Large deformation of uniaxially loaded slender microbeams on the basis of modified couple stress theory: Analytical solution and Galerkin-based method

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ABSTRACT

Large deformation regime of micro-scale slender beam-like structures subjected to axially pointed loads is of high interest to nanotechnologists and applied mechanics community. Herein, size-dependent nonlinear governing equations are derived by employing modified couple stress theory. Under various boundary conditions, analytical relations between axially applied loads and deformations are presented. Additionally, a novel Galerkin-based assumed mode method (AMM) is established to solve the highly nonlinear equations. In some particular cases, the predicted results by the analytical approach are also checked with those of AMM and a reasonably good agreement is reported. Subsequently, the key role of the material length scale on the load-deformation of microbeams is discussed and the deficiencies of the classical elasticity theory in predicting such a crucial mechanical behavior are explained in some detail. The influences of slenderness ratio and thickness of the microbeam on the obtained results are also examined. The present work could be considered as a pivotal step in better realizing the postbuckling behavior of nano-/micro- electro-mechanical systems consist of microbeams.

1. Introduction

Micro-scaled beam-like structures have many applications in modern technology including microactuators [1–4], micro-electromechanical systems [5–8], microsensors (both physical and chemical) [9–11], microtribological apparatus [12–14], and optical micro-devices [15,16]. For optimal design of these micromechanical systems, the mechanical behavior of the microstructure should be rationally predicted. When these microsystems are exploited as load bearer microdevices, their capabilities in safe transferring the applied load should be carefully understood. To this end, appropriate analytical/numerical models would increase our knowledge on such a fact. Actually, these models enable us to scrutiny the roles of influential factors on large deformation regimes and possible instabilities of such microdevices under externally applied load.

Recent studies of investigators [17–19] have shown that mechanical behaviors of microstructures depend on the lateral dimension of the micro-beams/-plates. As such a dimension approaches to the material length scale parameter, the size-dependency of mechanical behavior of microstructure becomes more obvious. The classical elasticity theory (CET) cannot display this fact since its main postulate is that the state of stress of each point only relies on the state of stress of that point. To overcome this drawback of the CET, several advance theories have been developed in the previous century including nonlocal continuum field theory of Eringen [20,21], surface elasticity theory of Gurtin-M Murdoch [22,23], and couple stress theory of Mindlin, Toupin, and Koiter [24–26]. The first two theories are commonly exploited for investigating statics, buckling, and vibrations of nanostructures [27–37]. The main reason of this fact is that the effect of material scale parameter as well as surface properties of these models on mechanical behavior would generally vanish as the dimensions of the structures become in the order of microns. The above-mentioned last theory has been extensively used for statics, buckling, postbuckling, and vibrations of microstructures including microbeams, microplates, and microshells [38–41].

Theoretically, if an axially pointed force is exactly exerted on the centre of cross-sectional area of a microbeam, no bending moment is generated within the structure. By increasing the applied force and reaching to its critical value, the microbeam would arrive at the buckling state. The critical buckling load is commonly predicted by the theory of elasticity/stability of linearly elastic solids. In real situations, due to existence of a small eccentricity, a bending moment field would also be generated within the microstructure, and thereby, the microbeam would bend due to the applied load. By increasing the
load, the state of equilibrium becomes far away from the elastic buckling and it would arrive at the postbuckling state. Based on the CET, length and bending rigidity of the microstructure plus to the shear effect are among the crucial factors influence on the large deformation regime of elastic solids under extreme loads. However, at the micro-scale, the material length effect factor also plays a vital role in such postbuckling curves. Until now, various mechanical problems associated with large deformations of beams in the context of the CET including nonlinear statics, postbuckling, and nonlinear vibrations have been examined comprehensively [42–47]. In most of these works, large deformations of beam-like structures were studied analytically. However, research works on developing closed-form solutions for large deformations of microbeams via the modified couple stress theory (MCST) are so rare.

Concerning large deformation of microstructures and their postbuckling behavior, Ansari et al. [48] studied thermal postbuckling of functionally graded microbeams using Timoshenko beam model based on the MCST. To analyze the governing equations, differential quadrature method (DQM) was employed. By exploiting MCST, Xia et al. [49] examined static bending, free vibration, and postbuckling of slider microbeams using Euler-Bernoulli beam model. The extensibility of the neutral axis was incorporated into the model, and the postbuckling behavior and nonlinear flexural frequencies of pinned-pinned microbeams were obtained via multiple scale method. Ke et al. [50] investigated dynamic stability of functionally graded microbeams via non-classical Timoshenko beam model on the basis of the MCST. By implementing the Hamilton’s principle, the governing equations were obtained and solved using DQM. Mohammadi and Mahzoon [51] examined the influence of thermal field on postbuckling behavior of simply supported Euler-Bernoulli with movable and immovable ends. For this purpose, the governing equations were constructed based on the MCST accounting for the von-Karman nonlinear term, and the corresponding non-classical boundary conditions were introduced. Recently, Wang et al. [52] analyzed nonlinear bending and postbuckling of extensible microbeams based on MCST using shooting technique. The influence of the material length scale parameter and Poisson’s ratio on the bending and thermal postbuckling behaviors of microbeams were explained and discussed. Farokhi et al. [53] examined nonlinear free vibration of pre-stressed viscous micro-scaled beams under transversely harmonic loading. The nonlinear von Karman term is incorporated into the axial strain, the damping of the lateral motion was considered via a viscous model, and the only equation of motion of transverse vibration was obtained in the context of the MCST. By application of Galerkin-based assumed mode method, the governing equation was discretized in the spatial domain and then, the resulting ordinary differential equations were solved using pseudo-arc-length continuation technique. In another work, Ansari et al. [54] suggested an analytical solution for postbuckling and nonlinear frequencies of microbeams using Euler-Bernoulli beam theory. The von Karman nonlinear term of axial strain was considered, and the problem was analyzed for fully pinned and fully clamped microbeams. Ke et al. [55] studied nonlinear free transverse vibrations of microbeams with variation of material properties across the thickness using the MCST and the Timoshenko beam model. By adopting DQM and direct iterative methodology, the governing equations for nonlinear frequencies of microbeams of various boundary conditions and different indexes of material properties. Additionally, the MCST have been extensively employed for free linear and nonlinear vibrations of microplates as well as their postbuckling behavior [56–61].

A close survey of the literature reveals that no analytical model has been established to show large deformation of highly deformable microstructures under statically applied loads. In most of undertaken works [50,54,51,53,55], only the nonlinear von Karman strain has been included in the formulations of the problem. It means that the suggested models would be only applicable to the initial stage of postbuckling of microbeams. Additionally, such models were restricted to particular boundary conditions. To bridge such scientific gaps and to explain the role of material length factor on large deformation of microbeams, the author was encouraged to establish an analytical model to explore the problem in a more general framework. The newly developed model can be successfully extended and be employed for checking the efficiency of numerical models in predicting large displacements and corresponding stresses within microstructures.

Herein, mechanical response of highly deformable micro-scaled beam-like structures subjected to a horizontally pointed force is of concern. Using Euler-Bernoulli beam mode, the nonlinear governing equations of the microbeam are constructed in the framework of the modified couple stress theory. An analytical solution is proposed and analytical expressions of axial load bearing capacity and large displacements are given for simply supported, fully clamped, clamped-shear free, and cantilevered microbeams. To ensure regarding the given formulations on the mechanical response of microbeams, a novel Galerkin-based approach is also developed. A close survey of the obtained results reveals the high accuracy of both suggested numerical solution and the analytical approach. The deformed shapes of the microbeam under various end conditions are demonstrated and the role of the material length scale factor on the plotted results is carefully displayed. Additionally, the effects of influential factors on large mechanical response of microbeams are discussed.

2. Nonlinear governing equations in the context of MCST

To include the material length scale in the constitutive equations, Yang et al. [62] reduced the material’s constants of the couple stress theory of Mindlin [24] to only one. Such a newly developed theory is commonly called modified couple stress. According to this theory, the elastic strain energy of the microstructure, $U_e$, and the work done by the applied uniaxial compressive load of magnitude $N$, $U_p$, is given by:

$$U_e = \frac{1}{2} \int_{\Omega} (\sigma_i \epsilon_j + m_i \delta_i) d\Omega,$$

$$U_p = - \frac{1}{2} \int_{\Omega} N \left( \frac{d\varepsilon}{ds} \right)^2 ds,$$

where $\Omega$ denotes the volume of the microstructure, $d\Omega$ is its infinitesimal volume of the deformed microbeam $(= A_0 ds$ where $A_0$ is the cross-sectional area and $ds$ is the curvilinear length of the infinitesimal volume), $\epsilon_i$ is the stress tensor, $m_i$ represents the couple stress tensor, $\epsilon_i$ is the strain tensor, and $\chi_i$ is the symmetric part of the curvature tensor. These two later parameters represent rate of deformations which are given by:

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \chi_{ij} = \frac{1}{2} \left( \frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i} \right) \Theta = \nabla \times \mathbf{u},$$

where $u = u_x \mathbf{e}_x + u_y \mathbf{e}_y + u_z \mathbf{e}_z$ is the displacement vector, $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are the unit base vectors of the cartesian coordinate system, $\Theta$ is the rotation vector, and, $\Theta$ and $\mathbf{v}$ represent the partial and nabla signs, respectively. The stresses and the deviatoric part of the couple stresses are defined by:

$$\sigma_{ij} = \lambda_b \epsilon_{ij} \delta_{ij} + 2G_b \epsilon_{ij},$$

$$m_{ij} = 2G_b l^2 \epsilon_{ij},$$

where $l$ denotes the material length scale parameter, $\lambda_b$ and $G_b$ are the Lame’s constants which are stated in terms of elastic Young modulus ($E_b$) and Poisson’s ratio ($\nu_b$) as follows:

$$\lambda_b = \frac{E_b \nu_b}{(1 + \nu_b)(1 - 2\nu_b)}; \quad G_b = \frac{E_b}{2(1 + \nu_b)}.$$

Now if the slender microbeam is modeled based on the hypotheses of the Euler-Bernoulli beam theory, the displacement field of the postbuckled microbeam in the cartesian coordinate system are written as:
where $z$ is the distance from the neutral axis, $w = w(s)$ is the transverse displacement, $u = u(s)$ denotes the horizontal displacement of the neutral axis, and $\phi = \phi(s)$ is the angle of rotation of the neutral axis. By virtue of inextensibility condition of the neutral axis of the microbeam, it could be readily concluded:

$$\frac{du}{dx} = \sin(\phi), \quad \frac{dv}{dx} = \cos(\phi) - 1.$$  \hspace{1cm} (6)

By introducing Eq. (5) to Eq. (2), and then, substituting the resulting expressions of Eq. (2) into Eq. (3), and finally substituting the resulted statements of Eq. (3) into Eqs. (1b) and (7), the total elastic strain energy associated with the flexural deformation of the highly deformable slender microbeam would be provided by:

$$U_t = \frac{1}{2} \int_0^l \left( E^b_s b_s + \frac{1}{4} G_s A_s \alpha^2 (1 + \cos(\phi))^2 \right) \frac{d\phi}{ds} \frac{d\phi}{ds} \, ds,$$  \hspace{1cm} (7)

where $I_b$ is the moment inertia of the cross-sectional area of the microbeam, and the effective Young’s modulus is given by:

$$E^b_s = E_s (1 - \nu_b)(1 + \nu_b)(1 - 2\nu_b).$$

Using Hamilton’s principle in view of Eqs. (1b) and (7), the governing equation pertinent to the post-buckling of slender microbeams on the basis of the MCST takes the following form:

$$E^b_s b_s \cos(\frac{\phi}{2}) G_s A_s \alpha^2 \frac{d\phi}{ds} \frac{d\phi}{ds} + N \sin(\phi) = 0.$$  \hspace{1cm} (8)

At the initial stage of postbuckling (i.e., small rotation-large displacement), it can be rationally stated: $\cos(\frac{\phi}{2}) \approx 1$, and Eq. (8) is reduced to:

$$(E^b_s b_s + G_s A_s \alpha^2) \frac{d^2\phi}{ds^2} + N \sin(\phi) = 0.$$  \hspace{1cm} (9)

Finding an exact solution to Eq. (8) is a very problematic task; however, analytical methods like as asymptotic solutions (for example, multiple variable technique is implemented: $\chi = \chi(\delta)$, $\delta = \frac{\phi}{2}$, $\alpha = \frac{\alpha}{2}$, $\beta = \frac{\beta}{2}$, $\gamma = \frac{\gamma}{2}$). The obtained equation pertinent to the post-buckling of highly stretchable microbeams is given by:

$$E^b_s b_s + G_s A_s \alpha^2 \frac{d^2\phi}{ds^2} + N \sin(\phi) = 0.$$  \hspace{1cm} (10)

To analyze the problem regardless of the geometry and mechanical data of the microbeam, the given governing equation and boundary conditions should be presented in the dimensionless form. Now consider the following dimensionless quantities:

$$\bar{\tau} = \frac{\tau}{l^2} \quad \bar{\phi} = \phi(s) \quad \bar{s} = \frac{s}{l} \quad \bar{u} = \frac{u}{l} \quad \bar{w} = \frac{w}{l} \quad \bar{\chi} = \frac{N l^2}{E^b_s b_s + G_s A_s \alpha^2}.$$  \hspace{1cm} (11)

The left side of Eq.(18) denotes the complete elliptic integral of the first kind which is commonly presented by K(β) and its Taylor series

$$M_b(\phi) = \left( E^b_s b_s + \cos(\frac{\phi}{2}) G_s A_s \alpha^2 \right) \frac{d\phi}{ds}$$  \hspace{1cm} (12)

$$\frac{d\phi}{ds}(s = 0) = 0.$$  \hspace{1cm} (13)

with the following boundary conditions:

$$\bar{\phi}(1) = \alpha, \quad \frac{d\bar{\phi}}{d\bar{s}}(1) = 0.$$  \hspace{1cm} (14)

Eq. (13) represents a nonlinear equation due to appearance of $\sin(\phi)$. For low levels of slope, we have $\sin(\phi) \approx \phi$, and Eq. (13) is reduced to that of elastic buckling of microbeams. To solve Eq. (13) for the slope field, this is premultiplied by $d\bar{\phi}$. After taking the integral of the resulted relation, one can arrive at:

$$\frac{1}{2} \left( \frac{d\bar{\phi}}{d\bar{s}} \right)^2 + 2\bar{\chi} \sin(\bar{\phi}) + C = 0.$$  \hspace{1cm} (15)

where $C$ is an integration constant. To determine such a constant, the boundary conditions in Eq. (14) is enforced to Eq. (15), so, $C = -2\bar{\chi} \sin^2(\frac{\bar{\phi}}{2})$. As a result,

$$d\bar{s} = \frac{1}{2\bar{\chi} \left( \sin^2(\frac{\bar{\phi}}{2}) - \sin^2(\frac{\bar{\phi}}{2}) \right)} \, d\bar{\phi}.$$  \hspace{1cm} (16)

It is assumed that the length of the neutral axis of the microbeam would not alter due to the applied force (i.e., the contribution of the internal axial load in changing the neutral axis length is assumed to be fairly negligible). Thereby,

$$\int_0^l \frac{d\bar{\phi}}{\sqrt{1 - \bar{\beta}^2 \sin^2(\bar{\phi})}} = 1.$$  \hspace{1cm} (17)

To facilitate evaluation of the given integral in Eq. (17), the change of variable technique is implemented: $\sin(\bar{\phi}) = \beta \sin(\gamma)$ where $\bar{\beta} = \sin(\bar{\gamma})$. Therefore, it is readily obtained: $d\bar{\phi} = \frac{\beta}{\sqrt{1 - \bar{\beta}^2 \sin^2(\gamma)}} \, d\gamma$, and Eq. (17) is reduced to:

$$\int_0^\frac{\pi}{2} \frac{d\gamma}{\sqrt{1 - \beta^2 \sin^2(\gamma)}} = \bar{\chi}.$$  \hspace{1cm} (18)

The left side of Eq. (18) denotes the complete elliptic integral of the first kind which is commonly presented by K(β) and its Taylor series

$u = u(s) - z(\phi), \quad u_1 = 0, \quad u_2 = w(s).$  \hspace{1cm} (5)
expansion is provided by:
\[
K(\beta) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \beta^{2n}.
\]  

By introducing Eqs. (18)–(12) in view of Eq. (19), the nonlinear relation between the applied force and the maximum slope accounting for the material length scale factor is obtained as follows:
\[
N = \frac{1}{E_b} \left( E^b b^b + G_b A_b b \right) \left( \sin \left( \frac{\alpha}{2} \right) \right).
\]  

By neglecting the material length scale, Eq. (20) is simplified to the postbuckling relation of macrobeams acted upon by compressive axial loads:
\[
E = \frac{E^b b^b}{b} \left( \sin \left( \frac{\alpha}{2} \right) \right).
\]

Let define a material scale ratio by \( R = (N - N(l = 0))/N(l = 0) \). For each value of \( \alpha \),
\[
R = \frac{(1 + \gamma_0)(1 - 2\phi_b^2)}{2(1 - \gamma_0^2)} \left( \frac{1}{\gamma_0} \right)^2.
\]

where \( \gamma_0 \) is the gyration radius of the microbeam's cross-section. Clearly, Eq. (22) displays that the role of the material length scale on large deformation of the microbeam becomes highlighted as its gyration radius lessens.

To compute the large displacements of the postbuckled microbeam, Eq. (6) is exploited. These relations, in the dimensionless form, are given by: \( d\vec{w} = \sin(\vec{\phi}) d\gamma \) and \( d\alpha = (\cos(\vec{\phi}) - 1) d\gamma \). By substituting Eq. (16) to these relations and using the given change of variable in the previous part, the dimensionless large displacements of the cantilevered microbeam are evaluated as:
\[
\sigma = 2 \left( \frac{E}{\bar{K}} \left( \sin \left( \frac{\alpha}{2} \right) \right) \frac{d\vec{w}}{d\gamma} \right) - \gamma.
\]

\[
\sigma = \frac{2 \sin \left( \frac{\alpha}{2} \right)}{K \left( \sin \left( \frac{\alpha}{2} \right) \right)^2} \left( 1 - \sqrt{1 - \left( \frac{\sin \left( \frac{\alpha}{2} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right)^2} \right).
\]

where \( \gamma = \arcsin \left( \frac{\sin \left( \frac{\alpha}{2} \right)}{\sin \left( \frac{\alpha}{2} \right)} \right) \) and \( E(\beta, \gamma) \) denotes the elliptic integral of the second kind which is defined by:
\[
E(\beta, \gamma) = \int_0^\gamma \sqrt{1 - \beta^2 \sin^2(\gamma)} \, d\gamma.
\]

and the complete elliptic integral of the second kind and its Taylor's series about \( \beta = 0 \) is provided by:
\[
E(\beta) = E(\beta, \gamma) = E \left( \frac{\beta}{2} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \beta^{2n}.
\]

According to Eqs. (23a) and (23b), it is concluded that the maximum horizontal and transverse displacements would occur at \( \gamma = \pi \) or \( \Phi = \alpha \) (i.e., \( \gamma = 1 \) or free end point of the CF microbeam). These maximum displacements are readily calculated as follows:
\[
\mu_{\text{max}} = 2\beta \left( \frac{\sin \left( \frac{\alpha}{2} \right)}{K \left( \sin \left( \frac{\alpha}{2} \right) \right)^2} - 1 \right).
\]

\[
K \left( \sin \left( \frac{\alpha}{2} \right) \right) = \frac{2\beta \sin \left( \frac{\alpha}{2} \right)}{K \left( \sin \left( \frac{\alpha}{2} \right) \right)}.
\]

3.2. Large deformation analysis of SS microbeams

Consider an SS microbeam whose simply supported ends could freely move horizontally. Such a microbeam is subjected to horizontally compressive loads at its ends, and finding explicit expressions of large displacements as well as load-deformation regime of the postbuckling state is of high interest. With regard to the symmetric nature of geometry and loading of the microstructure, the resulted deformation field would be symmetric with respect to the normal plane passes through the mid-span point. By considering the origin of the rectangular coordinate system at the left support, and assuming that the slope of the deformed microbeam would be equal to \( \alpha \) and \( -\alpha \) at the left and the right supports, respectively, the dimensionless governing equation associated with the postbuckling of the SS microbeam takes the following form:

\[
\frac{d^2\vec{w}}{d\gamma^2} + \chi^2 \sin(\vec{\phi}) = 0; \quad 0 \leq \gamma \leq 1.
\]

such that the following conditions should meet:
\[
\vec{w}(\pi = 0) = \frac{d\vec{w}}{d\gamma}(0) = 0, \quad \Phi \left( \frac{3}{2} \right) = 0.
\]

Through solving Eq. (27) with the given boundary conditions in Eq. (28), one can arrive at:
\[
\frac{d\vec{w}}{d\gamma} = \begin{cases} -\frac{2\beta \cos(\gamma) d\gamma}{\sqrt{1 - \beta^2 \sin^2(\gamma)}}, & 0 \leq \gamma \leq \frac{\pi}{2}, \\
\frac{2\beta \cos(\gamma) d\gamma}{\sqrt{1 - \beta^2 \sin^2(\gamma)}}, & \frac{\pi}{2} \leq \gamma \leq 0. 
\end{cases}
\]

Now consider the following change of variable: \( \sin(\gamma) = \beta \sin(\frac{\alpha}{2}) \), where \( \beta = \sin(\frac{\alpha}{2}) \). Therefore,
\[
d\vec{w} = \begin{cases} \frac{2\beta \cos(\gamma) d\gamma}{\sqrt{1 - \beta^2 \sin^2(\gamma)}}, & 0 \leq \gamma \leq \frac{\pi}{2}, \\
\frac{2\beta \cos(\gamma) d\gamma}{\sqrt{1 - \beta^2 \sin^2(\gamma)}}, & -\frac{\pi}{2} \leq \gamma \leq 0. 
\end{cases}
\]

Let assume that the length of the neutral axial of the microbeam does not change during deformation. Thereby, \( \int_0^{\pi/2} d\vec{w} = 1 \). By introducing Eqs. (30)–(29), it is derived:
\[
\int_0^{\pi/2} d\vec{w} = \int_0^{\pi/2} \frac{dy}{\sqrt{1 - \beta^2 \sin^2(\gamma)}} \int_0^{\pi/2} \frac{dy}{\sqrt{1 - \beta^2 \sin^2(\gamma)}} = 1.
\]

or
\[
\int_0^{\pi/2} \frac{dy}{\sqrt{1 - \beta^2 \sin^2(\gamma)}} = \frac{\pi}{2}.
\]

or
\[
K \left( \sin \left( \frac{\alpha}{2} \right) \right) = \frac{\pi}{2}.
\]

As a result, the postbuckling load of the SS microbeam in terms of the maximum slope would be:
\[ N = \frac{4}{l_b^2}(E_s^b l_b + G_A l_B^2)K^2 \sin \left( \frac{\alpha}{2} \right) \]  
\[ (34) \]

Additionally, by following the provided procedure in Section 3.1, the maximum horizontal and transverse displacements are derived as:
\[ u_{\text{max}} = \left\{ \frac{E}{K(\sin(\frac{\alpha}{2}))^2} \right\} \left( \sin \left( \frac{\alpha}{2} \right) \right) - 1. \]  
\[ (35a) \]
\[ w_{\text{max}} = \frac{b_s \sin \left( \frac{\alpha}{2} \right)}{K(\sin(\frac{\alpha}{2}))}. \]  
\[ (35b) \]

### 3.3. Large deformation analysis of CC microbeams

Consider a CC microbeam whose left support is immovable while its right end could freely move along horizontal direction. For such a microstructure, the slope at the ends and the mid-span points are zero while transverse displacements of the supports are zero. It could be readily proved that the inflection points of the postbuckled microbeam would occur at \( \tau = \frac{1}{4} \) and \( \frac{1}{2} \). In view of the symmetry of deformation regime, if the slope of the microbeam would be equal to \( \alpha \) at \( \tau = \frac{1}{2} \), the slope of the right inflection point (i.e., \( \tau = \frac{3}{4} \)), is \(-\alpha\). To assess postbuckling behavior of CC microbeams, the slope field can be evaluated by solving the following nonlinear equation:
\[ \frac{d^2 \bar{\phi}}{d\tau^2} + \chi^2 \sin(\bar{\phi}) = 0; \quad 0 \leq \tau \leq 1, \]  
\[ (36) \]

with the following geometrical conditions:
\[ \bar{\phi}(0) = 0, \quad \bar{\phi}(1) = 0, \quad \bar{\phi}'(\frac{1}{4}) = \alpha, \quad \frac{d\bar{\phi}}{d\tau}(\frac{1}{4}) = 0. \]  
\[ (37) \]

By following up the given procedure in the previous subsections, the postbuckling load of the CC microbeam is obtained:
\[ N = \frac{16}{l_b^2}(E_s^b l_b + G_A l_B^2)K^2 \sin \left( \frac{\alpha}{2} \right). \]  
\[ (38) \]

and the maximum displacements of the postbuckled microbeam are readily evaluated by:
\[ u_{\text{max}} = 2b_s \left\{ \frac{E}{K(\sin(\frac{\alpha}{2}))^2} \right\} \left( \sin \left( \frac{\alpha}{2} \right) \right) - 1. \]  
\[ (39a) \]
\[ w_{\text{max}} = \frac{b_s \sin \left( \frac{\alpha}{2} \right)}{K(\sin(\frac{\alpha}{2}))}. \]  
\[ (39b) \]

### 3.4. Large deformation analysis of CSF microbeams

Consider an axially loaded microbeam whose left support is clamped and immovable while its right end is shear free. Both ends have been prevented from any rotation. According to the geometry of deformation, the inflection point would be at the midspan point and lets its slope being \( \alpha \). Hence, the postbuckling behavior of CSF slender microbeams under horizontally applied load is governed by:
\[ \frac{d^2 \bar{\phi}}{d\tau^2} + \chi^2 \sin(\bar{\phi}) = 0; \quad 0 \leq \tau \leq 1. \]  
\[ (40) \]

with the following conditions:
\[ \bar{\phi}(0) = 0, \quad \bar{\phi}(1) = 0, \quad \bar{\phi}'(\frac{1}{2}) = \alpha, \quad \frac{d\bar{\phi}}{d\tau}(\frac{1}{2}) = 0. \]  
\[ (41) \]

By following the given trends in Sections 3.1 and 3.2, the analytical postbuckling load as well as the maximum displacements of the postbuckled CSF microbeam take the following form:
\[ N = \frac{4}{l_b^2}(E_s^b l_b + G_A l_B^2)K^2 \sin \left( \frac{\alpha}{2} \right). \]  
\[ (42) \]

and
\[ u_{\text{max}} = 2l_b \left\{ \frac{E}{K(\sin(\frac{\alpha}{2}))^2} \right\} \left( \sin \left( \frac{\alpha}{2} \right) \right) - 1. \]  
\[ (43a) \]
\[ w_{\text{max}} = \frac{2l_b \sin \left( \frac{\alpha}{2} \right)}{K(\sin(\frac{\alpha}{2}))}. \]  
\[ (43b) \]

### 4. Application of Galerkin-based assumed mode method

In the assumed mode method (AMM), the unknown field of the problem is discretized in terms of admissible modes that at least satisfy the geometrical conditions of the problem:
\[ \bar{\phi}(\tau) = \sum_{j=1}^{NM} \sigma_j \psi_j(\tau), \quad \bar{\phi}'(\tau) = \sum_{j=1}^{NM} \delta \psi_j(\tau), \]  
\[ (44) \]

where \( \delta \) is the variational symbol, \( \sigma_i \) denote the unknown dimensionless coefficient associated with the \( j \)th mode, \( NM \) is the number of mode shapes, and \( \psi_j(\tau) \) represents the \( j \)th admissible mode shape.

Using Eq. (7), the variation of the elastic strain energy of the largely deformed microbeam accounting for only flexural deformation is expressed by:
\[ \delta U_e = \int_0^1 \left( E_s^b l_b + G_A l_B^2 \cos^2 \left( \frac{\phi}{2} \right) \right) \frac{d(\phi)}{d\tau} \frac{d\phi}{d\tau} - \frac{1}{4} G_A l_B^2 \sin(\phi) \frac{d\phi}{d\tau} \frac{d\phi}{d\tau} \]  
\[ (45) \]

Additionally, the variation of the work done by the applied force on the postbuckled microbeam takes the following form:
\[ \delta U_p = \int_0^1 N \delta \phi \cos \delta \phi d\tau. \]  
\[ (46) \]

For more convenience in analyzing the problem, we prefer to employ the dimensionless values of \( \delta U_e \) and \( \delta U_p \). These are presented by:
\[ \delta \mathcal{U}_e = \int_0^1 \left( 1 + R \cos^2 \left( \frac{\phi}{2} \right) \right) \frac{d(\phi)}{d\tau} \frac{d\phi}{d\tau} - \frac{1}{4} R \sin(\phi) \left( \frac{d\phi}{d\tau} \right)^2 d\tau, \]  
\[ (47) \]

and
\[ \delta \mathcal{U}_p = \int_0^1 \mathcal{N} \delta \phi \cos \delta \phi d\tau. \]  
\[ (48) \]

where \( \mathcal{N} = N l_b^2 (E_s^b l_b) \) and the parameter \( R \) is provided by Eq. (22). To construct the governing equations in a discretized format based on the AMM, the Hamilton’s principle is employed: \( \int_0^1 (\delta \mathcal{U}_e - \delta \mathcal{U}_p) d\tau = 0 \). By substituting Eq. (44) into the recent relation, the following \( NM \) non-linear-coupled equations are obtained:
In order to solve Eq. (49) for the unknown parameters $\pi_i$, $i = 1, 2, ..., N_M$, Newton’s methodology is exploited. Therefore, $K_{i}^N \Delta \pi_i = f_{i}^N$, 

\begin{align}
 f_{i}^N &= \int_{0}^{1} \left[ \left( 1 + \frac{R}{2} \left( 1 + \cos(\theta) \right) \right) \frac{\partial \phi}{\partial \sigma} \frac{\partial \phi}{\partial \sigma} - \frac{R}{4} \psi \sin(\theta) \left( \frac{\partial \phi}{\partial \sigma} \right)^2 \right] d\sigma \\
 & - \int_{0}^{1} \psi \mathbf{N} \sin(\theta) d\sigma.
\end{align}

\begin{equation}
(50)
\end{equation}

In order to prove Eq. (49) for the unknown parameters $\pi_i$, $i = 1, 2, ..., N_M$, Newton’s methodology is exploited. Therefore, $K_{i}^N \Delta \pi_i = f_{i}^N$, 

\begin{equation}
(50)
\end{equation}

where

\begin{align}
 f_{i}^N &= \int_{0}^{1} \left[ \frac{1}{2} \left[ (\psi \phi) - \psi \psi \mathbf{N} \cos(\theta) \right] \mathbf{N} \mathbf{N} - \frac{R}{4} \psi \sin(\theta) \left( \frac{\partial \phi}{\partial \sigma} \right)^2 \right] d\sigma, \quad (51a) \\
& - \psi \mathbf{N} \sin(\theta) d\sigma. \\
\end{align}

\begin{equation}
(51a)
\end{equation}

$\Delta \pi_i = \sigma_i^{k+1} - \sigma_i^k$ 

\begin{equation}
(51c)
\end{equation}

where $\sigma_i^k$ denotes the value of $\pi_i$ at the $k$th iteration. For a given postbuckling load $N_i$, the search for an accurate slope field is continued until the condition $\sum_{j=1}^{N_M} (\Delta \pi_j)^2 / \sum_{j=1}^{N_M} (\sigma_j^k)^2 < 10^{-10}$ is met.

5. Results and discussion

Consider an epoxy microbeam with the following properties: $E_b = 1.44$ GPa, $\nu_b = 0.38$, $l_b = 3$ mm, $l = 17.6$ mm, $b = 10$ mm, $h = 100$ $\mu$m where $b$ and $h$ represent dimensions of the rectangular-shaped cross-section. The given value of material length scale parameter is based on the experimentally observed data of Lam et al. [63]. In this part, postbuckling behavior of microbeams accounting for material length scale parameter is going to be examined carefully. Additionally, special attention is paid to the deficiencies of the CET in capturing the predicted results by the MCST. In all demonstrated results, the plots obtained based on the MCST and those of the CET have been specified by solid and dotted-dashed lines, respectively.

5.1. A comparison study

In order to check the predicted results by the suggested analytical model, we perform a verification study. To this end, the obtained postbuckling paths by the suggested analytical models are compared with those of assumed mode method (AMM). The results of the AMM have been provided for two cases: Case 1: $\cos^2(\frac{\theta}{2}) \approx 1$, and, Case 2: $\cos^2(\frac{\theta}{2}) \neq 1$.

The predicted maximum horizontal and transverse displacements by the AMM in the case 1 and those of suggested analytical approach in terms of the applied force have been demonstrated in Figs. 2(a) and (b). For AMM analysis, the first three modes have been considered, and for evaluation of the integrals in Eqs. (51a) and (51b), the length of the microbeam is subdivided into 500 identical cells with 9 Gaussian points in each cell. For postbuckling analysis of SS and CF microbeams using AMM, $\psi(\theta) = \sin((2i-1)\pi(1-\xi))$ and $\sin((0.5(2i-1)\pi\xi))$ are used, respectively. The presented results by the solid lines are pertinent to the presented analytical model while those obtained based on the AMM have been presented by the dashed lines with circle markers. As it is seen in Figs. 2(a) and (b), there exists a fairly good agreement between the predicted postbuckling curves by the AMM and those of the suggested analytical model for most levels of the applied loads. For the considered microbeam with SS ends, the relative discrepancies between the results of the AMM and those of the suggested analytical model would generally increase by growing of the applied force. For all considered levels of the applied load, such discrepancies are lower than 10.5%. Regarding the obtained results for CF microbeams (see Fig. 2(b)), the relative discrepancies between the maximum horizontal displacements based on the AMM and those of the analytical model are commonly lower than 0.05% while such discrepancies for maximum transverse displacements are about 0.3%.

We are also interested in exploring that how the proposed analytical solution for the approximate version of the nonlinear equation could capture the results of the proposed AMM for the exact version of the governing equation, namely case 2. In Fig. 3(a) and (b), the predicted maximum displacements of SS and CF microbeams as a function of magnitude of load have been plotted. As it is obvious from the demonstrated results, the predicted maximum horizontal displacements by the AMM are underestimated by those of the analytical solution. By increasing the applied load, the relative discrepancies between the maximum horizontal displacements of these two methods would reduce. The maximum relative error between the above-mentioned results occurs just prior to postbuckling initiation (i.e., elastic buckling) and it is approximately equal to 7.5%. Regarding the predicted maximum transverse displacements of microbeams, those of AMM are underestimated by the analytical solution up to a certain value of the applied load. For loads greater than this particular value, the predicted maximum transverse displacements by the AMM are overestimated by those of the analytical solution and their relative discrepancies would grow by increasing of the applied load. For both SS and CF microbeams and for the considered range of the applied load, the discrepancies between such results are lower than 4.5%.

5.2. Load-displacement of microbeams with various ends

In Figs. 4(a)–(d), the plots of axial load in terms of maximum displacements have been provided based on the suggested analytical solution for SS, CC, CSF, and CF boundary conditions. As it is seen, the plots of load-maximum transverse displacement consist of two apparent branches. In the first branch, the maximum transverse displacement grows in terms of the magnitude of the applied force. Such a fact is predicted by both CET and MCST. For a given force, the obtained maximum values of transverse displacements based on the CET are greater than those of the MCST. This is mainly related to the positive incorporation of the material length scale into the bending rigidity of the microbeam. By increasing of the applied load, the relative discrepancies between the results of the CET and those of the MCST would decrease. Actually, in the first branch, the role of the material length scale parameter on the deformation would generally reduce as the axial load increases. Such a fact holds true for all considered boundary conditions. In the second branch, the maximum deflection would lessen by growing of the applied load and the relative discrepancies between the results of the MCST and those of the CET would commonly increase. Concerning the graphs of load in terms of maximum horizontal displacement, the predicted maximum horizontal displacement would increase as the applied load increases. Generally, the obtained maximum horizontal displacements based on the MCST are lower than those obtained based on the CET. As it was explained earlier, the main reason of this fact is that the bending rigidity of the microbeam based on the MCST is greater than that of the CET.

5.3. Large deformed shapes of microbeams with various ends

An interesting study is performed to show the postbuckled shapes of microbeams of various end conditions under different levels of the applied load using analytical solution. In Figs. 5(a)–(d), the plots of the highly deformed microbeam under various loads correspond to $a = 50, 80$, and $100^\circ$ based on the MCST have been demonstrated for SS, CC, CSF, and CF boundary conditions. In the case of SS microbeam (see Fig. 5(a)), the corresponding forces to the above-mentioned values of $a$ in order are...
Fig. 2. The predicted postbuckling paths of the microstructure by the suggested analytical method and AMM by considering the length scale: (a) SS microbeam, (b) CF microbeam; (–○–) AMM in the case of $\cos^2 \left( \frac{\pi}{2} \right) = 1$, (−−) analytical method.

Fig. 3. The predicted postbuckling paths of the microstructure by the suggested analytical method and AMM by considering the length scale: (a) SS microbeam, (b) CF microbeam; (–○–) AMM in the case of $\cos^2 \left( \frac{\pi}{2} \right) \neq 1$, (−−) analytical method.
equal to 2910.110, 3414.343, and 3981.726 μN. Based on the CET, a close scrutiny reveals that these forces lead to \( \alpha = 65.002, 89.638, \) and 107.846 °. For microbeams with CC end conditions on the basis of MCST (see solid lines in Fig. 5(b)), the applied forces pertinent to \( \alpha = 50, 80, \) and 100 ° are 11,640.484, 13,666.872, and 16,038.177 μN, respectively. Such forces cause deformation of microbeams with \( \alpha = 62.221, 89.400, \) and 107.110 ° based on the CET. Regarding CSF microbeams (see Fig. 5(c)), the suggested model based on the MCST predicts that axial forces of

![Analytical postbuckling curves for highly stretchable microbeams under various boundary conditions: (a) SS, (b) CC, (c) CSF, (d) CF; ([––]) CET, (—) MCST; (□) \( N_{w_{\text{max}}}, \) (○) \( N_{u_{\text{max}}}, l_b = 3 \text{ mm}).

Fig. 5. The postbuckled shapes of microbeams for different levels of the applied load: (a) SS, (b) CC, (c) CSF, (d) CF; ([––]) CET, (—) MCST; (□) \( \alpha = 50°, \) (▵) \( \alpha = 80°, \) (○) \( \alpha = 90°, \) \( l_b = 3 \text{ mm}).

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5.4. Effect of thickness on mechanical response of microbeams

In this part, the influence of thickness on the nonlinear response of uniaxially loaded microbeams of various end conditions is going to be investigated via the suggested analytical solution. In Figs. 6 and 7, the plots of applied load in terms of maximum values of horizontal and transverse displacements have been presented for three levels of the microbeam’s thickness (i.e., \( h = 50, 100, \) and \( 150 \) \( \mu \text{m} \)). The required load to cause a specified displacement would magnify as the thickness increases. For higher levels of microbeam’s thickness, the slopes of displacement-load curves are commonly lower. In other words, for higher values of thickness, variation of the applied load is more influential on the variation of both horizontal and transverse displacements. For all considered levels of the thickness ratio, the required load to produce a particular displacement based on the MCST is higher than that obtained based on the CET. The relative discrepancies between these loads based on the MCST and those of the CET are commonly not affected by the variation of the thickness ratio. For example, further study indicates that such discrepancies are about 6.7% for all considered values of thickness ratio and all given boundary conditions.

5.5. Effect of slenderness ratio on mechanical response of microbeams

We are also interested in the role of the slenderness ratio on the nonlinear deformation of axially loaded microbeams under various boundary conditions. In Figs. 8 and 9, the graphs of load in terms of maximum displacements have been demonstrated for three levels of the slenderness ratio (i.e., \( \lambda = 100, 150, \) and \( 200 \)) and various boundary conditions by using analytical solution. According to the plotted results, the axial load bearing capacity of the microbeam would reduce as its slenderness ratio grows. Such a fact is valid for all considered boundary conditions. For microbeams with higher slenderness ratio, variation of the applied load is more influential on the variation of both horizontal and transverse displacements. For all considered levels of the slenderness ratio, the required load to produce a particular displacement based on the MCST is higher than that obtained based on the CET. The relative discrepancies between these loads based on the MCST and those of the CET are commonly not affected by the variation of the slenderness ratio. For example, further study indicates that such discrepancies are about 6.7% for all considered values of slenderness ratio and all given boundary conditions.

6. Concluding remarks

In the context of modified couple stress theory, large deformations of axially loaded microbeams are examined under simply supported, clamped-clamped, clamped-shear free, and clamped-free boundary conditions. The nonlinear governing equation is derived using Euler-Bernoulli beam model accounting for the material length scale factor. By introducing the geometrical conditions of the microbeam with various ends, the explicit expressions of axially applied load as well as horizontal and vertical displacements in terms of the maximum slope of the microbeam are obtained. A Galerkin-based approach is also developed to study nonlinear load-displacement of microbeams. In several particular cases, the analytically obtained results are verified.
with those predicted by the assumed mode method and a good achievement is reported. Under various levels of the applied load, the whole deformation regime of highly deflected-rotated microbeams are demonstrated based on both classical elasticity and modified couple stress theories. The nonlinear load-maximum displacements plots are displayed and the roles of the thickness, slenderness ratio, and material length scale factor on the mechanical behavior of microbeams are explained and discussed. The drawbacks of the classical elasticity

Fig. 7. Role of the thickness on the plots of \( N-u_{\text{max}} \): (a) SS, (b) CC, (c) CSF, (d) CF; (\(-\cdot\)) CET, (\(-\cdot\cdot\)) MCST; (\(\square\)) \( h = 50 \), (\(\triangle\)) \( h = 100 \), (\(\bigcirc\)) \( h = 150 \) μm; \( b = 3 \) mm.

Fig. 8. Role of the slenderness ratio on the plots of \( N-u_{\text{max}} \): (a) SS, (b) CC, (c) CSF, (d) CF; (\(-\cdot\)) CET, (\(-\cdot\cdot\)) MCST; (\(\square\)) \( \lambda = 100 \), (\(\triangle\)) \( \lambda = 150 \), (\(\bigcirc\)) \( \lambda = 200 \).
theory in predicting nonlinear behavior of the microbeam under various end conditions are also studied.

Rationally, the presented models could be exploited for evaluating nonlinear static response of highly stretchable slender microbeams. Surely, for those microbeams with moderate and low levels of slenderness ratio, the modified couple stress theory in conjunction with shear deformable beam models (i.e., Timoshenko or higher-order) should be employed to capture the shear deformation as well. Additionally, this work has been inherently established based on one material length scale parameter. It is anticipated that development of more complete models with three material length scale factors would increase our knowledge on large deformation of such micro-scale structures. To capture postbuckling behavior of highly stretchable microbeams more accurately, these crucially displayed issues could be followed by interested investigators.

References


