Thermo-mechanical analysis of functionally graded plate-like nanorotors: A surface elasticity model

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Abstract
Nanorotors could have many applications in nanotechnology, however, their thermo-elasto-static responses due to rotary motion as well as externally applied loads have not been examined yet. This paper deals with thermo-elastic fields of rotating functionally graded nanoscale plate-like rotors using the surface elasticity theory of Gurtin and Murdoch. By dividing the nanoplate into adequate number of annular rings, the analytical expressions of elastic fields within these rings are appropriately derived. Due to the symmetry of both thermal loading and the exerted centrifugal force, the elastic field is symmetric and each ring has only two unknown parameters. To determine these parameters, appropriate boundary conditions at the interfaces of the rings as well as the non-classical conditions at the innermost and outermost surfaces of the nanoplate are defined and enforced. In a particular case, the obtained results are checked with those of another work and a good agreement is achieved. The roles of the environment’s temperature, angular velocity, power-law index, and surface energy effect on the resulted thermo-elastic fields are carefully studied. This work can be regarded as a crucial step in nanomechanical assessing of advanced composite nanorotors.

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1. Introduction
Nanoplates offer exceptional opportunities for integrating physics-based ideas to control their properties for fabrication of new nanodevices. Because of their shapes, nanoplates generally have brilliant electrical, magnetic, thermal, and mechanical properties. Such advanced properties provide them for a wide range of applications including physical and chemical sensors [1–3], nanocoatings [4,5], nanocomposites [6,7], photovoltaics [8,9], metamaterials [10,11], microelectromechanical systems [12], nanoelectromechanical systems [13], and nanorotors [14]. From applied mechanics points of view, the later application of the nanoplates is of high interest in the present work.

Nanostructures commonly exhibit higher performance with respect to their bulk counterparts [15–17]. The inherent properties of nanostructures can be tailored by controlling their size, shape, composition, and crystallinity. Herein, the roles of the first and third factors on the thermo-elastic fields of rotating nanoplates are also of particular concern. Actually, for more efficient controlling the mechanical response of the rotating nanoplate, the properties of its constitutive materials are allowed to vary gradually across the radial direction. However, the material properties are assumed to be constant through the thickness of the nanoplate. These materials are functionally graded ones that have been widely exploited in members of macroscale rotors such as spinning disks and blades. To date, deformations of rotating blades and plates made-up of functionally graded materials have been extensively examined [18–24]. However, a brief survey of the literature reveals that the elastic fields of rotating functionally graded nanoplates (RFGNPs) in thermal environments have not been addressed yet.

When dimensions of a structure reduce, ratio of the surface layer to the volume of the bulk increases. Therefore, for an externally loaded structure, the ratio of the elastic strain energy of the surface layer to that of the bulk would grow by reduction of dimensions. The classical continuum mechanics (CCM) cannot capture such a phenomenon. To consider such an important effect in formulations of the CCM, Gurtin and Murdoch [25–27] proposed the surface elasticity theory. In the newly established theory, surface layer is appropriately added to the bulk. In fact, the surface is a thin layer with negligible thickness, which tightly adhered to the underlying bulk zone. Thereby, the displacements of the surface layer are identical to those of the bulk at its vicinity; however, the constitutive relations of the surface layer are completely different from those of the bulk. For instance, in the case of an isotropic surface layer, the stresses are related to the strains by three constants of the surface, namely residual surface stress plus to two Lamé’s constants. Until now, the surface elasticity theory of Gurtin and Murdoch has been broadly applied to many engineering and
physics-based problems of nanowires [28–38], nanotubes [39–43], and nanoplates [44–48].

Due to the rotary motion of the nanoplate, a centrifugal force would exert on each element of the nanoplate. Further, existence of temperature gradient between the innermost and outermost surfaces of the nanoplate leads to generation of thermal stresses within the nanoplate. These two driving forces correspond to the in-plane deformation of the thermally affected plate-like nanorotors. Finding an analytical solution to displacements and stresses in-plane deformation of the thermally affected plate-like nanoplates [44–48] within the nanoplate. These two driving forces would exert on each element of the nanoplate. Further, existence of temperature gradient between the innermost and outermost surfaces of the nanoplate, and nozzle due to the inter-atomic bonds is not considered. Commonly, the nonlocality effect leads to softening behavior of the thermally affected plate-like nanostructures [49–51].

3. Thermal field analysis

3.1. Governing equations

In view of the first and second assumptions plus to the axisymmetric geometry and loading conditions of the problem, the governing equation of the thermal field of the bulk reads:

\[ \frac{1}{r} \frac{d}{dr} \left( r k \frac{dT}{dr} \right) = 0; \quad r_{in} < r < r_{out}, \tag{1} \]

where \( k(r) \) is the thermal conductivity of the constitutive materials of the functionally graded nanoplate, and \( T = T(r) \) is the thermal field. The following boundary conditions are considered for the inner and outer surfaces of the rotating nanoplate:

\[ -k_{in} \frac{dT}{dr} + h_{in} (T - T_{en,in}) = 0; \quad r = r_{in}, \tag{2a} \]
\[ k_{out} \frac{dT}{dr} + h_{out} (T - T_{en,out}) = 0; \quad r = r_{out}, \tag{2b} \]

where \( k_{in} \) and \( k_{out} \) represent the thermal conductivity of the inner and outer surfaces of the nanoplate, respectively, \( h_{in} \) and \( h_{out} \) in order denote the coefficients of relative heat transfer between the inner and outer surfaces of the nanoplate and the adjacent environment, and \( d \) denotes the differential symbol. By considering a power-law relation for the coefficient of the thermal conductivity:

\[ k(r) = k_{in} + (k_{out} - k_{in}) \left( \frac{r - r_{in}}{r_{out} - r_{in}} \right)^p, \tag{3} \]

where \( p \) is the power-law index that specifies variation of the material properties of the functionally graded materials along the radial direction. Generally, finding an explicit solution to Eq. (1) in view of Eq. (3) is not an easy task. In the following, an efficient approach is developed and the thermal field within the RFGNP is determined.

Let us divide the nanoplate into \( N \) annular subdomains such that the material properties within each subdomain could be considered uniform. Under such circumstances, using Eq. (1), the governing equation of the thermal field within the \( i \)th subdomain takes the following form:

\[ r_{avg,i} \frac{dT_i}{dr} + \left( k_i + r_{avg,i} \frac{dk_i}{dr} \right) \frac{dT_i}{dr} = 0; \quad r_{i-1} \leq r \leq r_i, \tag{4} \]

where \( T_i = T_i(r) \) and \( k_i = k_i(r_{avg,i}) \) in order are the thermal field and the thermal conductivity of the \( i \)th subdomain, respectively, and \( r_{avg,i} \) represents the mean radius of the \( i \)th annular subdomain.
A solution to Eq. (4) is readily sought as follows:
\[
T_i(t) = A_i + B_i e^{-\alpha r_i t} \quad r_i - 1 \leq t \leq r_i,
\]
\[
c_i = \frac{1}{T_{i+1}} \int_{r_i}^{1} \frac{1}{k} dr \quad r \Rightarrow r_{i+1},
\]
(5)
where \(A_i\) and \(B_i\) are the unknowns that can be found by satisfying the boundary conditions of all considered subdomains.

3.2. Boundary conditions

The requirement of continuity of the thermal field as well as the heat flux at the interface of the \(i\)th and \((i+1)\)th subdomains yields:
\[
T_i(t_i) = T_{i+1}(t_i); \quad i = 1, 2, ..., N - 1,
\]
(6a)
\[
k_i(T_i(T_{i+1}) = k_{i+1}(T_{i+1}(T_{i})).
\]
(6b)
On the other hand, the boundary conditions in Eqs. (2a) and (2b) in order correspond to the interior and exterior surfaces of the first and \(N\)th subdomains. Hence, these conditions can be rewritten as follows:
\[
-k_i \frac{dT_i}{dr} = h_i(T_i - T_{in,m}) = 0; \quad r = r_{in},
\]
(7a)
\[
k_{out} \frac{dT_{N}}{dr} = h_{out}(T_N - T_{out,m}) = 0; \quad r = r_{out}.
\]
(7b)

3.3. Development of a semi-analytical approach

The conditions in Eqs. (6a), (6b), (7a), and (7b) display 2N relations. In order to determine the unknowns of the problem, namely \(A_i\) and \(B_i\); \(i = 1, 2, ..., N\), Eq. (5) is substituted into these relations. Therefore,
\[
k_i \frac{dT_i}{dr} = h_i(A_i + B_i e^{-\alpha r_i T_{in,m} - T_{in,m}}) = 0, \quad (8a)
\]
\[
k_{out} \frac{dT_{N}}{dr} = h_{out}(A_N + B_N e^{-\alpha r_{out,m} - T_{out,m}}) = 0,
\]
(8b)
\[
A_i + B_i \ln(-c_i r_i) = A_{i+1} + B_{i+1} \ln(-c_{i+1} r_i), \quad (8c)
\]
\[
B_i = B_{i+1} + c_i.
\]
(8d)
Eqs. (8a)-(8d) present a set of 2N linear equations that can be rewritten in a more compact form as follows:
\[
K \mathbf{c} = \mathbf{f},
\]
(9)
where the transpose of vector \(\mathbf{c}\), and the nonzero elements of \(K\) and \(\mathbf{f}\) are stated by:
\[
K_{11} = h_{in}, K_{12} = h_{in} \exp(-c_1 r_{in}), K_{21} = h_{in} \exp(-c_1 r_{in}),
\]
\[
K_{(2i+1)-1} = 1, \quad K_{(2i+2)i} = \exp(-c_i r_i), \quad K_{(2i+2)i} = -1, \quad K_{(2i+2)-1} = 0,
\]
\[
K_{(2i+1)-2} = \exp(-c_{i+1} r_i), \quad K_{(2i+2)(2i+2)} = 0, \quad K_{(2i+2)(2i+2)} = c_i \exp(-c_{i+1} r_i), \quad i = 1, 2, ..., N - 1,
\]
\[
K_{(2N)(2N)} = h_{out}, \quad K_{(2N)(2N)} = h_{out} \exp(-c_N r_{out}),
\]
(10a)
\[
f_1 = h_{in} T_{in,m}, \quad f_{2N} = h_{out} T_{out,m}.
\]
(10b)
By solving Eq. (9) for \(\mathbf{c}\), the unknown coefficients will be readily obtained. Subsequently, in view of Eq. (5), the thermal fields within the subdomains of the nanoplate can be determined.

4. Thermo-elastic field analysis

4.1. Governing equations

By recalling the first, second, and third assumptions, the only governing equation in terms of the stress components is expressed by:
\[
d\sigma_{rr} - \sigma_{\theta \theta} + \rho \alpha^2 = 0,
\]
(11)
where \(\alpha = r a^2\) is the radial acceleration, \(\sigma_{rr}\) and \(\sigma_{\theta \theta}\) represent the radial and tangential components of the stress field, respectively, and \(\rho = \rho(r)\) is the density of the constitutive materials of the functionally graded nanoplate such that:
\[
\rho(r) = \rho_{in} + \left(\rho_{out} - \rho_{in}\right) \left(\frac{r - r_{in}}{r_{out} - r_{in}}\right)^p,
\]
(12)
where \(\rho_{in}\) and \(\rho_{out}\) denote the density of the functionally graded nanoplate at the innermost and outermost surfaces, respectively. In the context of linear elastic deformation, the radial and hoop strains, \(\varepsilon_{rr}\) and \(\varepsilon_{\theta \theta}\), are expressed by:
\[
\varepsilon_{rr} = \frac{du_r}{dr} - \alpha \rho T, \quad \varepsilon_{\theta \theta} = \frac{u_\theta}{r} - \alpha \theta T,
\]
(13)
in which \(u_r\) denotes the radial displacement field, and \(\alpha\) is the coefficient of thermal expansion which is given by:
\[
\alpha = \alpha_{T_{in}} + \left(\alpha_{T_{out}} - \alpha_{T_{in}}\right) \left(\frac{r - r_{in}}{r_{out} - r_{in}}\right)^p,
\]
(14)
in which \(\alpha_{T_{in}}\) and \(\alpha_{T_{out}}\) are the coefficients of thermal expansion of the innermost and outermost surfaces of the FGPNP, respectively. In view of the fourth assumption, the generalized Hook’s law explains the stress–displacement relations of the elastic nanoplate as:
\[
\sigma_{rr} = \frac{E(r)}{1 - \nu^2(r)} \left[\frac{du_r}{dr} + \nu(r) \frac{u_\theta}{r} - \alpha(r) (1 + \nu(r)) T\right],
\]
(15a)
\[
\sigma_{\theta \theta} = \frac{E(r)}{1 - \nu^2(r)} \left[\frac{u_\theta}{r} + \nu(r) \frac{du_r}{dr} - \alpha(r) (1 + \nu(r)) T\right].
\]
(15b)
Young’s modulus \((E)\) and Poisson’s ratio \((\nu)\) of the constitutive materials of the functionally graded nanoplate are given by:
\[
E(r) = E_{in} + \left(E_{out} - E_{in}\right) \left(\frac{r - r_{in}}{r_{out} - r_{in}}\right)^p,
\]
(16a)
\[
\nu(r) = \nu_{in} + \left(\nu_{out} - \nu_{in}\right) \left(\frac{r - r_{in}}{r_{out} - r_{in}}\right)^p,
\]
(16b)
where \(E_{in} E_{out}\) and \(\nu_{in} \nu_{out}\) in order represent, respectively, Young’s modulus and Poisson’s ratio’s of the innermost/outermost surfaces of the nanoplate. By substituting Eqs. (15a) and (15b) into Eq. (11), it is obtainable:
\[
\frac{d}{dr} \left\{ \frac{E(r) \frac{du_r}{dr} + \nu(r) u_\theta - (1 + \nu(r))T \alpha(r)T'}{1 - \nu^2(r)} \right\} + \frac{E(r)}{r(1 + \nu(r))} \frac{du_r}{dr} - \frac{u_\theta}{r} + \rho \alpha^2 = 0.
\]
(17)
This equation can be rewritten in a more compact form as follows:
\[
r^2 \frac{d^2 u_r}{dr^2} + r (1 + \gamma) \frac{du_r}{dr} - (1 - \nu(r)) u_\theta - Tr^2 + m^2 r^3 = 0,
\]
(18)
where
\[
\gamma = \frac{r (1 - \nu^2(r))}{E(r)} \frac{d}{dr} \left[ \frac{E(r) \alpha(r) T'}{1 - \nu^2(r)} \right],
\]
\[
\omega^2 = \frac{\rho(r) \alpha^2 (1 - \nu(r))}{E(r)}, \quad T = \frac{1 - \nu^2(r)}{E(r)} \frac{d}{dr} \left[ \frac{E(r) \alpha(r) T'}{1 - \nu^2(r)} \right].
\]
(19)
Eq. (18) furnishes us regarding thermo-elasto-static deformation
of RFGPNs. For an arbitrary variation of the material behavior in the spatial domain of the nanoplate, seeking an analytical solution to this equation could be a very problematic job. Just for particular cases, for example, macro-scale plates with homogeneous materials or plates with variation of material property of the form: $[\rho r(r) = \rho_{\text{in}}(r/r_{\text{out}})^3]$, the explicit solution to Eq. (18) is available in the literature [20,52,53]. In the next part, an efficient methodology is proposed to determine the elastic field within RFGPNs acted upon by a thermal field.

4.2. Development of a semi-analytical approach

Let us divide the nanoplate into N annular subdomains, just like those used for the thermal field analysis. Now assume that the material properties would be constant in each subdomain. Hence, in view of Eq. (18), the governing equation of the ith subdomain reads:

$$r^2 \frac{d^2 u_{ri}}{dr^2} + r \left(1 + \nu_i \right) \frac{du_{ri}}{dr} - (1 - \nu_i \gamma_i) u_{ri} = T_i r^2 - m_i^2 r^2; \quad r_{i-1} \leq r \leq r_i,$$

where $u_{ri}$ is the radial displacement of the ith subdomain, and other parameters of the ith subdomain in Eq. (20) are defined by:

$$\gamma_i = \frac{r_{\text{ave}}(1 - L^2(r_{\text{ave}}))}{E(r_{\text{ave}})}} \frac{d}{dr} \left[ \frac{E(r)}{1-L^2(r)} \right] \bigg|_{r=r_{\text{ave}}},$$

$$m_i = \frac{\nu_i \gamma_i}{\gamma_i^2 + 4(1-\nu_i \gamma_i)} + \frac{\nu_i}{\gamma_i^2 + 4(1-\nu_i \gamma_i)}.$$

Eq. (20) is a nonhomogeneous linear second-order ordinary differential equation. By evaluating the private and general solutions, a complete solution to this equation is sought as follows:

$$u_{ri}(r) = C_i r^2 + D_i r^4 + \Gamma_i r^3 + A_i r^2.$$

where $C_i$, $D_i$, and $\Gamma_i$ are unknown coefficients, and the parameters are given by:

$$\lambda_i = \frac{1}{2} \left( -\gamma_i + \sqrt{\gamma_i^2 + 4(1-\nu_i \gamma_i)} \right), \quad \mu_i = \frac{1}{2} \left( -\gamma_i - \sqrt{\gamma_i^2 + 4(1-\nu_i \gamma_i)} \right),$$

$$\Gamma_i = \frac{1}{\gamma_i (2+\nu_i)} \left( \frac{A_i}{2+\nu_i} + \frac{B_i}{2+\nu_i} \right).$$

Using Eqs. (15a) and (15b), the radial and hoop stress fields within the ith subdomain of the rotating nanoplate are evaluated as:

$$\sigma_{\tau,i} = E(r) \left[ C_i (1 + \nu_i) \rho_i^3 - D_i (1 + \nu_i) \rho_i^3 + \Gamma_i (3 + \nu_i) r^2 \right] r \bigg|_{r=r_i},$$

$$\sigma_{\theta,i} = E(r) \left[ C_i (1 + \nu_i) \rho_i^3 - D_i (1 + \nu_i) \rho_i^3 + \Gamma_i (3 + \nu_i) r^2 \right] + A_i (2 + \nu_i) r - \alpha \left[ (1 + \nu_i) T(r) \right] r \bigg|_{r=r_i}.\tag{24a}$$

$$\sigma_{\tau,i} = E(r) \left[ C_i (1 + \nu_i) \rho_i^3 - D_i (1 + \nu_i) \rho_i^3 + \Gamma_i (3 + \nu_i) r^2 \right] r \bigg|_{r=r_i},$$

$$\sigma_{\theta,i} = E(r) \left[ C_i (1 + \nu_i) \rho_i^3 - D_i (1 + \nu_i) \rho_i^3 + \Gamma_i (3 + \nu_i) r^2 \right] + A_i (2 + \nu_i) r - \alpha \left[ (1 + \nu_i) T(r) \right] r \bigg|_{r=r_i}.\tag{24b}$$

4.3. Boundary conditions

4.3.1. Interface boundary conditions

The requirement of continuity of the radial displacement as well as the radial stress at the interface of the ith and (i+1) th subdomains leads to:

$$u_{r,i+1}(r_i) = u_{r,i}(r_i); \quad i = 1, 2, \ldots, N-1.$$

$$\sigma_{\tau,i+1}(r_i) = \sigma_{\tau,i}(r_i); \quad i = 1, 2, \ldots, N-1.\tag{25a}$$

4.3.2. Surface boundary conditions

4.3.2.1. A brief introduction to Gurtin–Murdoch elasticity theory

When the radius of the nanoplate reduces to several nanometers, the effect of energy of the surface layer should be appropriately taken into account to capture near to exact mechanical behavior of the nanostructure. In fact, the surface layer is a very thin layer with negligible thickness which has been tightly bonded to the bulk zone. According to the Gurtin–Murdoch elasticity model [25–27], the constitutive relations of the surface layer are provided by:

$$\sigma_{\tau,i} = \tau_0 + 2 (\mu_0 - \tau_0) \epsilon_{\tau,i} + (\lambda_0 + \tau_0) \epsilon_{\gamma,i} \delta_{\tau,i} + \tau_0 \frac{\partial u_{\tau,i}}{\partial \delta_{\tau,i}}$$

$$+ 2 \alpha \beta \gamma \delta_{\tau,i} + \lambda_0 \epsilon_{\gamma,i} \delta_{\tau,i} + \lambda_0 \epsilon_{\gamma,i} \delta_{\tau,i} + \lambda_0 \epsilon_{\gamma,i} \delta_{\tau,i} + \alpha \beta \gamma \delta_{\tau,i} + \lambda_0 \epsilon_{\gamma,i} \delta_{\tau,i},$$

where $\alpha$, $\beta$, and $\gamma$ are the coefficients that have to be appropriately taken into account in the static equilibrium equations of the above-mentioned surface element.

4.3.2.2. Free–free condition. In the case of free–free condition, both innermost and outermost surfaces of the nanoplate are traction free. Consider an infinitesimal element on the inner and outer surfaces of the annular nanoplate. Since the surrounding environment of the nanoplate does not exert any pressure on its outer surface, application of Newton’s second law to this element leads to:

$$\sigma_{\tau,i} = \sigma_{\tau,i}^\alpha = \sigma_{\tau,i}^\alpha;\tag{27a}$$

$$-\sigma_{\tau,i}^\beta = \sigma_{\tau,i}^\beta;\tag{27b}$$

where $\sigma_{\tau,i}^\alpha$ and $\sigma_{\tau,i}^\beta$ are the hoop stress of the innermost and outermost surface layers, respectively. In constructing Eqs. (27a) and (27b), it is assumed that the density of the surface layer is negligible. It indicates that the exerted centrifugal force on this layer has been neglected. For high levels of the angular velocity of the rotating nanoplate, particularly in the absence of the thermal field, such effect could become important and should be appropriately taken into account in the static equilibrium equations of the above-mentioned surface element.

In the case of plane stress, by neglecting the radial strain within the innermost and outermost surface layers, the surface hoop stress is expressed by:

$$\sigma_{\tau,i}^\alpha = \tau_{i,0} + 2 (\mu_{i,0} - \tau_{i,0}) \epsilon_{\tau,i} + 2 \mu_{i,0} - 1 \left( \lambda_{i,0} + \tau_{i,0} \right) \epsilon_{\gamma,i} \delta_{\tau,i} + \tau_{i,0} \frac{\partial u_{\tau,i}}{\partial \delta_{\tau,i}}$$

$$- 2 \alpha \beta \gamma \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i} + \alpha \beta \gamma \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i},\tag{28a}$$

$$\sigma_{\tau,i}^\beta = \tau_{i,0} + 2 (\mu_{i,0} - \tau_{i,0}) \epsilon_{\tau,i} + 2 \mu_{i,0} - 1 \left( \lambda_{i,0} + \tau_{i,0} \right) \epsilon_{\gamma,i} \delta_{\tau,i} + \tau_{i,0} \frac{\partial u_{\tau,i}}{\partial \delta_{\tau,i}}$$

$$- 2 \alpha \beta \gamma \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i} + \alpha \beta \gamma \delta_{\tau,i} + \lambda_{i,0} \epsilon_{\gamma,i} \delta_{\tau,i},\tag{28b}$$

where $\tau_{i,0}(r_{0,0})$, $\mu_{i,0}(r_{0,0})$, $\lambda_{i,0}(r_{0,0})$, and $\alpha_{i,0}(r_{0,0})$ are the material properties of the innermost(outermost) surface layer.

4.3.2.3. Fixed–free condition. In the case of fixed–free condition, the inner surface is prohibited from any radial deformation, and the outermost surface is traction free. Therefore:

$$u_{r,i}(r_{i,0}) = 0;\tag{29a}$$

$$-\sigma_{\tau,i}(r_{i,0}) = \sigma_{\tau,i}(r_{i,0}).\tag{29b}$$
4.4. Enforcement of boundary conditions

4.4.1. Free-free condition

In the case of free-free conditions, by substituting Eqs. (22), (24a), and (24b) into Eqs. (27a), (25a), (25b), and (27b), respectively, one can arrive at the following relations:

\[
\begin{align*}
\frac{E_{in}}{1 - \nu_{in}^2} \left[ C_{i}(\lambda_{in} + \nu_{in}) r_{in}^{i-1} + D_i (\nu_{in} + 1 + \nu_{in}) r_{in}^{i-1} + \frac{\mu_{in} + 1}{2} r_{in}^{i-1} - \frac{\mu_{in}}{2} r_{in}^{i-1} \right] & = \frac{1}{r_{in}} \left[ 2(\lambda_{in} + \nu_{in}) T_{in} - \lambda_{in} r_{in} + 2 \lambda_{in} - \nu_{in} r_{in} \right], \\
\frac{E_{out}}{1 - \nu_{out}^2} \left[ C_{i}(\lambda_{out} + \nu_{out}) r_{out}^{i-1} + D_i (\nu_{out} + 1 + \nu_{out}) r_{out}^{i-1} + \frac{\mu_{out} + 1}{2} r_{out}^{i-1} - \frac{\mu_{out}}{2} r_{out}^{i-1} \right] & = \frac{1}{r_{out}} \left[ 2(\lambda_{out} + \nu_{out}) T_{out} - \lambda_{out} r_{out} + 2 \lambda_{out} - \nu_{out} r_{out} \right].
\end{align*}
\]

By solving the set of 2N equations in Eqs. (30a)--(30d) for \( \mathbf{e} \), the unknown coefficients of the elastic fields are evaluated. Subsequently, by using Eqs. (22), (24a), and (24b), the radial displacement as well as the radial and hoop stresses within the sub-domains of the RFGNP with free-free boundary condition are obtained.

4.4.2. Fixed-free condition

By substituting Eqs. (22), (24a), and (24b) into Eqs. (29a), (25a), (25b), and (29b), the equations associated with the fixed-free condition are obtained. It is readily found that Eqs. (30b), (30c), and (30d) remain unchanged, however, Eq. (30a) is modified in the following form:

\[
\begin{align*}
C_{i}(\lambda_{in} + \nu_{in}) r_{in}^{i-1} + D_i (\nu_{in} + 1 + \nu_{in}) r_{in}^{i-1} + \frac{\mu_{in} + 1}{2} r_{in}^{i-1} & = \frac{1}{r_{in}} \left[ 2(\lambda_{in} + \nu_{in}) T_{in} - \lambda_{in} r_{in} + 2 \lambda_{in} - \nu_{in} r_{in} \right], \\
C_{i}(\lambda_{out} + \nu_{out}) r_{out}^{i-1} + D_i (\nu_{out} + 1 + \nu_{out}) r_{out}^{i-1} + \frac{\mu_{out} + 1}{2} r_{out}^{i-1} & = \frac{1}{r_{out}} \left[ 2(\lambda_{out} + \nu_{out}) T_{out} - \lambda_{out} r_{out} + 2 \lambda_{out} - \nu_{out} r_{out} \right],
\end{align*}
\]

5. Results and discussion

5.1. Some validations of the proposed model

5.1.1. A comparison study

In order to check the accuracy of the proposed model, the predicted elastic fields are compared with those of another work in a special case. Let us consider a rotating functionally graded plate whose material properties vary across the radial direction by the following power-law relations: \( \sigma(\rho) = E(\rho/r_{min})^{\alpha}, \rho(\rho) = \beta(\rho/r_{min})^{\beta} \), and \( \tau(\rho) = \alpha(\rho/r_{min})^{\gamma} \). For such a macro-scale structure, Peng and Li [54] suggested an analytical methodology to determine its thermo-elastic fields. We employ their suggested solution to study the accuracy of the proposed semi-analytical model. In the case of \( p = -0.5, r_{min} = 5.0, r_{out} = 0.2 \text{ m}, \omega = 600 \text{ rad/s}, T_{min} = 0, \) and \( T_{out} = 1000 \text{ °C} \) for a rotating plate whose outermost surface is made of Zirconia, the predicted dimensionless thermo-elastic fields as well as the thermal field for both fixed-free and free-free boundary conditions have been plotted in Fig. 2(a) and (b), respectively. The material properties of Zirconia have been given in Ref. [54]. The dimensionless radial displacement (\( u_\rho \)), radial stress (\( \sigma_{\rho\rho} \)), and hoop stress (\( \sigma_{\rho\theta} \)) are defined by:

\[
\begin{align*}
\sigma_{\rho\rho} & = \frac{E_{out}}{r_{out}(1 + \nu_{out})} \left[ E_{out} \alpha_{\rho\rho}(\rho/r_{out})^{\gamma} + (1 - \nu_{out}) \rho_{\rho\rho}(\rho/r_{out})^{\beta} \right] u_\rho, \\
\sigma_{\rho\theta} & = \frac{E_{out}}{r_{out}(1 + \nu_{out})} \left[ E_{out} \alpha_{\rho\theta}(\rho/r_{out})^{\gamma} + (1 - \nu_{out}) \rho_{\rho\rho}(\rho/r_{out})^{\beta} \right].
\end{align*}
\]

The predicted results by the proposed model and those of Peng and Li have been demonstrated by empty circles and solid lines, respectively. As it is obvious from Fig. 2(a) and (b), the proposed model can successfully capture the results of the Peng and Li. Note that the plotted results are for a macro-scale plate whose innermost and outermost radii are in the order of 10³ nm. At such a scale, the effect of the surface energy on deformation of the
**Fig. 2.** Comparison of the predicted thermo-elastic field of rotating functionally graded annular plate by the proposed model and those of Peng and Li [54] for (a) fixed–free, and (b) free–free boundary conditions; (○) present work, (−) Peng and Li [54].

**Fig. 3.** Convergence check of the proposed numerical model for fixed–free RFGNP; (…) exact solution without the surface effect, (−−−) proposed model without the surface effect, (−) proposed model by considering the surface effect; $r_{in} = 2$ nm, $r_{out} = 6$ nm, $\omega = 600$ rad/s, $T_{in} = 0$, $T_{out} = 1000$ °C, $p = 1.5$. 
structure is surely negligible. Because of this fact, the material properties of the surface layers of the plate have not been given.

5.1.2. A convergence study

To ensure regarding stability and efficiency of the proposed model, a convergence check study is performed. For this purpose, consider a RFGNP whose material properties vary in accordance with the given power-law relations in the previous part. The geometry and mechanical properties of the nanostructure whose outer surface has been made of aluminum are as follows:

\[ E_{\text{out}} = 70 \times 10^{9} \text{ Pa}, \; \nu_{\text{in}} = \nu_{\text{out}} = 0.3, \; \rho_{\text{in}} = 2700 \text{ kg/m}^3, \; \alpha_{\text{in}} = 23.1 \times 10^{-6}/^\circ\text{C}, \; k_{\text{in}} = 209 \text{ W/(m C)}, \; \mu_{\text{in}} = -5.4251 \text{ N/m}, \; \lambda_{\text{in}} = 3.4939 \text{ N/m}, \; r_{\text{in}} = 0.5689 \text{ N/m}, \; r_{\text{in}} = 2 \text{ nm}, \; r_{\text{out}} = 6 \text{ nm}, \; \text{and} \; p = 1.5. \]

The nanoplate rotates with angular velocity \( \omega = 600 \text{ rad/s} \), and is subjected to the following thermal field of the RFGNP: (a) fixed-free, (b) free-free; \( T_{\text{en, in}} = 0 \) and \( T_{\text{en, out}} = 1000 \text{ }^\circ\text{C} \). The maximum values of the radial displacement, radial stress, and hoop stress fields are calculated and demonstrated by the dotted lines in Fig. 3. Additionally, the plots of the predicted maximum thermo-elastic fields by the proposed model as a function of the number of subdomains are presented in this figure.

Regarding the demonstrated results of the proposed model in Fig. 3, the obtained results by considering the surface effect and those predicted without considering the surface energy effect in order are plotted by the solid and dashed lines, respectively. The demonstrated results in Fig. 3 display that the discrepancies between the results of the exact solution and those of the proposed model without considering the surface energy effect would reduce as the number of subdomains increases. Further, the predicted thermo-elastic fields accounting for the surface effect are generally underestimated by those obtained without consideration of the surface effect. The maximum relative discrepancies between those obtained without the surface effect are about 2, 5.8, and 7.7 percent, respectively. In the upcoming parts, the role of the surface energy effect on the thermo-elastic fields of the nanostructure is methodically examined through various numerical studies.

5.2. Parametric studies

We are interested in investigating the effects of the thermal loading, angular velocity, radius of the nanoplate, variation of the material properties, and surface energy effect on the mechanical behavior of rotating nanoplates. To this end, consider a nanoplate with the following data: \( E_{\text{in}} = 76 \times 10^{9} \text{ Pa}, \; \nu_{\text{in}} = 0.26, \; \rho_{\text{in}} = 10.500 \text{ kg/m}^3, \; \alpha_{\text{in}} = 18.9 \times 10^{-6}/^\circ\text{C}, \; k_{\text{in}} = 429 \text{ W/(m C)}, \; \mu_{\text{in}} = 0.11 \text{ N/m}, \; \lambda_{\text{in}} = 1 \text{ N/m}, \; r_{\text{in}} = 0.89 \text{ N/m}, \; E_{\text{out}} = 70 \times 10^{9} \text{ Pa}, \; \nu_{\text{out}} = 0.3, \; \rho_{\text{out}} = 2700 \text{ kg/m}^3, \; \alpha_{\text{out}} = 23.1 \times 10^{-6}/^\circ\text{C}, \; k_{\text{out}} = 209 \text{ W/(m C)}, \; \mu_{\text{out}} = -5.4251 \text{ N/m}, \; \lambda_{\text{out}} = 3.4939 \text{ N/m}, \; \text{and} \; r_{\text{out}} = 0.5689 \text{ N/m}. \]

With the lack of information on the thermal coefficient expansion of the surface layer, the value of this quantity is set equal to that of the bulk at its vicinity. In all numerical studies in this part, \( N = 100 \), and the obtained results with and without considering the surface effect in order have been presented by the solid and dashed lines, respectively.

5.2.1. Role of the environment's temperature

An important parametric study is carried out to show the effect of the environment’s temperature and surface energy effect on the thermo-elastic field of the RFGNPs. In Fig. 4(a) and (b), the plots of the radial displacement, radial stress, and hoop stress fields of the nanostructure are provided for different levels of the environment’s temperature (i.e., \( T_{\text{en, in}} = 100, 200, 400, \text{ and } 800 {^\circ}\text{C} \)). The results have been demonstrated for fixed-free and free-free RFGNPs with \( \omega = 600 \text{ rad/s} \) and \( p = 2.5 \). As it is obvious in these figures, the values of all radial displacement, radial stress, and hoop stress at each point of the nanostructure would grow as the temperature increases. For both fixed-free and free-free RFGBPs, the predicted radial displacement and radial stress fields by
considering the surface effect are generally underestimated by those obtained without consideration of the surface effect. Such a fact also holds true for positive hoop stresses. However, the absolute value of the negative hoop stresses by consideration of the surface effect are commonly overestimated by those predicted without considering the surface effect. A close scrutiny of the demonstrated results indicates that the effect of the surface energy on the elastic fields of the RFGNPs with the free–free boundary condition is more obvious with respect to those of the fixed–free RFGNPs. Generally, the relative discrepancies between the radial displacements of both fixed–free and free–free RFGNPs accounting for the surface effect and those without considering the surface energy effect would reduce as one becomes closer to the outer surface of the nanoplate. Nevertheless, the relative discrepancies between the radial stress fields by considering the surface effect and those of without consideration of the surface effect would grow as one approaches to the outer surface. Concerning the hoop stress fields, the surface energy effect has the most influence on the obtained results of those points close to the point that the sign of the hoop stress field changes. As we move far away from this point, the influence of the surface energy effect on the hoop stress field would lessen.

5.2.2. Role of the angular velocity

Effect of the angular velocity of the nanoplate on its thermo-elastic field is the subject of another interesting study that should be carefully investigated. To this end, the plots of the elastic field of the RFGNPs with fixed–free and free–free boundary conditions have been provided in Fig. 5(a) and (b) for different levels of the angular velocity, respectively. Since only the effect of the centrifugal forces due to the angular velocity of the nanoplate on its elastic field is of high interest, the temperatures of both inside and outside environments are set equal to zero. As it is seen in Fig. 5(a) and (b), for both fixed–free and free–free boundary conditions, the radial displacements, radial stresses, and hoop stresses within the RFGNP would magnify as the angular velocity of the nanoplate increases. A close scrutiny of the obtained results reveals that the elastic fields of the nanoplate in terms of its angular velocity commonly vary in a parabolic manner. Concerning the effect of the surface energy on the elastic fields of the RFGNP only due to its rotary motion, such a fact is more obvious in the case of the free–free boundary condition. For various values of the angular velocity of the nanoplate, the relative discrepancies between the elastic fields accounting for the surface energy effect and those obtained without consideration of the surface effect are approximately identical. In other words, variation of the angular velocity of the nanoplate is trivially influential on the effect of the surface energy. For both boundary conditions, the effect of the surface energy on the elastic fields of the regions closer to the outer surface is more apparent. In most of the cases, irrespective of the angular velocity of the nanoplate, the maximum relative discrepancies between the elastic fields of the RFGNP by considering the surface effect and those without consideration of the surface effect occur on the outer surface of the nanoplate. Regarding the location of occurrence of the maximum elastic fields in the case of fixed–free, further studies show that the maximum values of the radial displacement, radial stress, and hoop stress take place at about \( r_{in} + 0.865(r_{out} - r_{in}) \), \( r_{in} \), and \( r_{in} + 0.285(r_{out} - r_{in}) \), respectively. However, in the case of free–free boundary condition, the above-mentioned maximum values in order take place at about \( r_{in} \), \( r_{in} + 0.305(r_{out} - r_{in}) \), and \( r_{in} \).

5.2.3. Role of the power-law index

A crucial study has been performed to explain how the material variation across the radial direction could affect the resulted thermo-elastic fields within RFGNPs. In Fig. 6(a) and (b), the
plotted results of the thermo-elastic field of the RFGNP for four levels of the power-law index of the functionally graded materials (i.e., \( p = 0, 0.3, 1, \) and \( 5 \)) have been demonstrated. These results are given for a rotating nanoplate with \( \omega = 600 \text{ rad/s} \) which is subjected to \( T_{\text{en, out}} = 1000 \text{ C} \). For both fixed–free and free–free boundary conditions, the relative discrepancies between the predicted radial displacements accounting for the surface effect and those without consideration of the surface effect would lessen as one moves from the inner surface to the outer surface of the nanoplate. For all considered power-law indexes, the CCM could predict the radial displacements by the surface elasticity theory of Gurtin and Murdoch with relative errors lower than 3 and 4 percent for fixed–free and free–free conditions, respectively. Concerning the radial stress fields of both fixed–free and free–free
RFGNPs, the role of the surface energy effect on the obtained results would commonly magnify as one approaches to the outer surface of the nanoplate. In the cases of the fixed–free and free–free boundary conditions, the CCM could capture the predicted radial stresses by the surface elasticity model with relative errors lower than 26 and 36 percent, respectively. The maximum relative discrepancies between the results of these two theories happen in the case of \( p=0 \) (i.e., homogeneous material). Since determining the role of the material variation on the influence of the surface energy on the obtained results based on Fig. 6(a) and (b) is not an easy task, other useful plots should be provided that clearly show such effects. In Fig. 7(a) and (b), the graphs of the maximum thermo-elastic fields of the RFGNP as a function of the power-law index have been presented for both fixed–free and free–free boundary conditions. Generally, the predicted maximum thermo-elastic fields by considering the surface energy effect are underestimated by those obtained without consideration of the surface effect. According to Fig. 7(a), the predicted thermo-elastic fields by the CCM and those of the surface elasticity model would reduce as the power-law index increases. Additionally, the relative discrepancies between the predicted maximum radial displacements, radial stresses, and hoop stresses would magnify with the power-law index. It implies that the role of the surface energy effect on the maximum thermo-elastic fields would increase as the power-law index grows. Such a fact also holds true for the radial displacements and hoop stresses of the RFGNP with free–free boundary condition. However, the relative discrepancies between the maximum radial stresses by considering the surface effect and those obtained without consideration of the surface effect would reduce as the power-law index increases from 0 to 0.3. For the power-law index greater than 0.3, such discrepancies would grow as the power-law index magnifies.

6. Conclusions

A semi-analytical solution is developed to study steady-state mechanical response of rotating functionally graded nanoplates in thermal environments using the surface elasticity approach. A generalized power-law relation is assumed for variation of the material properties across the radial direction. The convergence check of the proposed methodology and comparison of the obtained results with those of another work in a special case are performed and remarkable results are reported. Subsequently, the effects of the surface energy, temperature of the outside environment, angular velocity of the nanoplate, and variation of the material properties on the thermo-elastic fields of the nanostructure would grow as the power-law index increases.

In the present work, elasto-thermo-static response of functionally graded nanorotors was studied in the context of the surface elasticity theory using a semi-analytical methodology. Studying such a problem by considering both nonlocality and surface effects would also increase our knowledge regarding the nanomechanical problem. Using advanced continuum field theories, examining elastic fields of nanorotors with bidirectional functionally graded materials is still an open problem. If the angular velocity of the nanorotor or the environment temperature varies with time, the elastic field of the nanorotor would be time-dependent and knowing how the trend of the temperature as well as the dynamic angular velocity could affect the resulted elastic fields is of great importance. These issues represent the unknown realms of the mechanical behavior of functionally graded nanorotors that should be paid attention to by the applied mechanics community.

References
