a certain value. For diameters greater than this value, the above-mentioned relative discrepancies would commonly magnify with the diameter of the nanobeam. For the considered functionally graded nanobeam subjected to \( \Delta T=1300 \) °C, the role of the surface energy on the free vibration of the nanostructure becomes highlighted by decreasing of the diameter. Furthermore, the above-mentioned relative discrepancies reach to their maximum values at the critical diameters.

7.2.4. Natural frequencies vs. power-law index of materials

We are also interested in the role of the power-law index of material in the free dynamic response. Actually, we want to explore how variation of the material properties could influence on the free transverse vibrations of the nanostructure. For this purpose, the predicted fundamental frequencies by various models in terms of the power-law index of material have been plotted in Fig. 7(a–c) for three levels of the slenderness ratio (i.e., \( \lambda=10, 20, \) and \( 40 \)). As it is seen, the predicted fundamental frequencies decrease as the power-law index of material increases. Such a fact holds true for all considered levels of the slenderness ratio. A brief survey of the plotted results shows that the maximum relative discrepancies between the results of the RBT and those of the HOBT occur at \( p \approx 0.9 \) and 1.1 in the cases of \( \lambda \approx 10 \) and 20, respectively. These values in order are 32.2 and 11.9%. Furthermore, the maximum relative discrepancies between the predicted results of the TBT and those of the HOBT reach to their maximum values (i.e., 2.9 and 1.8% percent) at 1.04 and 1.3 for \( \lambda \approx 10 \) and 20, respectively. Further scrutiny of the demonstrated results reveals that the RBT is not a trustable model at all when analyzing of free transverse vibration of AFGNNBs with the slenderness ratio lower than 10 is of concern. In the case of \( \lambda=40 \), the relative discrepancies between the results of the RBT or the TBT and those of the HOBT would commonly magnify as the power-law index of materials increases. For \( p_m=5 \), the RBT(TBT) overestimates the predicted fundamental frequency.
of the HOBT with relative error about 12.8(2.5)%. Irrespective of the considered slenderness ratio, the relative discrepancies between the obtained results based on the SET and those of the CET would commonly grow with the power-law index of materials.

7.2.5. Natural frequencies vs. power-law index of geometry

This part explains how variation of the cross-section of the AFGNNB based on a power-law formula could influence on its free dynamic response. To this end, the plots of the predicted fundamental frequencies based on the RBT, TBT, and HOBT as a function of the power-law index of geometry are provided in Fig. 8(a–c) for three levels of the slenderness ratio (i.e., $\lambda=10$, 20, and 40). The results are given for nanobeams with $p_m=p_c=2$ under $\Delta T=300^\circ$C when they do not interact with their elastic environment (i.e., $K_t=K_r=0$). According to the plotted results, the fundamental frequency would reduce with the power-law index of geometry. In the cases of $\lambda=10$ and 20 (see Fig. 8(a) and (b)), the plotted results consist of two major parts: an obvious descending branch with steep slope and a fairly descending part with mild slope. In the case of $\lambda=40$ (see Fig. 8(c)), the predicted fundamental frequencies by various models would harshly reduce as the power-law index of geometry reduces. For special levels of power-law index of geometry, the fundamental frequencies become zero. These particular values are called critical power-law index of geometry. Generally, the RBT overestimates the fundamental frequencies of the HOBT. The maximum relative discrepancies between RBT’s results and those of the HOBT occur at $p_g \approx 0.3$ in the cases of $\lambda=10$ and 20 which are, respectively, equal to 41 and 16%. Additionally, the maximum relative discrepancies between the results of the TBT and those of the HOBT (i.e., 4.3 and 2.6%) happen at $p_g \approx 0.1$ and 0.3, respectively, for $\lambda=10$ and 20. In the case of $\lambda=40$, the relative discrepancies between the predicted fundamental frequencies by the RBT or the TBT and those of the HOBT would usually magnify as the power-law index of geometry grows. The maximum values of discrepancies are observed at the critical power-law index of geometry.

Concerning the role of the surface energy effect, the predicted results by the suggested models based on the SET are commonly overestimated by those obtained based on the CET in the cases of $\lambda=10$ and 20. However, in the cases of $\lambda=40$ and $p_g > 1$, the predicted results by the SET are underestimated by those of the CET and the discrepancies between these values reach to their maximum values at the critical power-law index of geometry. In the case of $\lambda=10$, the relative discrepancies between the predicted fundamental frequencies based on the SET and those of the CET magnify with the power-law index of geometry and reach to their maximum values at $p_g=5$. However, in the case of $\lambda=20$, the above-mentioned relative discrepancies take their peak point at $p_g \approx 1.8$. 

Fig. 7. Variation of the fundamental frequency in terms of the power-law index of the material property for different levels of the slenderness ratio: (a) $\lambda=10$, (b) $\lambda=20$, (c) $\lambda=40$; ($\Delta T=300^\circ$C, $p_g=1$, $K_t=K_r=0$; (−−) without surface effect, (−−) with surface effect; (□) RBT, (○) TBT, (△) HOBT.)
Fig. 8. Variation of the fundamental frequency in terms of the power-law index of the geometry property for different levels of the slenderness ratio: (a) \( \lambda=10 \), (b) \( \lambda=20 \), (c) \( \lambda=40 \); \( \Delta T=300 \) \( ^\circ \)C, \( p_{m}=p_{s}=2, K_{1}=K_{2}=0; \) (\( \bigcirc \)) without surface effect, (\( \bullet \)) with surface effect; (\( \bigodot \)) RBT, (\( \bigcirc \)) TBT, (\( \triangle \)) HOBT.

8. Conclusions

Free transverse vibrations of functionally graded nanobeams with varying cross-section under a thermal field were studied in the context of the surface elasticity theory of Gurtin–Murdock. The material properties of both bulk and the surface layer vary across the length of the nanobeam. The equations of motion were constructed based on the RBT, TBT, and HOBT accounting for the surface effect. Since finding an exact or even an analytical solution to the resulting relations was not an easy task, a meshless technique was proposed. The natural frequencies of the nanostructure were evaluated and the roles of the temperature gradient as well as the crucial geometry and material properties on the dominant frequencies were explained. The roles of surface energy effect as well as the shear deformation on the predicted results were examined. The capabilities of the suggested models based on the RBT and TBT in predicting the free dynamic response of the nanostructure on the basis of the HOBT were displayed and discussed in some detail. Furthermore, the critical values of the slenderness ratio, temperature gradient, nanobeam’s diameter, and power-law index of geometry were numerically evaluated. It was shown that the influences of both shear deformation and surface energy on the vibration behavior of the nanostructure reach to their maximum possible limits at these critical values.

References


