Application of nonlocal higher-order beam theory to transverse wave analysis of magnetically affected forests of single-walled carbon nanotubes

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1. Introduction

Vertically aligned single-walled carbon nanotubes (SWCNTs) display exceptional nanostructure composed of nanotubes directed along their longitudinal axes. There exist many experimental approaches to align groups of SWCNTs in a specific direction including thermal chemical vapor deposition (CVD) [1], plasma enhanced CVD [2], electrophoretic deposition [3], and mechanical strain [4,5]. Due to the brilliant mechanical and physical properties of SWCNTs, vertically aligned nanotubes have been now recognized as potential elements for a diverse range of applications such as field emission nanodevices [6], carbon fiber ropes [7], nanosensors (both physical and chemical) [8], adhesive nanofilms [9], transistors [10,11], fuel cells [12], energy harvesters and supercapacitors [13,14]. If the mechanical properties of consisting SWCNTs of the group could be somehow modified, it is anticipated that the efficiency of the made nanodevice would be increased. It is experimentally validated that the mechanical properties of SWCNT-based nanocomposites could be improved by excreting a magnetic field [15]. When a magnetic field is applied on such nanostructures, not only the SWCNTs are aligned along the direction of the magnetic field [16,17], but also their mechanical behavior is enhanced [15,18]. To date, no comprehensive mechanical model has been developed to explain vibrations of and characteristics of elastic sound waves in vertically aligned forest SWCNTs (FSWCNTs) with three-dimensional configuration.

Recently, it has been proved that the mechanical properties of arrays of single-walled carbon nanotubes (SWCNTs) are enhanced along the direction of the applied magnetic field, however, no study has been performed to explain characteristics of waves within magnetically affected forests of SWCNTs with three-dimensional configuration. Using nonlocal higher-order beam theory, characteristics of traveling transverse waves in vertically aligned jangles of SWCNTs in the presence of a longitudinal magnetic field are going to be explored carefully. Both nonlocal discrete and continuous models are established and the capabilities of the continuous model in capturing the characteristics of waves by the discrete model are declared. The newly established continuous model would be very useful in examining vibrations of magnetically affected nanosystems with high population in which the discrete model suffers from huge amount of computational effort and labor costs. Roles of magnetic field strength, small-scale parameter, radius of SWCNTs, intertube distance, longitudinal wavenumber, and number of constitutive SWCNTs of the nanosystem on dispersion curves, phase velocities, and group velocities are investigated. The influences of nonlocality and magnetic field strength on these characteristics of elastic waves are also highlighted.

Keywords: Transverse waves Forest of single-walled carbon nanotubes Longitudinal magnetic field Nonlocal higher-order beam theory Discrete model Continuous model

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age, nonlocal elasticity theory (NET) is suggested by Eringen [25–27]. The main feature of this advanced theory in elastic solids is the nonlocality of stress fields. The kernel function and the small-scale parameter are among significant factors that influence on the strength of nonlocality. For example, by choosing the Dirac delta function as the kernel function, the nonlocal stresses as well as the nonlocal equations of motion are reduced to their local counterparts. Wang and Hu [28] examined flexural wave propagation in armchair (5,5) and (10,10) SWCNTs for a wide range of wave numbers by exploiting the nonlocal beam theories and the molecular dynamic approach (MDA). The obtained results displayed that the local models cannot appropriately capture the dispersion curves of SWCNTs acted upon by transverse waves with high wavenumbers. It was also noticed that the nonlocal Timoshenko beam theory could give a better prediction for the dispersion curves of both SWCNTs than the nonlocal Euler–Bernoulli beam theory. By employing Mindlin-type shell model as well as the MDA, Arash and Ansari [29] investigated free transverse vibrations of both cantilevered and fully clamped SWCNTs. They showed that the predicted fundamental frequencies by the MDA are generally overestimated the results of the local shell-based model, particularly for lower length to diameter ratios; however, the predicted results by the nonlocal shell model on the basis of the small-scale parameter equal to 1.7 and 2 nm could reasonably reproduce the fundamental frequencies of fully clamped and cantilevered SWCNTs, respectively, based on the MDA. More detailed explanations on the importance of nonlocality in examining dynamic response of nanotubes could be found in Ref. [30]. Further, calibration of the nonlocal factor (i.e., small-scale parameter) for vibrational problems of carbon nanotubes has been displayed by several scholars [31,32]. In this study, the nonlocal governing equations pertinent to each tube and the whole nanostructure are established by employing higher-order beam model in the context of the nonlocal continuum theory of Eringen.

So far, various aspects of vibrations of individual SWCNTs including free vibration [33–39], free and forced vibrations due to moving inside fluid flow [40–44] and moving nanoparticles [45,46], free vibration of nanotubes and nanobeams in the presence of a longitudinal magnetic field [47–50], forced vibrations of magnetically affected SWCNTs conveying fluid flow [51–54], nonlinear vibrations [55,56], mechanical-sensing analysis of SWCNTs [57–59], chaos in embedded SWCNTs and conveying-fluid nanotubes [60–62], and axial dynamic buckling of SWCNTs [63] have been studied. In Refs. [60–63], generalized multi-symplectic method [64,65], an appropriate methodology for examining local dynamic behaviors of the infinite dimensional dynamical systems, was employed. Additionally, free and forced vibrations of ensembles of vertically aligned SWCNTs with membrane or jungle configurations [66–70] as well as their axial buckling [71,72] have been addressed by exploiting nonlocal Rayleigh, Timoshenko, and higher-order beam theories. However, sound waves characteristics in magnetically affected jungles of SWCNTs have not been displayed and investigated.

This paper deals with characteristics of elastic sound waves within vertically aligned FSWCNTs subjected to a longitudinal magnetic field using nonlinear higher-order beam model (NHOBM). To construct the nonlocal equations of motion of the magnetically affected nanosystem, the expressions of kinetic energy, the elastic strain energy, and the work done on the nanosystem by the longitudinal magnetic field, are appropriately stated. Subsequently, the nonlocal discrete governing equations are derived by employing Hamilton’s principle. For a magnetically affected nanosystem consists of N nanotubes, there exists 4N coupled equations; thereby, wave analysis of a highly populated nanosystem is associated with a huge amount of calculations. In order to reduce labor and computational costs, an appropriate continuous model is developed and its effectiveness would be explored through various numerical examples. By exploiting such a helpful nonlocal continuous model, the number of equations of motion of the nanosystem describe its transverse vibration would reduce to four, irrespective of the number of SWCNTs. Based on the newly developed continuous model, the nonlocal dispersion relation is explicitly given and the effects of influential factors on characteristics of transverse flexural and shear waves are studied for various levels of the magnetic field strength and small-scale parameter. This work could be regarded as a basis for better understanding the mechanical behavior of more complex systems like as FSWCNTs with randomly distributed tubes which are acted upon by magnetic fields. The influences of randomness and three-dimensionality effect of the magnetic field on vibrations of FSWCNTs and study characteristics of transverse sound waves within them could be considered as hot topics for future works.

2. Statement of the problem and evaluation of vdW forces

2.1. The nanomechanical problem

A vertically aligned FSWCNTs consists of $N_x$ and $N_z$ nanotube along the $y$ and $z$ directions is considered (see Fig. 1(a)). This nanosystem is subjected to a longitudinal magnetic field of strength $H_z$ and study its capability in transferring transverse waves is of high concern from mechanical points of view. According to the equivalent continuum structure model, each nanotube is replaced by a circular cylindrical shell of inner and outer radii of $r_1$ and $r_2$ such that its mean radius, $r_m$ is equal to the radius of the nanotube and its thickness is $t_0=0.34$ nm. By exertion a longitudinal magnetic field on the nanosystem, a transverse magnetic force would be applied on the $(m, n)$th nanotube as [47,48]:

$$f_m = f_m(x_m) + f_m(y_m),$$

where $f_m(x_m)$ is the magnetic force density, and $f_m(y_m)$ represents the deflections of the $(m, n)$th tube along the $y$ and $z$ directions, respectively, $\eta$ denotes the magnetic permeability of the nanotube, and $H_z$ is the magnetic field strength. Each tube is dynamically interacted with its adjacent one via van der Waals (vdW) forces as explained in the following part. These intertube forces play a vital role in dynamical analysis of the vertically aligned FSWCNTs.

2.2. Evaluation of vdW forces between doubly parallel vibrating infinite SWCNTs

According to the Lennard–Jones potential function between two neutral atoms:

$$\Phi(\lambda) = 4\varepsilon\left(\frac{\sigma^6}{\lambda^6} - \frac{\sigma^12}{\lambda^{12}}\right),$$

where $\sigma = \frac{\sigma_0}{\sqrt{2}}$ denotes the corresponding distance at which the potential function takes the zero value, $r_m$ represents the distance at which the potential function reaches its minimum value, $\lambda$ is the distance between $i$th atom and $j$th atom, and $\epsilon$ represents the potential well depth. The vdW force between atoms $i$ and $j$ is evaluated from the following relation:

$$f = -\frac{d\Phi}{d\lambda} e_i,$$

where $d$ is the differential sign, $e_i$ represents the unit vector associated with the position vector $\lambda$. By considering $(x_1, r_m\cos \varphi_1, r_m\sin \varphi_1)$ and $(x_2, r_m\cos \varphi_2, d + r_m\sin \varphi_2)$ as the coordinates of the $i$th atom from the first tube and the $j$th atom from the second tube, the position vector is expressed as follows:

$$\lambda = (x_2 - x_1) e_i + (r_m(\cos \varphi_2 - \cos \varphi_1) - \Delta V) e_j + (r_m(\sin \varphi_2 - \sin \varphi_1) + d - \Delta W) e_z,$$

where $\Delta W(x, t) = W_1(x, t) - W_2(x, t)$. $\Delta V(x, t) = V_1(x, t) - V_2(x, t)$. $W_1(x, t)/V_1(x, t)$ and $W_2(x, t)/V_2(x, t)$ denote the transverse displacements of these tubes along the z/y axis, $e_z, e_y, e_x$ represent the unit base vectors of the considered rectangular coordinate system. By substituting Eq. (3) into Eq. (2), the components of the total vdW force in Cartesian coordinate system are given by:

$$F_z = \frac{24\varepsilon a^2 C_{NT}^2}{\sigma^2 L} \left[ \int_0^L \int_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \left[ \frac{2}{3} \left( \frac{\sigma}{2} \right)^{12} - \left( \frac{\sigma}{2} \right)^6 \right] \right]$$
Magnetically affected forest of SWCNTs

Fig. 1. (a) Schematic representation of a vertically aligned FSWCNTs acted upon by a longitudinal magnetic field and (b) a continuum-based model for transverse wave analysis within magnetically affected FSWCNTs.
where \( \sigma_{CNT} = \frac{4\sqrt{3}}{\sqrt{\pi}} \) denotes the carbon atom's surface density, \( a \) represents the bond length of carbon–carbon, and \( L \) is an arbitrary length of the nanotube. By linear modeling of variation of vdW force due to the relative transverse displacements of two adjacent tubes, the transverse components of variation of the total vdW force are evaluated in the following form:

\[
\Delta F_z = C_{\perp} \Delta V, \\
\Delta F_x = C_{\parallel} \Delta W,
\]

where the coefficients of vdW forces are as:

\[
C_{\perp}(r_m, d) = \frac{256 \epsilon r_m^2}{9a^2} \int_0^{2\pi} \int_0^{2\pi} \left\{ \frac{\sigma_1}{2} \left[ Y_1 \lambda_{13} - 14 Y_4 \lambda_{15} (r_m (\cos \varphi_2 - \cos \varphi_1))^2 \right] - \frac{\sigma_2}{3} \left[ Y_1 \lambda_{17} - 8 Y_4 \lambda_{19} (r_m (\cos \varphi_2 - \cos \varphi_1))^2 \right] \right\} \, d\varphi_1 \, d\varphi_2,
\]

(6a)

\[
C_{\parallel}(r_m, d) = -\frac{256 \epsilon r_m^2}{9a^2} \int_0^{2\pi} \int_0^{2\pi} \left\{ \frac{\sigma_1}{2} \left[ Y_1 \lambda_{13} - 14 Y_4 \lambda_{15} (d + r_m (\sin \varphi_2 - \sin \varphi_1))^2 \right] - \frac{\sigma_2}{3} \left[ Y_1 \lambda_{17} - 8 Y_4 \lambda_{19} (d + r_m (\sin \varphi_2 - \sin \varphi_1))^2 \right] \right\} \, d\varphi_1 \, d\varphi_2,
\]

(6b)

and

\[
Y_1 = \frac{2114}{1024}, \quad Y_2 = \frac{4296}{2048}, \quad Y_3 = \frac{58}{16}, \quad Y_4 = \frac{358}{128},
\]

(7a)

\[
\Lambda(\varphi_1, \varphi_2; r_m, d) = \sqrt{2r_m^2 (1 - \cos(\varphi_2 - \varphi_1)) + d^2 + 2r_m d (\sin \varphi_2 - \sin \varphi_1)}.
\]

(7b)

Based on Eq. (5), the vdW interactional forces between each pair of nearest tubes could be modeled via continuous elastic springs of constants \( C_{\perp} \) and \( C_{\parallel} \) (see Fig. 1(b)). It could be easily shown that the interactions between each two SWCNTs along the diagonal direction and perpendicular to that could be simulated via continuous springs of constants \( C_{\perp} \) and \( C_{\parallel} \), respectively, such that \( C_{\perp} = C_{\perp}(r_m, d \sqrt{2}) \) and \( C_{\parallel} = C_{\parallel}(r_m, d \sqrt{2}) \). In the following parts, the discrete and continuous equations of motion of the nanosystem according to the calculated vdW forces are derived using a higher-order beam model in the framework of the nonlocal continuum field theory of Eringen.

3. Construction of a nonlocal discrete model

3.1. Governing equations of magnetically affected FSWCNTs using discrete NHOBM

Using higher-order beam model of Reddy–Bickford [22–24], the kinetic energy, \( T \), and the strain energy of the magnetically affected FSWCNTs, \( U \), and the work done by the applied longitudinal magnetic field on the nanosystem, \( W \), in the framework of the nonlocal continuum theory of Eringen are given by:

\[
T(t) = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \int_0^\infty \left\{ \frac{d^4}{dt^4} (\varphi_{mn} - \alpha \dot{\varphi}_{mn})^2 + \frac{d^2}{dt^2} (\dot{\varphi}_{mn})^2 \right\} \, dx,
\]

(8a)

\[
U(t) = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^{N} \int_0^\infty \left\{ \frac{d^4}{dt^4} (\varphi_{mn} - \alpha \dot{\varphi}_{mn})^2 + \frac{d^2}{dt^2} (\dot{\varphi}_{mn})^2 \right\} \, dx,
\]

(8b)

\[
W(t) = \sum_{m=1}^{N} \sum_{n=1}^{N} \int_0^\infty \eta A_h H_0^2 \left( \frac{d^2}{dt^2} W_{mn} + \frac{d^4}{dt^4} W_{mn} \right) \, dx,
\]

(8c)

where \( \varphi_{mn} \) and \( W_{mn} \) are deflections along the y and z axes, \( \Psi_{mn} \) and \( \Psi_{mn} \) are the angles of deflections about the y and z axes, \( M_{bmn} \) and \( M_{bmn} \) and \( M_{bmn} \) and \( M_{bmn} \) in order denote nonlocal first-order/third-order moments about the y and z axes, \( Q_{bmn} \) and \( Q_{bmn} \) and \( Q_{bmn} \) represent the nonlocal shear forces of the (m, n)th SWCNT along the y and z axes, respectively, \( X_{mn} = \frac{d^2}{dt^2} (W_{mn} + V_{mn}) \), and \( Y_{mn} = \frac{d^2}{dt^2} (L_{mn} + F_{mn}) \).

In the context of the NHOBM, the nonlocal forces in the (m, n)th SWCNT would be stated as [72–74]:

\[
\dot{E} \left\{ M_{bmn} \right\} = J_2 \frac{d^2}{dt^2} \varphi_{mn} - \alpha J_4 \left( \frac{d^2}{dt^2} W_{mn} + \frac{d^4}{dt^4} W_{mn} \right),
\]

(9a)

\[
\dot{E} \left\{ M_{bmn} \right\} = J_2 \frac{d^2}{dt^2} \varphi_{mn} - \alpha J_4 \left( \frac{d^2}{dt^2} W_{mn} + \frac{d^4}{dt^4} W_{mn} \right),
\]

(9b)

\[
\dot{E} \left\{ Q_{bmn} \right\} + a \frac{d}{dx} \frac{d^4}{dx^4} \varphi_{mn} = \kappa \left( \Psi_{mn} + \frac{d}{dx} \frac{d^2}{dx^2} W_{mn} \right) + a J_4 \frac{d^2}{dx^2} \frac{d^2}{dx^2} \varphi_{mn} - a^2 J_4 \frac{d^2}{dx^2} \frac{d^2}{dx^2} \varphi_{mn}.
\]

(9c)

\[
\dot{E} \left\{ Q_{bmn} \right\} + a \frac{d}{dx} \frac{d^4}{dx^4} \varphi_{mn} = \kappa \left( \Psi_{mn} + \frac{d}{dx} \frac{d^2}{dx^2} W_{mn} \right) + a J_4 \frac{d^2}{dx^2} \frac{d^2}{dx^2} \varphi_{mn} - a^2 J_4 \frac{d^2}{dx^2} \frac{d^2}{dx^2} \varphi_{mn}.
\]

(9d)
where the new operator and parameters in recent relations are as:

\[ \Xi \{ \cdot \} = \{ \cdot \} - (e_0a)^2 \frac{d^2}{dx^2}, \quad a = 1/(3c_0^2), \quad \kappa = \frac{\int G_d(1 - 3a^2) \, dA}{\int A_d \, dA}. \]

\[ I_n = \int_{A_d} \kappa \, dA, \quad J_n = \int_{A_d} E_0 \, dA; \quad n = 0, 2, 4, 6, \]

(10)

where \( e_0 \) represents the small-scale parameter, \( E_0 \) is the Young's modulus, \( G_d \) is the shear elastic moduli, \( A_d \) is the cross-sectional area, and \( \kappa \) is the density of the nanobeam. By employing the Hamilton’s principle, \( \int \left[ (\delta T - \delta U + iW) \right] \, dx = 0 \), and by using Eqs. (9a)–(9d), the nonlocal discrete equations of motion of vertically aligned FSWCNTs under longitudinal magnetic fields on the basis of the NHOBM are derived as provided in Appendix A.

Now the following dimensionless quantities are taken into account:

\[ x = \frac{x}{l_0}, \quad \tau = \frac{a}{l_0} \sqrt{\int J_0 l_0 \mu}, \quad \xi = \frac{e_0a}{l_0}, \quad \bar{V}_m = \frac{V_m}{l_0}, \quad \bar{W}_m = \frac{W_m}{l_0}. \]

\[ \bar{\Psi}_{mn} = \frac{\Psi_{mn}}{\kappa \bar{V}_m}, \quad \bar{\Psi}_{am} = \frac{\Psi_{am}}{\kappa \bar{V}_m}. \]

\[ \bar{H}_x = H_x \sqrt{\frac{\eta \kappa l_0^2}{\kappa}}, \quad \bar{r}_1 = \frac{a_1 - a_2 l_t}{l_0^2}, \quad \bar{r}_2 = \frac{a_2 l_t}{l_0^2}, \quad \bar{r}_3 = \frac{\kappa l_0^2}{\kappa r_2}. \]

\[ \bar{r}_4 = \frac{a_1 - a_2^2}{a_1 J_0}, \quad \bar{r}_5 = \frac{a_1 - a_2^2}{a_1 J_0}, \quad \bar{r}_6 = \frac{\kappa r_5}{a_2 J_0}. \]

\[ \bar{C}_{d1} = \frac{C_{d(3)}}{a_2^2 J_0}, \quad \bar{C}_{d2} = \frac{C_{d(5)}}{a_2^2 J_0}. \]

(11)

By introducing Eq. (11) to Eqs. (A.1a)–(A.1d), the dimensionless equations of motion of the constitutive tubes of the FSWCNTs acted upon by the longitudinal magnetic field are expressed as given in Appendix B.

3.2. Wave analysis of magnetically affected FSWCNTs based on the discrete NHOBM

By employing a harmonic form for the transverse waves in the constitutive SWCNTs of the nanosystem as follows:

\[ < \bar{V}_m, \bar{W}_m, \bar{W}_m, \bar{V}_m, \bar{V}_m, \bar{V}_m >= < \bar{V}_m, \bar{V}_m, \bar{W}_m, \bar{W}_m, \bar{W}_m, \bar{W}_m > e^{i (\alpha x - \gamma z)}, \]

(12)

where \( i = \sqrt{-1}, \bar{V}_m, \bar{W}_m, \bar{W}_m, \bar{V}_m, \bar{V}_m \) represent the dimensionless amplitudes of the transverse waves, and \( \alpha \) and \( \gamma \) in order are the dimensionless frequency and longitudinal wavenumber. Now by substituting Eq. (12) into Eqs. (B.1a)–(B.1d); \( -\sigma^2 \bar{M} + \bar{K} \bar{X}_0 = 0 \) where \( \bar{M} \) and \( \bar{K} \) are calculated readily. The if and only if condition for existence of a non-trivial solution to such a set of linear equations is det \( -\sigma^2 \bar{M} + \bar{K} \) \( = 0 \). By solving this relation for the dimensionless frequencies, the frequency of the transverse waves for every value of the wavenumber is calculated by \( \omega = \frac{\sigma \gamma}{\alpha} \sqrt{\frac{l_0}{l_0}} \).

4. Construction of a nonlocal continuous model

4.1. Governing equations of magnetically affected FSWCNTs using continuous NHOBM

By exploiting Eqs. (A.1a)–(A.1d), the governing equations associated with the transverse motion of the \( (m,n) \)th interior SWCNT of magnetically affected ensemble are written as:

\[ \Xi \left\{ \left( I_2 - 2aI_4 + a^2 I_6 \right) \frac{d^2 \bar{V}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{V}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{V}_m}{dx^2}\right\} + \kappa \left( \bar{V}_m + \frac{\partial \bar{W}_m}{\partial x} \right) - (J_2 - 2aJ_4 + a^2 J_6) \frac{d^2 \bar{V}_m}{dx^2} + \right. \]

\[ + \left( aJ_2 - a^2 J_6 \right) \frac{d^2 \bar{V}_m}{dx^2} = 0, \quad (13a) \]

\[ \Xi \left\{ \left( I_2 - 2aI_4 + a^2 I_6 \right) \frac{d^2 \bar{W}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2}\right\} + \kappa \left( \bar{W}_m + \frac{\partial \bar{V}_m}{\partial x} \right) - (J_2 - 2aJ_4 + a^2 J_6) \frac{d^2 \bar{W}_m}{dx^2} + \right. \]

\[ + \left( aJ_2 - a^2 J_6 \right) \frac{d^2 \bar{W}_m}{dx^2} = 0, \quad (13b) \]

\[ \Xi \left\{ \left( I_2 - 2aI_4 + a^2 I_6 \right) \frac{d^2 \bar{V}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2}\right\} + \kappa \left( \bar{V}_m + \frac{\partial \bar{W}_m}{\partial x} \right) - (J_2 - 2aJ_4 + a^2 J_6) \frac{d^2 \bar{W}_m}{dx^2} + \right. \]

\[ + \left( aJ_2 - a^2 J_6 \right) \frac{d^2 \bar{W}_m}{dx^2} = 0, \quad (13c) \]

\[ \Xi \left\{ \left( I_2 - 2aI_4 + a^2 I_6 \right) \frac{d^2 \bar{W}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2} + \left( a^2 I_6 - aI_4 \right) \frac{d^2 \bar{W}_m}{dx^2}\right\} + \kappa \left( \bar{W}_m + \frac{\partial \bar{V}_m}{\partial x} \right) - (J_2 - 2aJ_4 + a^2 J_6) \frac{d^2 \bar{W}_m}{dx^2} + \right. \]

\[ + \left( aJ_2 - a^2 J_6 \right) \frac{d^2 \bar{W}_m}{dx^2} = 0. \quad (13d) \]

In order to establish an appropriate continuous model based on the discrete relations in Eqs. (13a)–(13d), we define continuous displacements \( v = v(x, y, z, t) \) and \( w = w(x, y, z, t) \) as:

\[ v_{mn}(x) \approx v(x, y, \alpha z, \omega t), \quad v_{mn}(x + d, y, z - d, t) \approx v(x, y, \alpha z - d, \omega t), \]

\[ v_{mn}(x - d, y, z + d, t) \approx v(x, y, \alpha z + d, \omega t), \quad v_{mn}(x - d, y, z - d, t) \approx v(x, y, \alpha z - d, \omega t), \]

\[ v_{mn}(x + d, y, z + d, t) \approx v(x, y, \alpha z + d, \omega t), \]

\[ v_{mn}(x + d, y, z - d, t) \approx v(x, y, \alpha z - d, \omega t). \]

\[ (14a) \]

\[ w_{mn}(x) \approx w(x, y, n, \omega t), \quad w_{mn}(x + d, y, z - d, t) \approx w(x, y, n + d, \omega t), \]

\[ w_{mn}(x - d, y, z + d, t) \approx w(x, y, n + d, \omega t), \quad w_{mn}(x - d, y, z - d, t) \approx w(x, y, n - d, \omega t), \]

\[ w_{mn}(x + d, y, z + d, t) \approx w(x, y, n + d, \omega t), \quad w_{mn}(x + d, y, z - d, t) \approx w(x, y, n - d, \omega t). \]

\[ (14b) \]

The transverse displacements of the adjacent tubes to the \((m,n)\)th nanotube could be approximated by some dominant terms of the Taylor series expansion as follows:

\[ v(x, y, n, \pm d, z_{mn}, \pm d, t) \]

\[ w(x, y, n, \pm d, z_{mn}, \pm d, t) \]
\[
\begin{align*}
\Xi = \sum_{i=1}^{6} \sum_{j=0}^{i} \left( \frac{\partial^{d}}{\partial \xi^{d}} \left[ x, y_{\text{ref}}, \frac{z_{\text{ref}}}{d} \right] \right) (\pm d)^{(d-1)/2},
\end{align*}
\]

and by using Eq. (11), the dimensionless form of these equations of motion are given by:

\[
\Xi \left\{ -r^2 \left( \frac{\partial \psi_{\parallel \xi}}{\partial \xi} + \frac{\partial \psi_{\parallel \eta}}{\partial \eta} \right) \right\} + \psi_{\parallel \xi} \left( \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} \right) - r^2 \left( \frac{\partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right) = 0, \tag{18a}
\]

\[
-\frac{r^2 \left( \frac{\partial \psi_{\parallel \xi}}{\partial \xi} \right)}{\partial \eta^2} - r^2 \left( \frac{\partial \psi_{\parallel \eta}}{\partial \eta} \right) - \frac{r^2 \partial^2 \psi_{\parallel \eta}}{\partial \xi^2} = 0,
\]

\[
\Xi \left\{ -r^2 \left( \frac{\partial \psi_{\parallel \xi}}{\partial \xi} + \frac{\partial \psi_{\parallel \eta}}{\partial \eta} \right) \right\} + \psi_{\parallel \xi} \left( \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} \right) - r^2 \left( \frac{\partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right) = 0.
\]

(18c)

(18d)

by combining Eqs. (14)-(16), and Eqs. (13a)-(13d), the continuous version of governing equations of FSWCNTs acted upon by a longitudinal magnetic field in accordance with the NHOBM are obtained as:

\[
\Xi \left\{ -r^2 \left( \frac{\partial \psi_{\parallel \xi}}{\partial \xi} + \frac{\partial \psi_{\parallel \eta}}{\partial \eta} \right) \right\} + \psi_{\parallel \xi} \left( \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} \right) - r^2 \left( \frac{\partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right) = 0,
\]

(17a)

(17b)

(17c)

(17d)

\[
\Xi \left\{ -r^2 \left( \frac{\partial \psi_{\parallel \xi}}{\partial \xi} + \frac{\partial \psi_{\parallel \eta}}{\partial \eta} \right) \right\} + \psi_{\parallel \xi} \left( \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} \right) - r^2 \left( \frac{\partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right) = 0.
\]

(18b)

where

\[
\Xi \left\{ \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} - \frac{r^2 \partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right\} + \psi_{\parallel \xi} \left( \frac{\partial^2 \psi_{\parallel \xi}}{\partial \eta^2} \right) - r^2 \left( \frac{\partial^2 \psi_{\parallel \eta}}{\partial \xi^2} \right) = 0.
\]

4.2. Wave analysis of magnetically affected FSWCNTs based on the continuous NHOBM

The harmonic transverse waves in magnetically affected FSWCNTs according to the NHOBM are assumed as follows:

\[
< \bar{\psi}_{\parallel \xi}, \bar{\psi}_{\parallel \eta}, \bar{\psi}_{\parallel \zeta} > =< \bar{\psi}_{\parallel \xi}, \bar{\psi}_{\parallel \eta}, \bar{\psi}_{\parallel \zeta} > e^{i \omega \left( \xi - \xi_{0} \right)},
\]

where \(< \bar{\psi}_{\parallel \xi}, \bar{\psi}_{\parallel \eta}, \bar{\psi}_{\parallel \zeta} > = \) is the dimensionless vector of the amplitudes, \(\omega\) represents the dimensionless frequency of the transverse waves.
where the values of $\zeta_i$ and $\eta_i$ are given in Appendix C. In Eq. (21), let the determinant of the coefficient matrix of the dimensionless amplitude vector equal to zero. Thereby, the characteristic relation associated with the transverse vibrations of the magnetically affected FSWCNTs based on the NHOBM is derived as:

$$P_5m^3 + P_6m^6 + P_7m^4 + P_8m^2 + P_9 = 0,$$

(22)

where the parameters $P_i$ are provided in Appendix D. Commonly, Eq. (22) has four positive roots plus four negative roots (in some special cases, four conjugate complex roots with low imaginary parts have been also reported). The positive roots which are represented by $\omega_i;i=1,...,4$, denote the dimensionless frequencies of the transverse waves in the magnetically affected nanosystem according to the NHOBM. The phase velocities, $v_{\phi_i}$, and the group velocities, $v_{g_i}$, pertinent to such frequencies are also defined by:

$$v_{\phi_i} = \frac{\omega_i}{k_x},$$

$$v_{g_i} = \frac{\partial \omega_i}{\partial k_x},$$

(23)

where

$$\frac{\partial \omega_i}{\partial k_x} = -\left(\frac{P_5}{8P_3m^3} + \frac{P_6}{4P_3m^3} + \frac{P_7}{4P_3m^3} + \frac{P_8}{2P_3m^3} \right).$$

(24)

5. Results and discussion

Consider magnetically affected FSWCNTs with the following data: $t_0=0.34$ nm, $r_0=1$ nm, $\rho_0=2500$ kg/m$^3$, and $V_f=0.2$. In this part, capabilities of the suggested continuous model in predicting the results of the discrete model are investigated. In the remainder, effects of the wavenumber, population of the nanosystem, intertube distance, radius of SWCNTs on the characteristics of transverse waves under various levels of the small-scale parameter and magnetic field strength are examined and discussed.

5.1. A study on the efficiency of the continuous model

To investigate the effectiveness of the newly established continuous model, a comparison study is performed. To this end, for a special transverse waves with $k_x = k_x = k$ and $\theta = \frac{\pi}{2}$, the predicted lowest frequencies by the proposed discrete model and those of the suggested continuous model are presented in Table 1 for different populations and various radii of SWCNTs. The results are provided for three levels of radius of SWCNTs (i.e., $r_0=1, 1.5$, and $2$ nm), four values of the magnetic field strength (i.e., $B_z=0, 0.1$, and $0.2$), and three levels of the longitudinal wavenumber: $\tilde{k}_x = \tilde{k}_y = \frac{2\pi}{N_x}$ and $\frac{2\pi}{N_y}$. Commonly, the predicted characteristics of both flexural and shear waves would reduce by increasing of the small-scale parameter. As it is seen, the influence of the small-scale parameter on characteristics of flexural and shear waves with larger numbers (or lower wavelengths) is more obvious. In fact, for transverse waves with lower wavelengths, the effect of shear deformation on vibrations of the nanosystem becomes highlighted. Due to the incorporation of the small-scale parameter into the shear strain energy of the nanosystem as well as because of approaching of the wavelength to the interatomic distances, the role of nonlocality on characteristics of waves with higher longitudinal wavenumbers becomes more crucial. Irrespective of

On the basis of the proposed continuous model, a more detailed investigation of the influences of the radius of SWCNTs, magnetic field strength, and population of the nanosystem on dispersion curves of both flexural and shear waves are provided in the following parts.

5.2. Role of the magnetic field strength on wave’s characteristics

In Fig. 2, the plots of frequencies, phase velocities, and group velocities of transverse waves in terms of the magnetic field strength have been demonstrated according to the suggested continuous model. The results have been provided for nanosystems with $r_0=1, 2$, and $3$ nm when they are exploited for transfeerring of transverse waves with $k_x = k_y = k_z = \frac{2\pi}{N_x}$ by considering three values for the small-scale parameter (i.e., $e_0=0, 1$, and $2$ nm). Irrespective of the radius of SWCNTs, the flexural frequencies as well as their corresponding phase velocities would grow by increasing of the magnetic field strength. The rate of such an increase relies on the magnetic field strength and radius of nanotubes. For lower levels of the strength of the magnetic field, the rate of increase of both flexural frequencies and phase velocities is more apparent for nanosystems with greater nanotube’s radius. Generally, the influence of the magnetic field strength is more obvious on characteristics of flexural waves. Regarding shear waves, up to a certain value of the radius, the shear frequencies and their corresponding phase velocities would reduce by growing the radius of nanotubes. For low levels of the magnetic field strength, the shear frequencies as well as shear phase velocities would slightly increase as the magnetic field strength grows. For high values of magnetic field, these characteristics of the shear waves would apparently magnify in terms of the magnetic field strength. Such a rate of increase is more obvious for those nanosystems with higher radius of SWCNTs.

In general, the predicted characteristics of both flexural and shear waves by the CET are greater than those obtained by the NET. For $r_0=1$ and $2$ nm, the relative discrepancies between the predicted flexural frequencies via the CET and those of the NET would generally lessen by growing of the magnetic field strength. In the case of $r_0=3$ nm, such discrepancies would reduce up to about $\tilde{B}_z=0.1$. For magnetic field strength greater than this value, the influence of the nonlocality on both flexural frequencies and their corresponding phase velocities would increase by increasing of the magnetic field strength. Concerning nonlocality effect on characteristics of shear waves, it could be observed that the relative discrepancies between the results of the CET and those of the NET would reduce as the magnetic field strength increases. Such a fact is more obvious for nanosystems with greater SWCNT’s radius which are under higher magnetic field strength.

5.3. Role of the small-scale parameter on wave’s characteristics

A crucial parametric study is performed to explain the effect of nonlocality on the characteristics of transverse waves. In Fig. 3, the plots of frequencies, phase and group velocities of both flexural and shear waves in terms of the small-scale parameter have been presented. The obtained results are for FSWCNTs under longitudinal magnetic field with $\tilde{B}_z=0$, $0.1$, and $0.2$. The nanosystem is aimed to transfer transverse waves with $\tilde{k}_x = \tilde{k}_y = \frac{2\pi}{N_x}$ and three levels of the longitudinal wavenumber: $\tilde{k}_z = \frac{2\pi}{N_0}$, $\frac{2\pi}{10}$, and $\frac{2\pi}{15}$. Commonly, the predicted characteristics of both flexural and shear waves would reduce by increasing of the small-scale parameter. As it is seen, the influence of the small-scale parameter on characteristics of flexural and shear waves with smaller wavenumbers (or lower wavelengths) is more obvious. In fact, for transverse waves with lower wavelengths, the effect of shear deformation on vibrations of the nanosystem becomes highlighted. Due to the incorporation of the small-scale parameter into the shear strain energy of the nanosystem as well as because of approaching of the wavelength to the interatomic distances, the role of nonlocality on characteristics of waves with higher longitudinal wavenumbers becomes more crucial.
Table 1
Prediction of the lowest frequencies of transverse waves by both discrete and continuous models for various levels of the magnetic field strength and population of the magnetically affected FSWCNTs

<table>
<thead>
<tr>
<th>ra (nm)</th>
<th>Pr</th>
<th>Nr = Nz = 5</th>
<th>Nr = Nz = 7</th>
<th>Nr = Nz = 9</th>
<th>Nr = Nz = 11</th>
</tr>
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<tr>
<td>Discrete model 1</td>
<td>0</td>
<td>2.299778</td>
<td>2.103168</td>
<td>2.026852</td>
<td>1.989990</td>
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<td></td>
<td>0.5</td>
<td>9.489761</td>
<td>9.449214</td>
<td>9.434390</td>
<td>9.427415</td>
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<tr>
<td></td>
<td>1.0</td>
<td>13.861858</td>
<td>13.859093</td>
<td>13.857944</td>
<td></td>
</tr>
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<td>14.189052</td>
<td>14.189046</td>
<td></td>
</tr>
<tr>
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<td>2.614764</td>
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<td>2.541342</td>
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<tr>
<td></td>
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<td>8.930901</td>
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</tr>
<tr>
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<td>10.598908</td>
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<tr>
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<td>10.721172</td>
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<td>2.131953</td>
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<td>2.017390</td>
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<td>9.230016</td>
<td>9.229989</td>
<td>9.229985</td>
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</table>

Fig. 2. Frequencies, phase velocities, and group velocities of transverse waves as a function of the magnetic field strength for various levels of the radius of the constitutive SWCNTs of the vertically aligned ensemble: ((o) ra=1, (□) ra=2, (△) ra=3 nm; (⋯) ea=0, (−−) ea=1, (→) ea=2 nm; kr = ky = kz = π/10; Nz = Np=100).
the values of the small-scale parameter and wavenumber, the characteristics of both flexural and shear waves would increase by growing the magnetic field strength; however, the influence of the magnetic field on characteristics of flexural waves is more apparent. Based on a brief survey of the obtained results, the CET predicts that the effect of magnetic field strength on flexural frequencies and their corresponding phase velocities of waves with lower wavenumbers is more obvious. Nevertheless, the obtained results via the suggested nonlocal continuous-based model indicate that the role of the magnetic field strength on the characteristics of both flexural and shear waves would strongly depend on the small-scale parameter. For higher values of both small-scale parameter and wavenumber, the influence of the strength of magnetic field on shear frequencies as well as their phase velocities would be more obvious.

5.4. Role of the radius of SWCNTs on wave’s characteristics

It is also of our interest to display the role of the radius of nanotubes on characteristics of transverse waves. For this purpose, the plotted results of frequencies, phase and group velocities of both flexural and shear waves in terms of radius of constitutive SWCNTs of the nanosystem have been presented in Fig. 3. The results are given for three levels of the magnetic field strength (i.e., $\vec{H}_z=0$, 0.25, and 0.5) and three values of the small-scale parameter (i.e., $\vec{e}_0\alpha=0$, 1, and 2 nm) in the case of $\vec{k}_x = \vec{k}_y = \vec{k}_z = \vec{e}_0 \alpha$. For all considered small-scale parameters, for low levels of the magnetic field strength (i.e., $\vec{H}_z \leq 0.25$), both flexural frequencies and corresponding phase velocities would increase by increasing of the radius of SWCNTs up to a certain level; For radius greater than this particular level, the aforementioned characteristics would reduce by growing radius of SWCNTs; however, for high values of magnetic field strength (i.e., $\vec{H}_z=0.5$), these characteristics of the flexural wave would reduce as the radius of nanotubes increases. Such a trend is more obvious for nanosystems with higher-levels of the small-scale parameter. Generally, the group velocities of flexural waves increase by growing of the radius of SWCNTs. Concerning shear waves, the plots of their corresponding frequencies and phase velocities take their locally minimum points at a particular value of radius. The plotted results obviously show that the location of these important points depends on both small-scale parameter and magnetic field. Additionally, the rate of variation of these characteristics as a function of radius is more apparent in the descending branch of the plots, particularly for lower levels of radius of SWCNTs. According to the plotted results, the group velocities of both flexural and shear waves would increase by an increase of the radius of nanotubes.

Regarding the role of the nonlocality on characteristics of the transverse waves, it should be noted that the influence of the small-scale parameter on flexural frequencies and phase velocities would commonly increase by increasing of the radius of SWCNTs, particularly for high strength of the applied magnetic field. For fairly low levels of the magnetic field strength, variation of the small-scale parameter has a slight influence on the relative discrepancies between the predicted shear frequencies by the CET and those obtained by the NET. Nevertheless, for high levels of the magnetic field strength (for example, $\vec{H}_z=0.5$), such relative discrepancies reduce by an increase of the radius of SWCNTs up to a certain value. For radii greater than this, the role of nonlocality on the shear frequencies would slightly grow as the radius of the nanotube increases. These explanations also hold true for phase velocities of shear waves.

5.5. Role of the intertube distance on wave’s characteristics

Understanding the influence of intertube distance on characteristics of both flexural and shear waves could provide valuable information on nanoengineering design and manufacturing of magnetically affected FSWCNTs for the considered jobs. To this end, the plots of frequencies,
phase velocities, and group velocities of transverse waves in terms of the
intertube distance have been demonstrated in Fig. 5. The obtained
results have been given for three levels of the strength of longitudinal
magnetic field (i.e., $H_l$=0.05, 0.1, and 0.15) for three values of the small-
scale parameter (i.e., $e_0$=0, 1, and 2 nm). The magnetically affected
nanosystem with 5 x 5 SWCNTs is aimed to be exploited for transfer-
ing transverse waves with $\vec{k}_x = \vec{k}_y = \frac{\omega}{v}$ and $\vec{k}_z = \frac{\omega}{v}$. Since the wave-
length of the longitudinal component of the waves has fairly a large
value (i.e., $\lambda=35$ nm), therefore, variation of the small-scale parameter
has a small influence on the characteristics of waves for different lev-
els of the intertube distances as well as various values of the magnetic
field strength. Thereby, the dotted, dashed, and solid lines are fairly co-
incident with each other. As it is observed from the plotted results in
Fig. 5, variation of the intertube distance leads to a slight variation of
the shear frequencies and shear phase velocities. By an increase of the
intertube distance, the flexural frequencies and corresponding phase ve-
clocities would harshly decrease until reaching to their locally minimum
values at a special intertube distance. Thereafter, by increasing of the in-
tertube distance, both frequencies and phase velocities of flexural waves
would mildly increase by increasing of the intertube distance. A close
scrutiny of these characteristics of transverse waves also reveals that
the relative discrepancies between the predicted results by the CET and
those of the NET would take their maximum values at such an intertube
distance. Concerning the trend of the plots of group velocities of flexu-
ral waves, their absolute maximum points occur at the aforementioned
special intertube distance. Additionally, variation of the magnetic field
strength has a slight influence on the variation of intertube distance as-
associated with such a peak point. The influence of variation of the small-

scale parameter on variations of shear frequencies as well as shear phase
velocities does not depend on the intertube distance.

5.6. Role of the longitudinal wavenumber on wave’s characteristics

One of the main properties of transverse waves within magnetically
affected FSWCNTs is the longitudinal wavenumber. It is a very crucial
task to be aware about the role of longitudinal wavenumber on frequen-
cies, phase and group velocities of both flexural and shear waves. For
this purpose, the predicted characteristics of waves in terms of the lon-
gitudinal wavenumber have been provided in Fig. 6 for three levels of
the magnetic field strength (i.e., $H_l=0$, 0.25, and 0.5) and three values
of the small-scale parameter (i.e., $e_0=0$, 0.5, and 1 nm) in the case
of $\vec{k}_x = \vec{k}_y = \frac{\omega}{v}$. According to the predicted results by the CET, the
flexural frequencies of the magnetically affected nanosystem would in-
crease in terms of the wavenumber such that the slopes of the plots per-
tinent to nanosystems under higher magnetic field strength are higher.
However, the NET predicts that the flexural frequencies increase almost
linearly with the wavenumber up to a particular value of the longitudi-
nal wavenumber. Subsequently, their values remain fairly unchanged as
the wavenumber varies. In other words, for longitudinal wavenumbers
greater than the above-mentioned value, variation of the wavenumber
has a slight influence on variation of the flexural frequencies on the ba-
sis of the NET. Also, the influence of the magnetic field strength on the
variation of characteristics of flexural waves is commonly more influen-
tial on wavenumbers lesser than the above-mentioned special value. It is
worth mentioning that the trends of the group velocities obtained by the
NET are completely different from those predicted by the CET. Actually,
the predicted group velocities of flexural waves based on the CET would
commonly increase by increasing of the wavenumber; however, those
obtained by the NET would generally take their peak points at particu-
lar levels of wavenumbers. Furthermore, for each level of the wavenum-
ber, both shear frequencies and pertinent phase velocities would be en-
hanced by growing of the magnetic field strength.
Fig. 5. Frequencies, phase velocities, and group velocities of transverse waves as a function of the intertube distance for various levels of the magnetic field strength and small-scale parameter \((\circ) \tilde{H}_x = 0.05, (\square) \tilde{H}_x = 0.1, (\triangle) \tilde{H}_x = 0.15; (...) e_0 a = 0, (\cdots) e_0 a = 0.5, (\cdots) e_0 a = 1 \text{ nm}; k_x = \frac{\pi}{50}, k_y = k_z = \frac{\pi}{2}, N_y = N_z = 5; e_0 a = 1 \text{ nm}).

Fig. 6. Frequencies, phase velocities, and group velocities of transverse waves as a function of the longitudinal wavenumber for various levels of the small-scale parameter and magnetic field strength: \((\circ) \tilde{H}_x = 0, (\square) \tilde{H}_x = 0.25, (\triangle) \tilde{H}_x = 0.5; (...) e_0 a = 0, (\cdots) e_0 a = 0.5, (\cdots) e_0 a = 1 \text{ nm}; k_x = \frac{\pi}{20}, N_y = N_z = 100).
A careful survey of the demonstrated results in Fig. 6 shows that the relative discrepancies between the characteristics of flexural waves based on the NET and those of the CET would magnify as the longitudinal wavenumber grows. This fact is commonly more obvious for magnetic-free nanosystems. In fact, by increasing of the wavenumber and by reducing the magnetic field strength, the role of the nonlocality on characteristics of flexural waves becomes highlighted.

5.7. Role of the population on wave’s characteristics

The population of FSWCNTs is one of the major properties of the magnetically affected nanosystems, however, its role on the transverse wave’s characteristics has not been examined yet. To this end, the plotted results of frequencies, phase and group velocities of the flexural and shear waves as a function of number of SWCNTs in the y-direction have been provided in Fig. 7. The results are given for three levels of the longitudinal wavenumber (i.e., \( \bar{k}_x = \frac{\pi}{50}\)) and \(\bar{k}_y\) and three levels of the magnetic field strength (i.e., \(H_y = 0, 0.1, \) and \(0.2\)) in the case of \(\bar{k}_x = \frac{\pi}{5}, \bar{k}_y = \frac{\pi}{5}, e_0 a=2\) nm. For a given longitudinal wavenumber, both flexural frequencies and corresponding phase velocities would reduce by growing population of the nanosystem. Such a rate of reduction is more obvious for lower population of nanosystem, particularly when it is acted upon by flexural waves with lower wavenumbers. For all considered values of the wavenumber, the group velocities would increase by increasing the number of SWCNTs. The rate of such an increase is also more obvious for lowly populated nanosystems under higher magnetic field and waves with lower wavenumbers. Concerning shear waves, both shear frequencies and their phase velocities would alter slightly in terms of the population of the nanosystem while the group velocities would increase as the number of nanotubes increases. The later fact is more apparent for magnetic-free nanosystems. Furthermore, variation of the magnetic field strength in the above-mentioned range would have a very tiny effect on variation of frequencies and phase velocities of shear waves. However, the magnetic field has an obvious effect on their group velocities.

A close scrutiny of the plotted results shows that the magnetic field strength is more influential on variation of both flexural frequencies and corresponding phase velocities of nanosystem with a higher number of SWCNTs. However, the role of the magnetic field strength on the change of group velocities of flexural waves would slightly reduce by growing the population of the nanosystem. Commonly, the influence of the magnetic field on frequencies and phase velocities of shear waves is not affected by the number of constitutive SWCNTs of the nanosystem.

6. Concluding remarks

Characteristics of transverse waves in vertically aligned FSWCNTs with uniformly distributed nanotubes subjected to a longitudinal magnetic field were studied. Using NHOBM, the nonlocal discrete equations of motion were constructed using Hamilton’s principle. Elastic wave analysis of these equations for highly populated FSWCNTs would take a lot of computational costs. As an alternative solution, a continuous model was established and the dispersion relation of the magnetically affected nanosystem was explicitly displayed. The main obtained results have been summarized as follows:

1. The efficacy of the newly developed continuous model in capturing the characteristics of waves of discrete models was proved through various instances. For various magnetic field strength and different population of the nanosystem, a reasonably good agreement between the predicted results by the continuous model and those of discrete one was reported.

2. The flexural frequencies and their corresponding phase velocities are enhanced by growing of the magnetic field strength. Such a fact is more obvious for vertically aligned FSWCNTs with higher radius of
SWCNTs. However, the shear frequencies as well as corresponding phase velocities would reduce with magnetic field strength up to a particular value. For magnetic field strength greater than this level, the proposed model predicts that these factors would somewhat grow.

3. Generally, the predicted flexural and shear frequencies as well as their phase and group velocities would lessen as the role of the nonlocality becomes highlighted. Such a fact is more apparent for waves with higher wavenumbers. According to the results of the proposed nonlocal continuous-based model, the influence of longitudinal magnetic field on characteristics of both flexural and shear wave does not only rely on the wavenumber of the transverse waves, but also on the small-scale factor.

4. By increasing of the intertube distance from that of the in-contact mode (i.e., $d = 2r_n + t$), the flexural frequencies and their corresponding phase (group) velocities of transverse waves would harshly decrease such that they would take their local minimum values at a special intertube distance. These values are fairly not affected by the magnetic field strength. For such intertube distances, the group velocities as well as the relative discrepancies between the predicted results by the classical elasticity theory and those of the nonlocal model would reach to their maximum values.

5. For fairly low longitudinal wavenumbers, the results of suggested nonlocal model displays that the flexural frequencies would almost increase linearly in terms of the wavenumber such that the slopes of plots associated with the higher magnetic field strength are more obvious. For wavenumbers greater than a particular value, the flexural frequencies remain fairly unchanged. Further, the role of nonlocality on both flexural and shear frequencies as well as their phase velocities would magnify as the wavenumber increases.

6. By growing the population of the nanosystem, flexural frequencies and corresponding phase velocities would reduce. This fact is more apparent for nanosystems with lower populations which are subjected to waves with lower longitudinal wavenumbers. Furthermore, the group velocities would increase as the population increases for all given longitudinal wavenumbers. Commonly, variation of the population has a trivial influence on the variation of shear frequencies and their corresponding phase velocities. However, the shear group velocities would somewhat increase as the number of constitutive tubes grows. By an increase of the population, the effect of the magnetic field strength on flexural frequencies and their corresponding phase velocities becomes highlighted.

**Appendix A. The nonlocal-discrete equations of motion**

\[ \begin{align*}
\lambda \left( I - 2aI_4 + a^2 I_6 \right) & \frac{\partial^2 \Psi_{mn}}{\partial t^2} + \left( a^2 I_6 - aI_4 \right) \frac{\partial^3 \Psi_{mn}}{\partial t^3} + aV_{mn} \frac{\partial^2 \Psi_{mn}}{\partial t \partial x} \\
& + \kappa \left( \Psi_{mn} + \frac{\partial V_{mn}}{\partial x} \right) - \left( I - 2aI_4 + a^2 I_6 \right) \frac{\partial^2 \Psi_{mn}}{\partial t^2} \\
& + \left( aI_4 - a^2 I_6 \right) \frac{\partial^3 \Psi_{mn}}{\partial t^3} = 0. \\
\end{align*} \]  

\[(A.1a)\]

\[ \begin{align*}
\lambda \left( I - 2aI_4 + a^2 I_6 \right) & \frac{\partial^2 \Psi_{mn}}{\partial t^2} - \left( a^2 I_6 - aI_4 \right) \frac{\partial^3 \Psi_{mn}}{\partial t^3} + aV_{mn} \frac{\partial^2 \Psi_{mn}}{\partial t \partial x} \\
& + \kappa \left( \Psi_{mn} + \frac{\partial V_{mn}}{\partial x} \right) - \left( I - 2aI_4 + a^2 I_6 \right) \frac{\partial^2 \Psi_{mn}}{\partial t^2} \\
& + \left( aI_4 - a^2 I_6 \right) \frac{\partial^3 \Psi_{mn}}{\partial t^3} = 0. \\
\end{align*} \]  

\[(A.1b)\]

**Appendix B. The dimensionless nonlocal-discrete equations of motion**

\[ \begin{align*}
\lambda \left( I - 2aI_4 + a^2 I_6 \right) & \frac{\partial^2 \bar{\Psi}_{mn}}{\partial \tau^2} + \left( a^2 I_6 - aI_4 \right) \frac{\partial^3 \bar{\Psi}_{mn}}{\partial \tau^3} + \bar{V}_{mn} \frac{\partial^2 \bar{\Psi}_{mn}}{\partial \tau \partial x} \\
& + \bar{\kappa} \left( \bar{\Psi}_{mn} + \frac{\partial \bar{V}_{mn}}{\partial x} \right) - \left( I - 2aI_4 + a^2 I_6 \right) \frac{\partial^2 \bar{\Psi}_{mn}}{\partial \tau^2} \\
& + \left( aI_4 - a^2 I_6 \right) \frac{\partial^3 \bar{\Psi}_{mn}}{\partial \tau^3} = 0. \\
\end{align*} \]  

\[(B.1a)\]
\[ +0.5 \mathcal{C}_{d1} \left( \mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1} \right) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{d2} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{d3} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} + \mathbf{W}_{\text{m}-1} - \mathbf{V}_{\text{m}-1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{d4} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} + \mathbf{W}_{\text{m}+1} - \mathbf{V}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{d5} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} + \mathbf{W}_{\text{m}+1} - \mathbf{V}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ - \gamma^2 \left( \frac{\partial \Psi_{m}}{\partial x} + \frac{\partial \Psi_{w}}{\partial y} \right)^2 - \gamma^2 \frac{\partial \Psi_{m}}{\partial x} \frac{\partial \Psi_{w}}{\partial y} = 0, \quad (B.1b) \]
\[ + \mathcal{C}_{c1} \left( \mathbf{W}_{\text{m}} - \mathbf{W}_{\text{m}+1} \right) (1 - \delta_N) + \left( \mathbf{W}_{\text{m}} - \mathbf{W}_{\text{m}+1} \right) (1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{c2} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{c3} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{c4} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ +0.5 \mathcal{C}_{c5} (\mathbf{V}_{\text{m}} - \mathbf{W}_{\text{m}} - \mathbf{V}_{\text{m}+1} + \mathbf{W}_{\text{m}+1}) (1 - \delta_N)(1 - \delta_N) \]
\[ - \gamma^2 \left( \frac{\partial \Psi_{m}}{\partial x} + \frac{\partial \Psi_{w}}{\partial y} \right)^2 - \gamma^2 \frac{\partial \Psi_{m}}{\partial x} \frac{\partial \Psi_{w}}{\partial y} = 0, \quad (B.1c) \]
\[ \eta_3 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ + \mathcal{C}_{dl} \left( \chi, \chi \right)^2 + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_6 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_2 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_5 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_8 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]

Appendix C. The values of \( \xi_i \) and \( \eta_i \)

\[ \xi_1 = 1 + \left( \chi, \chi \right)^2, \quad \xi_2 = - \gamma^2 \chi, \quad \xi_4 = 1. \]
\[ \xi_5 = 1 + \left( \chi, \chi \right)^2, \quad \xi_6 = - \gamma^2 \chi, \quad \xi_8 = 1. \]
\[ \eta_1 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ + \mathcal{C}_{d2} \left( \chi, \chi \right)^2 + \left( \chi, \chi \right)^2 + \mathcal{C}_{d2} (\chi, \chi) \]
\[ \eta_3 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_5 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]
\[ \eta_8 = \frac{1}{\mu \chi^2} + \left( \chi, \chi \right)^2 + \mathcal{C}_{d1} (\chi, \chi) \]

Appendix D. The values of \( P_i \)

\[ P_8 = \xi_1 \xi_2 \xi_5 \xi_6 \]
\[ P_6 = \xi_1 \xi_2 \xi_5 \xi_6 - \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_6 \xi_5 \xi_6 \]
\[ - \xi_1 \xi_2 \xi_5 \xi_6 + \xi_1 \xi_2 \xi_5 \xi_6 - \xi_1 \xi_6 \xi_5 \xi_6 
\]

(D.1)
References


