Nonlocal vibrations and potential instability of monolayers from double-walled carbon nanotubes subjected to temperature gradients

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A B S T R A C T

Using nonlocal elasticity theory of Eringen, free transverse thermo-elastic vibrations of vertically aligned double-walled carbon nanotubes (DWCNTs) with a membrane configuration is going to be explored methodically. Accounting for nonlocal heat conduction along the side walls of DWCNTs with allowance of heat dissipation from the outer surfaces, the nonlocal temperature fields within the constitutive tubes are assessed for a steady-state regime. The van der Waals interactional forces between atoms of each pair of DWCNTs as well as the innermost and outermost tubes are displayed by laterally continuous springs whose constants are appropriately evaluated. By establishing suitable discrete and continuous models on the basis of the Rayleigh and higher-order beam theories, nonlocal in-plane and out-of-plane vibrations of thermally affected nanosystems of monolayers of DWCNTs are examined and discussed. The critical values of the temperature change are stated explicitly and the role of influential factors on this crucial factor as well as the free vibration behavior are investigated in some detail. Through conducting a fairly comprehensive numerical study, the influences of the slenderness ratio, temperature change in the low and high temperatures, number of nanotubes, small-scale parameter, intertube distance, and stiffness of the surrounding environment on the nonlocal-fundamental frequency are explained. The obtained results from this work could provide crucial guidelines for the design of membranes or even jungles of vertically aligned DWCNTs as thermal interface nanostructures.

1. Introduction

A monolayer of vertically aligned double-walled carbon nanotubes (DWCNTs) is a nanosystem consists of parallel DWCNTs located adjacently in a flat plane. Actually it represents a single-layered membrane of tubes that could have many applications in nanotechnology including strain sensors [1,2], nano-electro-mechanical systems (NEMs) [3], and nano-scaled heat-sinks [4,5]. Several research groups have proposed ensembles of carbon nanotubes (CNTs) as the next generation of thermal interface materials [4–7]. Further composites from carbon nanotubes (CNTs) have been fabricated for engineering-based management and improvement of thermal conductivity [8,9]. With regard to these superior thermal properties, CNTs as well as their composites have been of focus of attention of interdisciplinary scholars, however, thermo-mechanical behavior of forests or even membranes made from CNTs has not been methodically displayed yet. To fill some part of this scientific gap, herein, we are interested in thermo-mechanical analysis of membranes of vertically aligned DWCNTs. The main objective of this work is to theoretically estimate their dominant frequencies for such nanosystems when they are subjected to longitudinally thermal gradients.

In a monolayer of vertically aligned DWCNTs, not only the atoms of the innermost and the outermost tubes of each DWCNT would interact dynamically with each other due to the existing van der Waals (vdW) forces, but also the atoms of two nearby DWCNTs would interact. More calculations reveal that the variation of vdW forces due to the in-plane deflections of DWCNTs is not the same as that of the out-of-plane deflections. This leads to dissimilar vibrational behavior of such nanosystems for the in-plane and out-of-plane cases. Herein a linear continuum-based vdW model is established for two adjacent DWCNTs and it is appropriately extended to thermo-mechanical modelling of vertically aligned DWCNTs embedded in an elastic matrix. It should be noticed that only the transverse motions of DWCNTs are taken into account in evaluating the variations of vdW forces. Further, the variations of such forces for both in-plane and out-of-plane vibrations are linearly linked to their relative in-plane and out-of-plane transverse displacements, respectively. In other words, the nonlinear terms in each direction as well as the coupled terms are excluded since linear analysis is of interest. The role of nonlinear intertube vdW forces on the local and nonlocal transverse vibrations of individual DWCNTs has been studied by Xu et al. [10] and Khosrozadeh and Hajabasi [11]. For more accurate prediction of nonlinear chaotic vibrations of thermally affected monolayers of vertically aligned DWCNTs as well as highly nonlinear forced vibration cases, consideration of the aforementioned

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effects would result in a more rational estimation of the model. This issue could be regarded as a hot topic for future works.

Through contacting the ends of the constitutive DWCNTs of a monolayer to heat sources, a nonlocal longitudinal heat flux could be generated within each tube due to the gradient of temperature at the ends. In fact, this is a nonlocal flux since the continuum theory of Eringen [12] displays that the thermal field at each point of the nanostructure could be affected by the temperatures of its nearby points. Due to the existence of heat exchange between the atoms of the outermost tubes of the monolayer with the surrounding environment, some part of the heat energy would be dissipated through the outer surface. Accounting for both nonlocality and energy dissipation across the length, we present a steady-state model to explain temperature profile along the constitutive tubes more accurately. Thereafter we proceed in calculating the resulted mechanical forces in the constitutive tubes due to the thermal loading. From theory of elastic beams’ points of view, major resulted forces within tubes would be axial thermal force and thermal bending moment. The axially thermal forces within tubes are incorporated into the transverse equations of motion when the nanosystem has been somehow restrained longitudinally at the nanotubes’ ends while the thermal bending moments have a tendency to bend nanotubes progressively. In the context of linear analysis, the axial thermal forces could influence on the natural frequencies of elastically restrained nanosystems, however, the thermal bending moments are highly incorporated into the pseudo-dynamic response of the nanosystem, and, generally, it cannot affect the frequencies of the beam-like nanosystem. The thermal bending moments could also influence on the interactional vdW forces between atoms of each pair of tubes. In the present study, temperature variation across the transverse direction of constitutive tubes does not exist, and thereby, no bending moment is generated within nanotubes.

Application of atomistic-based approaches to vibrations of monolayers of vertically aligned DWCNTs is surely compromised with a huge amount of labor costs and computational efforts. To conquer this milestone of atomic models, we employ the nonlocal continuum theory of Eringen [12–14] for modelling of both in-plane and out-of-plane dynamics of nanosystems. Until now this theory has been widely implemented for continuum-based modeling of single-, double-, and multi-walled carbon nanotubes. Concerning mechanical modeling of DWCNTs, free vibration [15–22], elastic wave analysis [23–29], nonlinear vibration [30], buckling and postbuckling [31–33], and vibrations in the presence of magnetic fields [34–36] have been addressed suitably. Regarding thermally affected individual DWCNTs, free transverse vibrations [37], instability of fluid-conveying [38], propagation of elastic waves [39], and buckling analysis [40] have been of concern of nanomechanics community during the past decade. Most of these works have been conducted in the context of the nonlocal continuum theory of Eringen by employing beam or shell models. Furthermore, transverse vibrations of membranes and forests of single-walled carbon nanotubes [41,42], their free dynamic analysis in the presence of longitudinal magnetic field [43,44], and their axial buckling [45,46] have been investigated using nonlocal Rayleigh, Timoshenko, and higher-order beam theories. This brief survey shows that the mechanical behavior of thermally affected membranes of vertically aligned DWCNTs has not been scrutinized yet. With regard to the vast applications of such nanosystems as mentioned in above, we are interested in examining their nonlocal vibrations using well-known beam theories.

In the present work, free nonlocal-transverse vibrations of elastically embedded single-layered membranes of vertically aligned DWCNTs acted upon by a longitudinal heat flux are reported. The continuum-based vdW forces for both statics and linear dynamics are displayed and the resulted elastic energy of such crucial forces is appropriately incorporated into the total elastic energy of the nanosystem. Using nonlocal elasticity theory of Eringen in conjunction with the Rayleigh and the higher-order beam theories, the equations of motion associated with both in-plane and out-of-plane vibrations of the thermally affected nanosystem are obtained. By implementing assumed mode method (AMM), the resulted equations are discretized and solved for natural frequencies on the basis of the nonlocal-discrete models. Subsequently, the appropriate nonlocal-continuous models are established and analyzed. The main feature of the newly developed models is their high capability in evaluating free dynamic response of thermally affected nanosystems with high numbers of DWCNTs. The efficacy of these newly developed models in capturing both in-plane and out-of-plane fundamental frequencies are displayed and proved through various numerical studies. A fairly comprehensive parametric study is then given and the roles of influential factors on the free dynamic response of the nanosystem are investigated and explained in some detail. By employing the newly sophisticated continuous-based models, the critical values of the temperature change for the nanosystem at hand are also derived explicitly, and the key roles of the slenderness ratio, small-scale parameter, intertube distance, population, and radius of DWCNTs on this crucial factor are revealed.

2. Defining the nanomechanical problem

Consider a membrane with $N$ vertically aligned DWCNTs whose lengths are $l_b$ and the distance between the major axes of each pair of DWCNTs is denoted by $d$ (see Fig. 1). The bottom and the top ends of
the constitutive tubes are subjected to the temperature change of $T_3$ and $T_2$, respectively, while the environmental temperature is represented by $T_{en}$. Each DWCNTs consists of doubly coaxial nanotubes of the same length such that the radii of the innermost and the outermost tubes in order are $r_m$, and $r_o$. A equivalent continuum structure (ECS) associated with each tube is a circular cylindrical shell of mean radius equal to the radius of the tube and the wall’s thickness, $t$, is set equal to 0.34 nm. The innermost and the outermost tubes interact statically and dynamically due to existing vdW forces between their constitutive atoms. Additionally, each tube interacts with that of the nearby DWCNTs due to the vdW forces. A detailed scrutiny of such static forces as well as purely dynamical part of vdW forces due to the lateral motion of neighboring tubes has been given in supplementary material, part A. Since flexural and shear transverse vibrations of thermally affected membranes from DWCNTs are of particular interest in this study, we model each tube via beam-like equivalent structures by considering the nonlocal effect. The deflections of the $i$th tube based on the nonlocal Rayleigh beam model (NRBM) in the $y$ and $z$ directions, $W^R_i(x,t)$ and $W^H_i(x,t)$, while the deflections and angles of deflections of such a tube on the basis of the nonlocal higher-order beam model (NHOBM) are represented by $W^R_i(x,t)/W^H_i(x,t)$ and $Ψ^R_i(x,t)/Ψ^H_i(x,t)$, respectively. Further, $E_y$, $G_y$, $I_y$, $A_y$, and $ρ_y$ in order represent the Young’s modulus, shear elastic modulus, moment inertia of cross-section of the ECS, cross-sectional area, and density of the $j$th tube from the DWCNTs ($j = 1, 2$). In general, the membrane is rested on an elastic foundation and it could be interacted with its surrounding medium due to existing bonds. By exploiting Pasternak spring model, the constants of continuous transverse and rotational springs are denoted by $K_x$ and $K_y$, respectively.

In this paper, we examine the dominant in-plane and out-of-plane vibrational behavior of the vertically aligned DWCNTs acted upon by temperature gradients between tubes’ ends. By employing Hamilton’s principle, nonlocal-discrete-based models using Rayleigh and higher-order beam theories are developed to provide us with the nonlocal-continuum-based equations of motion of the nanosystem. Subsequently, suitable nonlocal-continuous-based models are established and their capabilities are explained in some details.

### 3. Nonlocal temperature field analysis of nanotubes

According to the Fourier’s law for solid structures, the longitudinal heat conduction within a continuum-based DWCNTs accounting for both nonlocality and heat conduction through its innermost and outermost walls is displayed by:

$$\rho_b C_b \frac{dT}{dt} = -\frac{\partial q_i}{\partial x} - Q_{dis},$$

$$Q_{dis} = h_b P_b \left( T - T_{en} \right).$$  (1)

where $\theta$ is the partial sign, $t$ is the time factor, $T$ denotes the temperature field within the ECS associated with the nanotube, $T_{en}$ represents the temperature field of the environment, $C_b$ is the specific heat capacity, and $h_b$ is the heat transfer coefficient of the nanotube with its surrounding environment, $P_b$ is the sum of innermost and outermost pyramids of the ECS’s cross-section (i.e., $P_b = 2\pi(r_m + r_o)$), and $A_b$ is the whole cross-sectional area of both tubes (i.e., $A_b = 2\pi t_b(r_m + r_o)$).

In presenting Eq. (1), we have assumed that the temperature fields within both innermost and outermost tubes are the same and the lateral heat exchange between the constitutive tubes of the DWCNTs has been ignored.

In the context of the nonlocal continuum filed theory of Eringen [14], the nonlocal heat flux ($q_i^\theta$) is related to the local one ($q_i$) by a simple relation of the following form:

$$q_i^\theta = (\varepsilon_0 a)^2 \frac{\partial^2 q_i}{\partial x^2} = \tilde{q}_i.$$  (2)

where $\varepsilon_0 a$ represents the small-scale factor, $\tilde{q}_i = -k_b \frac{\partial T}{\partial x}$ is the longitudinal local heat flux, and $k_b$ is the thermal conductivity of the nanotube. By combining Eqs. (1) and (2), the nonlocal governing equation of heat transfer within the nanotube takes the following form:

$$\rho_b C_b \left( \frac{dT}{dt} - (\varepsilon_0 a)^2 \frac{\partial^2 T}{\partial x^2} \right) = h_b P_b \left( T - T_{en} \right) - (\varepsilon_0 a)^2 \frac{\partial^2 T}{\partial x^2}.$$  (3)

For steady-state heat conduction, Eq. (3) is reduced to:

$$\frac{d^2 T}{dx^2} - \bar{T} \left( T - \mu \frac{d^2 T}{dx^2} \right) = 0,$$  (4)

where

$$\bar{T}(\xi) = T(\xi) - T_{en}, \quad \mu = \frac{\varepsilon_0 a}{h_b}, \quad \bar{T} = \frac{h_b P_b r_b^2}{K_b A_b}.$$  (5)

Let’s assume the following boundary conditions for the ends of nanotubes:

$$T(0) = T_1 + T_{en}, \quad T(l_b) = T_2 + T_{en}.$$  (6)

By solving Eq. (4) for the given essential conditions in Eq. (6) in view of Eq. (5), the steady-state temperature field within all tubes of the nanosystem is evaluated by:

$$T(\xi) = \left( \frac{T_2 - T_1 \cosh(\beta) \sinh(\beta)}{\sinh(\beta)} + \tilde{T} \cosh(\beta) \right) \bar{T} \neq 0, \quad \tilde{T} \neq 0; \quad \bar{T} = 0; \quad (7)$$

where $\beta = \sqrt{\frac{\varepsilon_0 a}{h_b + \mu^2}}$. In Eq. (7), the second case (i.e., $\beta=0$) displays the circumstance that the lateral surfaces of both the innermost and outermost tubes have been fully insulated (i.e., no heat flux is allowed to be transferred through the lateral surface of the nanotube) while the first case is the more general one (i.e., it could be searched mathematically that the second case would be concluded from the first case using L’Hopital’s rule).

If both ends of each tube have been fixed along the longitudinal direction, the resulted axial thermal force within the $j$th tube could be readily calculated by:

$$N_j = -E_b A_b \int_0^{l_j} a_f(T) \bar{T}(\xi) \, d\xi,$$  (8)

where $a_f$ is the coefficient of longitudinal thermal expansion of CNTs which is generally temperature dependent. By assuming $a_f = a_f(T_1, T_2)$, and by substituting Eq. (7) into Eq. (8), the axial thermal forces within the constitutive nanotubes are explicitly provided by:

$$N_j = -E_b A_b a_f \bar{T} \left( T_1 + T_2 \right) \left( \cosh(\beta) - \cosh(\beta) \right) \frac{1}{2}; \quad \bar{T} \neq 0; \quad \bar{T} = 0.$$  (9)
4. Implementation of discrete NRBM model for thermally affected DWCNTs membranes

4.1. Establishment of nonlocal equations of motion

For the thermally affected membrane of vertically aligned DWCNTs modeled based on the NRBM, the kinetic energy (T\textsuperscript{K}) as well as the strain energy (U\textsuperscript{R}) by considering the nonlocality, the intertube vdW forces, and the interactional effect with the surrounding elastic medium are written as:

\[ T^K(t) = \frac{1}{2} \sum_{i=1}^{2N} \int_0^l \rho_b \left[ \frac{\partial V^R_i}{\partial t} + \frac{\partial W^R_i}{\partial t} \right]^2 + I_b \left[ \frac{\partial^2 V^R_i}{\partial t^2} + \frac{\partial^2 W^R_i}{\partial t^2} \right] \mathrm{d}x, \]  

(10a)

\[ U^R(t) = \frac{1}{2} \sum_{i=1}^{2N} \int_0^l \frac{\partial^2 V^R_i}{\partial x^2} (M^s_{by})^R - \frac{\partial^2 W^R_i}{\partial x^2} (M^s_{bz})^R \mathrm{d}x \]

\[ + \frac{1}{2} \sum_{i=2, \ldots}^{2N} \int_0^l \left[ C_{ij}(y_{ij}) (V^R_i - V^R_{i+1})^2 + C_{ij}(y_{ij}) (V^R_i - V^R_{i+2})^2 (1 - \delta_2) \right] \mathrm{d}x \]

\[ + \frac{1}{2} \sum_{i=2, \ldots}^{2N+1} \int_0^l \left[ C_{ij}(y_{ij}) (V^R_i - V^R_{i+1})^2 + C_{ij}(y_{ij}) (V^R_i - V^R_{i+2})^2 (1 - \delta_2) \right] \mathrm{d}x \]

\[ + \frac{1}{2} \sum_{i=1, \ldots}^{2N} \int_0^l \left[ C_{ij}(y_{ij}) (V^R_i - V^R_{i+1})^2 + C_{ij}(y_{ij}) (V^R_i - V^R_{i+2})^2 (1 - \delta_2) \right] \mathrm{d}x, \]  

(10b)

where \((M^s_{by})^R\) and \((M^s_{bz})^R\) represent the nonlocal bending moments about the y and z axes according to the Rayleigh beam theory, respectively, and \(\delta_y\) denotes the Kronecker delta tensor.

In the framework of the nonlocal elasticity theory of Eringen [12–14], the nonlocal bending moments within the ith tube are simply linked to their corresponding local values by the following relations [47–49]:

\[ (M^s_{by})^R (\varepsilon_0 a)^2 \frac{\partial^2 (M^s_{by})^R}{\partial x^2} = -E_b I_b \frac{\partial^2 W^R_i}{\partial x^2}, \]  

(11a)

\[ (M^s_{bz})^R (\varepsilon_0 a)^2 \frac{\partial^2 (M^s_{bz})^R}{\partial x^2} = -E_b I_b \frac{\partial^2 V^R_i}{\partial x^2}, \]  

(11b)

By employing the Hamilton’s principle, \(\int_0^1 (\partial T^K(t) - \partial U^R(t)) \mathrm{d}t = 0\), in view of Eqs. (10a) and (10b), the equations of motion of thermally affected membrane made of vertically aligned DWCNTs on the basis of the NRBM could be expressed by:

* The first thermally affected DWCNTs:

\[ \rho_b \left( A_{ij} \frac{\partial^2 V^R_i}{\partial t^2} - I_b \frac{\partial^2 V^R_i}{\partial x^2} \right) + C_{ij}(y_{ij}) \left( V^R_i - V^R_{i+1} \right) + C_{ij}(y_{ij}) \left( V^R_i - V^R_{i+2} \right) \]

\[ + K \frac{\partial^2 W^R_i}{\partial x^2} \left( 1 - \delta_2 \right) - N_T \frac{\partial^2 W^R_i}{\partial x^2} \]

\[ + E_b I_b \frac{\partial^2 V^R_i}{\partial x^2} = 0; \quad j = 1, 2, \ldots \]  

(12a)

\[ \rho_b \left( A_{ij} \frac{\partial^2 W^R_i}{\partial t^2} - I_b \frac{\partial^2 W^R_i}{\partial x^2} \right) + C_{ij}(y_{ij}) \left( W^R_i - W^R_{i+1} \right) + C_{ij}(y_{ij}) \left( W^R_i - W^R_{i+2} \right) \]

\[ + K \frac{\partial^2 V^R_i}{\partial x^2} \left( 1 - \delta_2 \right) - N_T \frac{\partial^2 V^R_i}{\partial x^2} \]

\[ + E_b I_b \frac{\partial^2 W^R_i}{\partial x^2} = 0, \]  

(12b)
The intermediate thermally affected DWCNTs:
\[
\begin{align*}
\rho_b \left( A_b \frac{\partial^6 V_{b,R}^{2-2n+3,j}}{\partial t^6} - I_b \frac{\partial^4 V_{b,R}^{2-2n+2,j}}{\partial t^4} \right) + C_{V_{b,R}^{2-2n+3,j}(2N-2n+3-j)} &+ C_{V_{b,R}^{2-2n+2,j}(2N-2n+3-j)} + C_{V_{b,R}^{2-2n+1,j}(2N-2n+3-j)} \\
&+ \left( K_b V_{b,R}^{2-2n+3,j} - K_b \frac{\partial^4 V_{b,R}^{2-2n+2,j}}{\partial t^4} \right)(1 - \delta_{j,j}) - N_{R,j} \frac{\partial^2 V_{b,R}^{2-2n+3,j}}{\partial t^2} = 0; \ n = 2, 3, \ldots, N - 1,
\end{align*}
\]
\[
\begin{align*}
\rho_b \left( A_b \frac{\partial^4 V_{b,R}^{2-2n+3,j}}{\partial t^4} - I_b \frac{\partial^2 V_{b,R}^{2-2n+2,j}}{\partial t^2} \right) + C_{V_{b,R}^{2-2n+3,j}(2N-2n+3-j)} &+ C_{V_{b,R}^{2-2n+2,j}(2N-2n+3-j)} + C_{V_{b,R}^{2-2n+1,j}(2N-2n+3-j)} \\
&+ \left( K_b V_{b,R}^{2-2n+3,j} - K_b \frac{\partial^2 V_{b,R}^{2-2n+2,j}}{\partial t^2} \right)(1 - \delta_{j,j}) - N_{R,j} \frac{\partial V_{b,R}^{2-2n+3,j}}{\partial t} = 0,
\end{align*}
\]
\[
\begin{align*}
\rho_b \left( A_b \frac{\partial^2 V_{b,R}^{2-2n+3,j}}{\partial t^2} - I_b \frac{\partial V_{b,R}^{2-2n+2,j}}{\partial t} \right) + C_{V_{b,R}^{2-2n+3,j}(2N-2n+3-j)} &+ C_{V_{b,R}^{2-2n+2,j}(2N-2n+3-j)} + C_{V_{b,R}^{2-2n+1,j}(2N-2n+3-j)} \\
&+ \left( K_b V_{b,R}^{2-2n+3,j} - K_b \frac{\partial V_{b,R}^{2-2n+2,j}}{\partial t} \right)(1 - \delta_{j,j}) - N_{R,j} \frac{\partial V_{b,R}^{2-2n+3,j}}{\partial t} = 0,
\end{align*}
\]
(13a)
(13b)
(14a)
(14b)

where $\mathcal{Z}[\cdot] = [1 - (\epsilon_x^2)^{\frac{1}{2}} \cdot \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial z}]$ represents the nonlocal operator. To investigate free vibration of the thermally affected nanosystem irrespective of the time and space dimensions, we introduce the following dimensionless parameters:
\[
\begin{align*}
\xi = \frac{x}{l_x}, \ \gamma = \frac{y}{l_y}, \ \eta = \frac{z}{l_z}, \ \bar{V}_i &= \frac{V_i}{V_{\text{R}}}, \ \bar{W}_j &= \frac{W_j}{W_{\text{R}}}, \ \bar{t} = \frac{t}{l_t}, \ \bar{\rho}_b = \frac{\rho_b}{\rho_{b_{\text{R}}}}, \ \bar{A}_b = \frac{A_b}{A_{b_{\text{R}}}}, \\
\bar{\rho}_b^2 = \frac{\rho_b^2}{\rho_{b_{\text{R}}}^2}, \ \bar{A}_b^2 = \frac{A_b^2}{A_{b_{\text{R}}}^2}, \ \bar{K}_b = \frac{K_b}{K_{b_{\text{R}}}}, \ \bar{F}_{b_{\text{R}}} = \frac{F_{b_{\text{R}}}}{F_{b_{\text{R}}}}, \ \bar{V}_{(\text{R})} &= \frac{V_{(\text{R})}}{V_{\text{R}}}, \ \bar{V}_{(\text{R})} \frac{\partial V_{(\text{R})}}{\partial \xi} &= \frac{C_{V_{(\text{R})}}}{E_b I_b}, \\
\bar{C}_{V_{(\text{R})}} &= \frac{C_{V_{(\text{R})}}}{E_b I_b}, \ \bar{I}_2 = (N - 1)d, \ \bar{\lambda}_1 = \frac{l_t}{l_t}, \ \bar{N}_{R,j} = \frac{N_{R,j}}{E_b I_b}.
\end{align*}
\] 
(15)

by mixing Eqs. (12a), (12b), (13a), (13b), (14a), and (14b) with Eq. (15), the dimensionless NRBMB-based equations of motion of thermally affected-vertically aligned DWCNTs embedded in an elastic matrix with plain membrane configuration take the following form:

• The first thermally affected DWCNTs:
\[
\begin{align*}
&\left( \epsilon_x^2 \delta_{j,j} + \delta_{j,j} \right) \frac{\partial^4 V_{b,R}^{2-2n+3,j}}{\partial t^4} - \left( \epsilon_x^2 \delta_{j,j+1} + \delta_{j,j+1} \right) \frac{\partial^2 V_{b,R}^{2-2n+3,j}}{\partial t^2} + C_{V_{b,R}^{2-2n+3,j}(2N-2n+3-j)} + C_{V_{b,R}^{2-2n+2,j}(2N-2n+3-j)} \\
&+ \left( \bar{K}_b V_{b,R}^{2-2n+3,j} - \bar{K}_b \frac{\partial^2 V_{b,R}^{2-2n+2,j}}{\partial t^2} \right)(1 - \delta_{j,j}) - \bar{N}_{R,j} \frac{\partial^2 V_{b,R}^{2-2n+3,j}}{\partial t^2} = 0, \\
\end{align*}
\]
(16a)
(16b)

• The intermediate thermally affected DWCNTs:
\[
\begin{align*}
&\left( \epsilon_x^2 \delta_{j,j} + \delta_{j,j} \right) \frac{\partial^2 V_{b,R}^{2-2n+3,j}}{\partial t^2} - \left( \epsilon_x^2 \delta_{j,j+1} + \delta_{j,j+1} \right) \frac{\partial V_{b,R}^{2-2n+3,j}}{\partial t} + C_{V_{b,R}^{2-2n+3,j}(2N-2n+3-j)} + C_{V_{b,R}^{2-2n+2,j}(2N-2n+3-j)} \\
&+ \left( \bar{K}_b V_{b,R}^{2-2n+3,j} - \bar{K}_b \frac{\partial V_{b,R}^{2-2n+2,j}}{\partial t} \right)(1 - \delta_{j,j}) - \bar{N}_{R,j} \frac{\partial V_{b,R}^{2-2n+3,j}}{\partial t} = 0; \ n = 2, 3, \ldots, N - 1,
\end{align*}
\]
(17a)
\[
\begin{align*}
\left( \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial t^2} + \frac{\partial \mathbf{W}_{2n}^{R}}{\partial t} \right) &= \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial x^2} + \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial y^2} + \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial z^2} + \mathbf{C} \left( \mathbf{W}_{2n}^{R} - \mathbf{W}_{2n}^{R} \right) + \mathbf{C} \left( \mathbf{W}_{2n+1}^{R} - \mathbf{W}_{2n+1}^{R} \right), \\
\text{subject to:} & \quad \mathbf{W}_{2n}^{R} = \mathbf{W}_{2n+1}^{R} = \mathbf{W}_{2n+2}^{R} = 0, \\
\end{align*}
\]

where \( \mathbf{W} \) is the displacement vector, \( \mathbf{C} \) is the damping matrix, and \( \mathbf{W}_{2n}^{R} \) and \( \mathbf{W}_{2n+1}^{R} \) are the nodal displacements at the current and neighboring nodes, respectively.

The Nth thermally affected DWCNTs:

\[
\begin{align*}
\left( \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial t^2} + \frac{\partial \mathbf{W}_{2n}^{R}}{\partial t} \right) &= \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial x^2} + \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial y^2} + \frac{\partial^2 \mathbf{W}_{2n}^{R}}{\partial z^2} + \mathbf{C} \left( \mathbf{W}_{2n}^{R} - \mathbf{W}_{2n}^{R} \right) + \mathbf{C} \left( \mathbf{W}_{2n+1}^{R} - \mathbf{W}_{2n+1}^{R} \right) + \mathbf{C} \left( \mathbf{W}_{2n+2}^{R} - \mathbf{W}_{2n+2}^{R} \right), \\
\text{subject to:} & \quad \mathbf{W}_{2n}^{R} = \mathbf{W}_{2n+1}^{R} = \mathbf{W}_{2n+2}^{R} = 0, \\
\end{align*}
\]

where \( \mathbf{W} \) is the displacement vector, \( \mathbf{C} \) is the damping matrix, and \( \mathbf{W}_{2n}^{R} \) and \( \mathbf{W}_{2n+1}^{R} \) are the nodal displacements at the current and neighboring nodes, respectively.

4.2. Evaluation of natural frequencies

Assume that the ends of all tubes of the vertically aligned DWCNTs be simple. Additionally, the outermost carbon nanotubes of the first and the Nth DWCNTs have been prevented from any lateral motion, i.e., \( \mathbf{V}_{2n}^{R}(\xi, \tau) = \mathbf{V}_{2n+1}^{R}(\xi, \tau) = 0, \mathbf{V}_{2n}^{R}(\xi, \tau) = \mathbf{W}_{2n}^{R}(\xi, \tau) = 0 \). We employ AMM for frequency analysis of vertically aligned membranes of DWCNTs embedded in an elastic matrix. Therefore, the deflections fields of the constitutive tubes of the membrane are discretized by:

\[
\begin{align*}
\mathbf{V}_{i}^{R}(\xi, \tau) &= \sum_{m=1}^{N_{xy}} \mathbf{V}_{i,m}^{R}(\tau) \sin(m \pi \xi), \\
\mathbf{W}_{i}^{R}(\xi, \tau) &= \sum_{m=1}^{N_{xy}} \mathbf{W}_{i,m}^{R}(\tau) \sin(m \pi \xi),
\end{align*}
\]

where \( \mathbf{V}_{i}^{R}(\tau) \) and \( \mathbf{W}_{i}^{R}(\tau) \) are the time-dependent quantities pertinent to the mth mode of vibration of the ith nanotube, \( N_{xy} \) and \( N_{xy} \) in order denote the numbers of modes taken into account for lateral vibrations of the nanosystem along the y and z directions. By introducing Eqs. (19 a) and (19 b) to Eqs. (16 a), (16 b), (17 a), (17 b), (18 a), and (18 b), the following set of second-order ordinary differential equations is obtained:

\[
d^2 \mathbf{x}^{R} \frac{d\tau}{d\tau} + \mathbf{\Gamma}^{R} \mathbf{x}^{R} = 0,
\]

where \( \mathbf{x}^{R} \) is defined as follows:

\[
\mathbf{x}^{R} = \begin{bmatrix} \mathbf{V}_{2m}^{R} \mathbf{W}_{2m}^{R} \mathbf{V}_{2m+1}^{R} \mathbf{W}_{2m+1}^{R} \cdots \mathbf{V}_{2N-2m}^{R} \mathbf{W}_{2N-2m}^{R} \end{bmatrix}^T \quad \Rightarrow \mathbf{V} \text{ or } \mathbf{W},
\]

and the elements of \( \mathbf{\Gamma}^{R} \) are provided in supplementary material, part B. To compute natural frequencies, we consider \( \mathbf{x}^{R}(\tau) = \mathbf{\xi}^{R} \exp(i \omega^{R} \tau) \) where \( i = \sqrt{-1} \), \( \mathbf{\xi}^{R} \) is the dimensionless vector of unknown parameters, and \( \omega^{R} \) denotes dimensionless natural frequency. By substituting this form into Eq. (20), the natural frequencies could be readily evaluated through solving the resulted set of equations.

5. Implementation of discrete NHOBM model for thermally affected DWCNTs membranes

5.1. Establishment of nonlocal equations of motion

Using higher-order beam theory of Reddy–Bickford [50,51] in conjunction with the nonlocal elasticity theory of Eringen [12–14], the kinetic energy (\( T^{R} \)) and the elastic strain energy (\( U^{R} \)) of the elastically embedded membrane of vertically aligned DWCNTs subjected to longitudinal
temperature gradient are written as follows:

\[
T^H(t) = \frac{1}{2} \sum_{i=1}^{2N} \int_{0}^{l_i} \left( I_0 \left( \left( \frac{\partial V^H_i}{\partial x} \right)^2 + \left( \frac{\partial W^H_i}{\partial x} \right)^2 \right) + I_1 \left( \frac{\partial W^H_i}{\partial t} \right)^2 \right) \, dx,
\]

(22a)

\[
U^H(t) = \frac{1}{2} \sum_{i=1}^{2N} \int_{0}^{l_i} \left[ \frac{\partial^2 W^H_i}{\partial x^2} \left( M^{W^H}_{i,1} \right)^H + \frac{\partial^2 V^H_i}{\partial x^2} \right] \, dx
\]

(22b)

where \( V^H_i \) and \( W^H_i \) are the deflections of the ith nanotube along the y and z axes, respectively, \( \Psi^H_i \) and \( \Psi^H_z \) in order denote the angle of deflections about the y-axis and z-axis, \( (Q^H_{yi})^H + a_i \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} \) and \( (Q^H_{zi})^H + a_i \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} \) represent the whole shear force within the ith tube along the y and z axes, respectively, \( (M^H_{yi})^H \) and \( (M^H_{zi})^H \) denote the nonlocal bending moment of the ith tube about the y-axis and the z-axis, respectively. By implementing the nonlocal elasticity theory of Eringen [12-14], these nonlocal internal forces are related to their corresponding local ones via the following relations [20,35,36,52]:

\[
\left( Q^H_{yi} + a_i \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} \right) - (\varepsilon_0 a_i)^2 \frac{\partial^2}{\partial x^2} \left( Q^H_{yi} + a_i \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} \right) = \kappa_i \left( \Psi^H_{yi} + \frac{\partial V^H_i}{\partial x} \right) + a_i J_4 \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} = \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} + \frac{\partial^2 V^H_i}{\partial x^2}.
\]

(23a)

\[
\left( Q^H_{zi} + a_i \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} \right) - (\varepsilon_0 a_i)^2 \frac{\partial^2}{\partial x^2} \left( Q^H_{zi} + a_i \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} \right) = \kappa_i \left( \Psi^H_{zi} + \frac{\partial W^H_i}{\partial x} \right) + a_i J_4 \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} = \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} + \frac{\partial^2 W^H_i}{\partial x^2}.
\]

(23b)

\[
M^H_{yi} - (\varepsilon_0 a_i)^2 \frac{\partial^2 M^H_{yi}}{\partial x^2} = J_{2y} \left( \frac{\partial \Psi^H_{yi}}{\partial x} \right) + a_i J_4 \left( \frac{\partial^2 \Psi^H_{yi}}{\partial x^2} + \frac{\partial^2 W^H_i}{\partial x^2} \right).
\]

(23c)

\[
M^H_{zi} - (\varepsilon_0 a_i)^2 \frac{\partial^2 M^H_{zi}}{\partial x^2} = J_{2z} \left( \frac{\partial \Psi^H_{zi}}{\partial x} \right) + a_i J_4 \left( \frac{\partial^2 \Psi^H_{zi}}{\partial x^2} + \frac{\partial^2 V^H_i}{\partial x^2} \right).
\]

(23d)

where \( a_i \), \( \kappa_i \), \( I_i \), and \( J_i \) are defined by:

\[
a_i = \frac{1}{3} \varepsilon_0 a_i, \quad \kappa_i = \int_{A_i} G_{xy} (1 - 3a_i z^2) \, dA,
\]

\[
I_i = \int_{A_i} \rho_i z^4 \, dA, \quad J_i = \int_{A_i} E_{xy} z^6 \, dA; \quad n = 0, 2, 4, 6.
\]

(24)

Now by employing Hamilton’s principle, \( \int_0^L (\delta T^H(t) - \delta U^H(t)) \, dt = 0 \), through using integration by part technique in conjunction with Eqs. (23a)-(23d), the nonlocal equations of motion that display transverse vibrations of the thermally affected-embedded membranes of DWCNTs subjected to a longitudinal heat flux on the basis of the NHOBM are displayed by:
• The first thermally affected DWCNTs

\[
\begin{align*}
&= \left\{ (I_{j2} - 2a_1I_{j1} + a_1^2I_{j0}) \frac{\partial^3\psi_H^{ij}}{\partial x^4} + (a_1^2I_{j0} - a_1I_{j2}) \frac{\partial^3\psi_H^{ij}}{\partial^2x^2} + (a_1I_{j2} - a_1^2I_{j0}) \frac{\partial^3\psi_H^{ij}}{\partial x^3} + k_r \psi_H^{ij}(1 - \delta_j) \right\} + k_j \frac{\partial W_H^{ij}}{\partial x} - (J_{j2} - 2a_1J_{j1} + a_1^2J_{j0})\frac{\partial^3\psi_H^{ij}}{\partial x^4} + (a_1J_{j2} - a_1^2J_{j0})\frac{\partial^3\psi_H^{ij}}{\partial x^3} = 0, \\
&= \left\{ (I_{j0} - 2a_1I_{j1} + a_1^2I_{j0}) \frac{\partial^2\psi_H^{ij}}{\partial^2x^2} + (a_1^2I_{j0} - a_1I_{j2}) \frac{\partial^2\psi_H^{ij}}{\partial x^3} + a_1J_{j2} - a_1^2J_{j0} \frac{\partial \psi_H^{ij}}{\partial x} = 0. \\
&= \left\{ (I_{j2} - 2a_1I_{j1} + a_1^2I_{j0}) \frac{\partial^3\psi_H}{\partial x^4} + (a_1^2I_{j0} - a_1I_{j2}) \frac{\partial^3\psi_H}{\partial^2x^2} + (a_1I_{j2} - a_1^2I_{j0}) \frac{\partial^3\psi_H}{\partial x^3} + k_r \psi_H(1 - \delta_j) \right\} + k_j \frac{\partial W}{\partial x} - (J_{j2} - 2a_1J_{j1} + a_1^2J_{j0})\frac{\partial^3\psi_H}{\partial x^4} + (a_1J_{j2} - a_1^2J_{j0})\frac{\partial^3\psi_H}{\partial x^3} = 0.
\end{align*}
\]
\[ \left\{ I_j - 2a_j I_0 + a_j^2 I_0 \right\} \frac{\partial \hat{\Psi}_j}{\partial \hat{z}^2} + \left\{ a_j^2 I_0 - a_j I_j \right\} \frac{\partial \hat{W}_j}{\partial \hat{z}^2} + \left\{ \frac{\partial \hat{\Psi}_j}{\partial \hat{z}} \right\} \frac{\partial \hat{W}_j}{\partial \hat{z}} + \left\{ \frac{\partial \hat{\Psi}_j}{\partial \hat{z}} \right\} \frac{\partial \hat{W}_j}{\partial \hat{z}} = 0, \]
• The intermediate thermally affected DWCNTs

\[ \begin{align*}
\mathbf{K}^{H}_{\text{2D}_2,2n+2} & = \mathbf{K}^{H}_{\text{2D}_2,2n+2} \nonumber \end{align*} \]

(30a)

\[ \begin{align*}
\mathbf{K}^{H}_{\text{2D}_2,2n+2} & = \mathbf{K}^{H}_{\text{2D}_2,2n+2} \nonumber \end{align*} \]

(30b)

\[ \begin{align*}
\mathbf{K}^{H}_{\text{2D}_2,2n+2} & = \mathbf{K}^{H}_{\text{2D}_2,2n+2} \nonumber \end{align*} \]

(30c)

\[ \begin{align*}
\mathbf{K}^{H}_{\text{2D}_2,2n+2} & = \mathbf{K}^{H}_{\text{2D}_2,2n+2} \nonumber \end{align*} \]

(30d)

• The Nth thermally affected DWCNTs

\[ \begin{align*}
\mathbf{K}^{H}_{N,2n+2} & = \mathbf{K}^{H}_{N,2n+2} \nonumber \end{align*} \]

(31a)

\[ \begin{align*}
\mathbf{K}^{H}_{N,2n+2} & = \mathbf{K}^{H}_{N,2n+2} \nonumber \end{align*} \]

(31b)

\[ \begin{align*}
\mathbf{K}^{H}_{N,2n+2} & = \mathbf{K}^{H}_{N,2n+2} \nonumber \end{align*} \]

(31c)
\[-\frac{\partial^2 \Psi^{H}}{\partial \xi^2} \frac{\partial \Psi^{H}}{\partial \xi} + \frac{\partial^2 \Psi^{H}}{\partial \xi^2} \frac{\partial \Psi^{H}}{\partial \xi} + \frac{\partial^2 \Psi^{H}}{\partial \xi^2} \frac{\partial \Psi^{H}}{\partial \xi} = 0,\]  

(31d)

in which

\[\gamma_1^2 = \frac{k_t I_0 t_b^4}{(I_2 - 2a_1 I_4 + a_1^2 I_6) a_1^2 I_6}, \quad \gamma_2^2 = \frac{(I_2 - 2a_1 I_4 + a_1^2 I_6)}{a_1^2 I_6}, \quad \gamma_4^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}, \quad \gamma_5^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}, \quad \gamma_6^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}, \]

\[\gamma_7^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}, \quad \gamma_9^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}, \quad \gamma_8^2 = \frac{k_t I_0 t_b^4}{a_1^2 I_6}.
\]

(32)

5.2. Evaluation of natural frequencies

For the thermally affected nanosystem modeled according to the NHOBM, it is assumed that all constitutive tubes have simple ends and the first and the last DWCNTs are prevented from any transverse motion (i.e., \(\Psi^{H}(\xi, \tau) = \Psi^{H}(\xi, \tau) = 0\)). By implementing the AMM, we consider the following admissible form for deformation fields of nanotubes:

\[\Psi^{H}(\xi, \tau) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Psi_{i,n}(\tau) \sin(n \pi \xi),\]  

(33a)

\[\Psi^{H}(\xi, \tau) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Psi_{i,n}(\tau) \cos(n \pi \xi),\]  

(33b)

\[\Psi^{H}(\xi, \tau) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Psi_{i,n}(\tau) \sin(n \pi \xi),\]  

(33c)

\[\Psi^{H}(\xi, \tau) = \sum_{n=1}^{N} \sum_{m=1}^{M} \Psi_{i,n}(\tau) \cos(n \pi \xi),\]  

(33d)

where \(\Psi_{i,n}(\tau), \Psi_{i,n}(\tau), \Psi_{i,n}(\tau), \Psi_{i,n}(\tau)\) denote the time-dependent parameters associated with deflections and angles of deflections. Now by substituting Eqs. (33a), (33b), (33c), and (33d) into Eqs. (29)–(31), one can arrive at the following set of linear second-order ordinary differential equations:

\[\ddot{\mathbf{m}}^H \dot{\mathbf{x}}^H + \mathbf{k}^H \mathbf{x}^H = 0,\]  

(34)

where \(\mathbf{x}^H\) is given by:

\[\mathbf{x}^H = \begin{bmatrix} \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H \\ \Psi_{i,2m}^H & \ldots & \Psi_{i,2m}^H & \ldots & \Psi_{i,2m}^H & \ldots & \Psi_{i,2m}^H & \ldots & \Psi_{i,2m}^H \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H & \ldots & \Psi_{i,1m}^H \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}^T,\]  

(35)

in which \([\cdot] = V \text{ or } W, [\cdot] = \Psi^2 \text{ or } \Psi^0\), and the elements of \(\mathbf{m}^H\) and \(\mathbf{k}^H\) are provided in supplementary material, part C. Finally, by following the given procedure in Section 4.2, the natural frequencies of the thermally affected membranes of vertically aligned DWCNTs are easily determined.

6. Development of continuous models for thermally affected DWCNTs membranes

As it was explained in the earlier parts, the dimensions of both mass and stiffness matrices for discrete models based on the NRBM or NHOBM would substantially increase by increasing the number of DWCNTs of the nanosystem. It implies that the evaluation of natural frequencies for highly populated nanosystems would compromise with great computational efforts. In order to reduce these expenditures, herein, we develop several continuous models based on the NRBM and NHOBM. To this end, let us to define continuous deformation fields of the nanosystem such that their values at the neutral axis of each tube would be equal to the deformation fields of the discrete models for the tube of concern (i.e.,
\( V_j^{(1)} = \psi_j^{(1)}(x, z, t) \), \( \psi_j^{(1)}(x, z, t) \), \( \Psi_j^{H} = \psi_j^{H}(x, z, t) \), \( \Psi_j^{H} = \psi_j^{H}(x, z, t) \jmath = 1, 2 \). For constructing the aforementioned continuous functions, it is noticed that they should satisfy the following conditions:

\[
\begin{align*}
V_j^{(1)}(x, z_{2n-2j}, t) & \approx V_j^{(1)}_{2n-2j}(x, t), \\
V_j^{(1)}(x, z_{2n-4j} - d, t) & \approx V_j^{(1)}_{2n-4j}(x, t), \quad j = 1, 2.
\end{align*}
\]

\[
\begin{align*}
V_j^{(1)}(x, z_{2n+2j}) & \approx V_j^{(1)}_{2n+2j}(x, t), \\
V_j^{(1)}(x, z_{2n+4j} + d, t) & \approx V_j^{(1)}_{2n+4j}(x, t), \quad j = 1, 2.
\end{align*}
\]

\[
\begin{align*}
\psi_j^{(1)}(x, z_{2n-2j}, t) & \approx \psi_j^{H}(x, z_{2n-2j}, t), \\
\psi_j^{(1)}(x, z_{2n-4j} - d, t) & \approx \psi_j^{H}(x, z_{2n-4j} - d, t), \quad j = 1, 2.
\end{align*}
\]

\[
\begin{align*}
\psi_j^{(1)}(x, z_{2n+2j}) & \approx \psi_j^{H}(x, z_{2n+2j}, t), \\
\psi_j^{(1)}(x, z_{2n+4j} + d, t) & \approx \psi_j^{H}(x, z_{2n+4j} + d, t), \quad j = 1, 2.
\end{align*}
\]

(36)

and the deformation fields at the neutral axis of the neighboring nanotubes are approximated by:

\[
[\star j](x, z_n \pm d, t) = \sum_{j=0}^{\infty} \frac{(z_n \pm d)}{r_i} \frac{\partial [\star j]}{\partial z_n}(x, z_n, t),
\]

(37)

where \([\star j] = v^{(1)} \) or \(w^{(1)}\) and \(m = 2n - 4 + j, \ 2n + j\).

In the following parts, the nonlocal-continuous-based models are developed on the basis of the local-discrete-based models established in the previous sections.

6.1. Continuous NRBM model for thermally affected DWCNTs membranes

6.1.1. Development of nonlocal-continuous equations of motion

By introducing Eq. (37) to Eq. (13 a) and (13b) in view of Eq. (36), the nonlocal-continuous equations of motion of thermally affected DWCNTs membranes acted upon by a longitudinal temperature gradient according to the NRBM take the following form:

\[
\begin{align*}
\rho_0 \left( A_j \frac{\partial v_j^R}{\partial t} - I_j \frac{\partial \psi_j^R}{\partial z_n} \right) + C_{\psi_j^{(1)},2j-1} \left( v_j^R - v_j^{(1)} \right) + C_{\psi_j^{(1)},2j} \left( v_j^R - v_j^{(1)} \right) & = 0, \\
\rho_0 \left( A_j \frac{\partial w_j^R}{\partial t} - I_j \frac{\partial \psi_j^R}{\partial z_n} \right) + C_{\psi_j^{(1)},2j-1} \left( w_j^R - w_j^{(1)} \right) + C_{\psi_j^{(1)},2j} \left( w_j^R - w_j^{(1)} \right) & = 0,
\end{align*}
\]

(38a)

\[
\begin{align*}
\rho_0 \left( A_j \frac{\partial^2 v_j^R}{\partial t^2} - I_j \frac{\partial \psi_j^R}{\partial z_n} \right) + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) & = 0, \\
\rho_0 \left( A_j \frac{\partial^2 w_j^R}{\partial t^2} - I_j \frac{\partial \psi_j^R}{\partial z_n} \right) + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) & = 0,
\end{align*}
\]

(38b)

and by introducing the dimensionless quantities in Eq. (15) to Eq. (38 a) and (38b), the dimensionless-nonlocal-continuous governing equations that display transverse vibrations of the thermally affected nanosystem based on the NRBM are derived as:

\[
\begin{align*}
\rho_0 \frac{\partial^2 v_j^R}{\partial t^2} - \rho_0 \frac{\partial^2 v_j^{(1)}}{\partial t^2} + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) & = 0, \\
\rho_0 \frac{\partial^2 w_j^R}{\partial t^2} - \rho_0 \frac{\partial^2 w_j^{(1)}}{\partial t^2} + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) & = 0,
\end{align*}
\]

(39a)

\[
\begin{align*}
\rho_0 \frac{\partial^2 v_j^R}{\partial t^2} - \rho_0 \frac{\partial^2 v_j^{(1)}}{\partial t^2} + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial v_j^R}{\partial t} - \frac{\partial v_j^{(1)}}{\partial t} \right) & = 0, \\
\rho_0 \frac{\partial^2 w_j^R}{\partial t^2} - \rho_0 \frac{\partial^2 w_j^{(1)}}{\partial t^2} + C_{\psi_j^{(1)},2j-1} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) + C_{\psi_j^{(1)},2j} \left( \frac{\partial w_j^R}{\partial t} - \frac{\partial w_j^{(1)}}{\partial t} \right) & = 0,
\end{align*}
\]

(39b)

6.1.2. Evaluation of natural frequencies

For elastically embedded membranes under longitudinal heat flux with simple ends at \( x = 0 \) and \( x = l \), whose edges have been fixed at \( z = 0 \) and \( z = (N_z - 1)d \), see Fig. 2, the following conditions should be met by deflections and nonlocal bending moments of nanotubes modeled according to the NRBM:

\[
\begin{align*}
\psi_j^R(\xi = 0, y, r) = \psi_j^R(\xi = 1, y, r) = 0, \\
\psi_j^R(\xi = 0, r) = \psi_j^R(\xi = 1, y, r) = 0,
\end{align*}
\]

(39)

\[
\begin{align*}
\psi_j^R(\xi = 0, y, r) = \frac{\rho_j^R}{(M_j^R)}(\xi = 1, y, r) = 0, \\
\psi_j^R(\xi = 0, r) = \frac{\rho_j^R}{(M_j^R)}(\xi = 1, y, r) = 0.
\end{align*}
\]

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Fig. 2. An illustration of the spatial domain of the nonlocal-continuous-based models employed for vibration analysis of thermally affected membranes consist of \( N \) vertically aligned DWCNTs.

\[
\begin{align*}
\left( \bar{M}_{b_5} \right)_{\xi = 0, \gamma, \tau} &= \left( \bar{M}_{b_5} \right)_{\xi = 1, \gamma, \tau} = 0, \\
\bar{w}^R_{j} \left( \xi, \gamma = 0, \tau = 1 \right) &= \varphi^R_{j} \left( \xi, \gamma = 1, \tau = 0 \right) = 0, \\
\varphi^R_{j} \left( \xi, \gamma = 0, \tau = 0 \right) &= \rho^R_{j} \left( \xi, \gamma = 1, \tau = 0 \right) = 0.
\end{align*}
\]

(40)

To solve Eq. (39 a) and (39 b) for natural frequencies in view of the given boundary conditions in Eq. (40), the continuous deflection fields of the thermally affected nanosystem are stated by:

\[
\begin{align*}
\bar{v}^R_{j} \left( \xi, \gamma, \tau \right) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{v}^R_{(m,n)} \sin(m \pi \xi) \sin(n \pi \gamma) e^{i \omega^j_{\xi \gamma} t}, \\
\bar{w}^R_{j} \left( \xi, \gamma, \tau \right) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{w}^R_{(m,n)} \sin(m \pi \xi) \sin(n \pi \gamma) e^{i \omega^j_{\xi \gamma} t},
\end{align*}
\]

(41a) (41b)

where \( i = \sqrt{-1} \), \( \bar{v}^R_{(m,n)} \) and \( \bar{w}^R_{(m,n)} \) represent the amplitudes of the \((m, n)\)th vibration mode, \( \omega^R_{j \xi \gamma} \) and \( \omega^R_{j \xi \gamma} \) denote the dimensionless natural frequencies pertinent to the out-of-plane and the in-plane vibration modes, respectively. By substituting Eqs. (41 a) and (41b) into Eqs. (39 a) and (39b), the characteristic equation and the natural frequencies of the nanosystem could be readily evaluated (see supplementary material, part D).

6.2. Continuous NHOBM model for thermally affected DWCNTs membranes

6.2.1. Development of nonlocal-continuous equations of motion

By virtue of Eq. (36) and through introducing Eq. (37) to Eq. (26 a)–(26d), the nonlocal-continuous equations that display transverse vibrations of thermally affected DWCNTs membranes embedded in an elastic matrix are obtained as follows:

\[
\begin{align*}
\left( I_{2j} - 2a_j I_{3j} + a_j^2 I_{4j} \right) \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} + \left( a_j^2 I_{3j} - a_j I_{4j} \right) \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} + K_j \psi^H_{\gamma j} (1 - \delta_{j1}) \\
+ \kappa_j \left( \psi^H_{\gamma j} + \frac{\partial \psi^H_{\gamma j}}{\partial \gamma} \right) - (I_{2j} - 2a_j I_{3j} + a_j^2 I_{4j}) \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} + (a_j I_{4j} - a_j^2 I_{3j}) \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} = 0,
\end{align*}
\]

(42a)

\[
\begin{align*}
&\frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} \left( a_j^2 I_{3j} - a_j I_{4j} \right) \frac{\partial \psi^H_{\gamma j}}{\partial \gamma} - a_j^2 I_{3j} \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} + C_{y_{j+2,1+1}} \left( \psi^H_{\gamma j} - \psi^H_{\gamma j+2} \right) + C_{y_{j+3,1+1}} \left( \psi^H_{\gamma j} - \psi^H_{\gamma j+3} \right) \\
&+ C_{y_{j+2,1+1}} \left( \phi^H_{\gamma j+2} + \phi^H_{\gamma j+3} - \phi^H_{\gamma j+1} \right) - \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} - \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} - \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} \left( a_j^2 I_{3j} - a_j I_{4j} \right) \frac{\partial \psi^H_{\gamma j}}{\partial \gamma} + K_j \psi^H_{\gamma j} (1 - \delta_{j1}) - N_{fg} \psi^H_{\gamma j} = 0
\end{align*}
\]

(42b)

\[
\begin{align*}
\frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} + \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} - a_j I_{4j} \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} + a_j^2 I_{3j} \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} = 0.
\end{align*}
\]

(42b)

\[
\begin{align*}
\left( I_{2j} - 2a_j I_{3j} + a_j^2 I_{4j} \right) \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} + \left( a_j^2 I_{3j} - a_j I_{4j} \right) \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} + K_j \psi^H_{\gamma j} (1 - \delta_{j1}) \\
+ \kappa_j \left( \psi^H_{\gamma j} + \frac{\partial \psi^H_{\gamma j}}{\partial \gamma} \right) - (I_{2j} - 2a_j I_{3j} + a_j^2 I_{4j}) \frac{\partial^2 \psi^H_{\gamma j}}{\partial \gamma^2} + (a_j I_{4j} - a_j^2 I_{3j}) \frac{\partial^3 \psi^H_{\gamma j}}{\partial \gamma^3} = 0,
\end{align*}
\]

(42c)
\[
\begin{align*}
\sum_{k=1}^{N_H} \left( \frac{\partial^2 H_{ij}}{\partial t^2} - \alpha_i \frac{\partial H_{ij}}{\partial t} + \beta_{ij} \frac{\partial^2 H_{ij}}{\partial x^2} + \gamma_{ij} \frac{\partial^4 H_{ij}}{\partial x^4} \right) &= 0, \\
\end{align*}
\]

Eq. (43a)–(43d) furnish us regarding dimensionless-nonlocal equations of motion of vertically aligned DWCNTs membranes subjected to a longitudinal temperature gradient according to the NHOBM. These are coupled equations and an efficient methodology should be implemented for assessing the natural frequencies of the nanosystem.

### 6.2.2. Evaluation of natural frequencies

It is assumed that the thermally affected nanosystem has simple supports at the ends of the constitutive nanotubes and the outermost tubes of the first and the last DWCNTs of the nanosystem have been provoked from any transverse motion; therefore,

\[
\begin{align*}
\bar{v}_H(\xi = 0, y, r) &= \bar{v}_H(\xi = 1, y, r) = 0, \\
\bar{w}_H(\xi = 0, y, r) &= \bar{w}_H(\xi = 1, y, r) = 0, \\
\end{align*}
\]

(44)

In order to examine natural frequencies of the thermally affected nanosystem according to Eq. (43a)–(43d) with the given conditions in Eq. (44), we take into account the following form for the deformation fields of the NHOBM:

\[
\begin{align*}
\bar{v}_i(\xi, y, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{v}^{\text{H}}_{(m\text{th})\text{y}}(m\pi\xi \sin(n\pi y)) e^{i\omega t}, \\
\bar{v}_j(\xi, y, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{v}^{\text{H}}_{(m\text{th})\text{y}}(m\pi\xi \sin(n\pi y)) e^{i\omega t}, \\
\bar{w}_i(\xi, y, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{w}^{\text{H}}_{(m\text{th})\text{y}}(m\pi\xi \sin(n\pi y)) e^{i\omega t}, \\
\bar{w}_j(\xi, y, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{w}^{\text{H}}_{(m\text{th})\text{y}}(m\pi\xi \sin(n\pi y)) e^{i\omega t},
\end{align*}
\]

(45a)–(45d)

where \(\bar{v}^{\text{H}}_{(m\text{th})\text{y}}\), \(\bar{v}^{\text{H}}_{(m\text{th})\text{y}}\), and \(\bar{w}^{\text{H}}_{(m\text{th})\text{y}}\) represent the amplitudes pertinent to the \((m, n)\)th vibration mode, \(\bar{v}^{\text{H}}_{(m\text{th})\text{y}}\) and \(\bar{v}^{\text{H}}_{(m\text{th})\text{y}}\) are the dimensionless frequencies associated with the vibrations in the \(y\) and \(z\) directions. Now, by substituting Eq. (45a)–(45d) into the linearly coupled Eq. (43a)–(43d), the characteristic equation of the thermally influenced DWCNTs membrane embedded in an elastic matrix is derived and the natural frequencies would be readily evaluated. The details of such calculations have been given in supplementary material, part E.
7. Results and discussion

Based on the given solutions for the nonlocal discrete and continuous models based on the Rayleigh and high-order shear deformable beam theories, the free thermo-elastic vibration characteristics of the nanosystem embedded in an elastic environment are going to be discussed here. The parameters used to calculate the in-plane and out-of-plane vibrations of thermally affected membranes of DWCNTs are as follows: \( r_m = 0.7 \text{ nm}, \nu_\text{p}=0.2, \varepsilon_{\text{th}}=2 \text{ nm}, E_{\text{b}} = E_{\text{p}}=1 \text{ TPa}, d = 2\varepsilon_{\text{th}}+\varepsilon_\text{s}, \rho_{\text{m}} = \rho_{\text{p}}=2300 \text{ kg/m}^3, \varepsilon_s=0.34 \text{ nm}. According to the previous studies [53], the thermal expansion coefficient of CNTs for high temperatures is different from that for low temperatures. This coefficient is considered to be \(-1.6 \times 10^{-6} \text{ } \text{ C}^{-1}\) for low temperatures and \(1.1 \times 10^{-6} \text{ } \text{ C}^{-1}\) for high temperatures. In all performed numerical studies, the nanosystem is isolated from the surrounding environment (i.e., \( K_c = K_0 = 0\) unless explicitly referred to. Concerning thermal conductivity and heat transfer coefficients of CNTs, Hsu et al. [54] proposed two-laser approach to measure heat transfer coefficient of SWCNTs bundles in air. It is reported that this factor would be in the range of \(1.5 \times 10^3 - 7.9 \times 10^4 \text{ W/m}^2\text{C}\) for nanotubes whose diameter ranges from 9.89 to 12.54 nm. Using micro-Raman spectroscopy technique, Wang et al. [55] reported that by increasing the diameter from 0.97 nm to 1.47 nm, the coefficient of heat transfer of suspended individual SWCNTs would grow from \(7.5 \times 10^4\) to \(8.9 \times 10^4 \text{ W/m}^2\text{C}\). By employing molecular dynamic, Che et al. [56] showed that the thermal conductivity of CNTs with diameter greater than 10 nm would be about 2980 W/m°C. In another review work, Han and Fina [57] proposed that the coefficient of thermal conductivity of CNT(11,11) would be in the range of 2000–7500 W/m°C when the environment temperature being in the range of 100–400°C. Based on this brief survey, we consider the coefficients of heat transfer and thermal conductivity of CNTs to be \(8.9 \times 10^4 \text{ W/m}^2\text{C}\) and 3500 W/m°C, respectively.

7.1. A comparison between the continuous-based results and the discrete-based ones

In Tables 1–3, the fundamental in-plane and out-of-plane frequencies of thermally affected membranes of DWCNTs according to the established continuous and discrete models are presented for different populations. The results are provided for slenderness ratios of 20, 30, 100, 1000, and 5000, at two temperature levels of 30 and 300°C. The given data in Tables 1 and 2 are given for the case of the nanosystem subjected to uniform temperature change (i.e., \( T_1 = T_2 = \Delta T \)) while those data in Table 3 are provided for nanosystem acted upon by temperature gradient (i.e., \( T_2 = T_1 + \Delta T \)) at uniform environmental temperature. The comparison of the continuous models’ results and those of the discrete models shows that the continuous model can estimate both in-plane and out-of-plane fundamental frequencies of discrete models’ results with acceptable accuracy. Further calculations reveal that the increase in the number of nanotubes does not affect the computational time of continuous models, so such models can be meritocratically used for exploring vibrational behavior of extremely populated nanosystems made from DWCNTs under temperature gradients.

As shown in Tables 1 and 3, the predicted results in the case of linear temperature variation are lower than those of the uniform temperature case, which is especially more apparent for the NRBM’s results. Generally, it can be concluded that the results of the two proposed models are closer to each other for the case of the linear temperature variation. Further studies show that increasing the temperature of the nanosystem from 30 to 300°C yields increasing of the fundamental frequencies. Actually, for nanosystems exposed to low temperatures, the resulted thermal tensile-forces within nanotubes would magnify with the temperature change and by increasing the geometrical stiffness, the total transverse stiffness of the thermally affected membrane would reduce. By increasing the number of DWCNTs from 5 to 50, the fundamental frequency decreases, which is more apparent in the case of in-plane vibration. It should be also noticed that the influence of population growth on the fundamental frequencies is higher in nanosystems with higher slenderness ratios. In addition, the rate of variation of frequency decreases with increasing the number of nanotubes in high-population nanosystems. The obtained results for both in-plane and out-of-plane vibrations show that the increase of the slenderness ratio up to a certain level would result in a decrease of the fundamental frequencies, and such a fact is much more obvious in nanosystems with more population. For slenderness ratios greater than above-mentioned certain value, the fundamental frequency commonly increases with the slenderness ratio.

In all cases, the obtained results based on the NHOBM are less than those of the NRBM. This fact is mainly attributed to the incorporation of the small-scale parameter into the shear stresses as well as the role of shear deformation in the total transverse stiffness of the constitutive nanotubes.
Table 2
Comparison of the in-plane fundamental frequencies of the discrete-based models and those of the continuous-based models in the case of uniform temperature change (low temperature case; $T_1 = T_2 = \Delta T$).

<table>
<thead>
<tr>
<th>Discrete models</th>
<th>$\Delta T = 30^\circ C$</th>
<th>$\Delta T = 300^\circ C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$N = 5$</td>
<td>$N = 10$</td>
</tr>
<tr>
<td>NRBM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1457.866</td>
<td>1177.587</td>
</tr>
<tr>
<td>30</td>
<td>1114.558</td>
<td>695.369</td>
</tr>
<tr>
<td>100</td>
<td>1157.499</td>
<td>529.889</td>
</tr>
<tr>
<td>1000</td>
<td>3925.708</td>
<td>1787.115</td>
</tr>
<tr>
<td>5000</td>
<td>8778.505</td>
<td>3996.270</td>
</tr>
<tr>
<td>NHORM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1413.057</td>
<td>1118.162</td>
</tr>
<tr>
<td>30</td>
<td>1108.136</td>
<td>683.780</td>
</tr>
<tr>
<td>100</td>
<td>1157.504</td>
<td>528.879</td>
</tr>
<tr>
<td>1000</td>
<td>3925.707</td>
<td>1787.115</td>
</tr>
<tr>
<td>5000</td>
<td>8778.504</td>
<td>3996.270</td>
</tr>
</tbody>
</table>

Table 3
Comparison of the out-of-plane fundamental frequencies of the discrete-based models and those of the continuous-based models in the case of existence of longitudinal temperature gradient (low temperature case; $T_1=0, T_2 = \Delta T$).

<table>
<thead>
<tr>
<th>Discrete models</th>
<th>$\Delta T = 30^\circ C$</th>
<th>$\Delta T = 300^\circ C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>$N = 5$</td>
<td>$N = 10$</td>
</tr>
<tr>
<td>NRBM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1104.879</td>
<td>1094.804</td>
</tr>
<tr>
<td>30</td>
<td>560.222</td>
<td>539.359</td>
</tr>
<tr>
<td>100</td>
<td>171.428</td>
<td>90.698</td>
</tr>
<tr>
<td>1000</td>
<td>140.836</td>
<td>63.910</td>
</tr>
<tr>
<td>5000</td>
<td>313.007</td>
<td>142.032</td>
</tr>
<tr>
<td>NHORM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>1040.569</td>
<td>1029.680</td>
</tr>
<tr>
<td>30</td>
<td>545.449</td>
<td>523.928</td>
</tr>
<tr>
<td>100</td>
<td>171.385</td>
<td>90.614</td>
</tr>
<tr>
<td>1000</td>
<td>140.836</td>
<td>63.910</td>
</tr>
<tr>
<td>5000</td>
<td>313.007</td>
<td>142.032</td>
</tr>
</tbody>
</table>

Table 2
Comparison of the in-plane fundamental frequencies of the discrete-based models and those of the continuous-based models in the case of uniform temperature change (low temperature case; $T_1 = T_2 = \Delta T$).

Table 3
Comparison of the out-of-plane fundamental frequencies of the discrete-based models and those of the continuous-based models in the case of existence of longitudinal temperature gradient (low temperature case; $T_1=0, T_2 = \Delta T$).

7.2. Another comparison study in a particular case

In this section, comparing of the results obtained from the proposed models with those of another work is of concern. Since thermo-elastic vibrations of a nanosystem consists of multiple DWCNTs have not been addressed yet, we restrict our verification study to a special case. Tounsi et al. [39] studied properties of elastic transverse waves in an individual DWCNTs in the presence of a uniform temperature field using nonlocal Euler–Bernoulli’s beam theory (NEBM). They proved the importance of the small-scale parameter as well as the temperature change on the characteristics of waves in such nanostuctures. For the sake of verification study, the following properties are considered: $\varepsilon_p = 2$ nm, $\rho_p = 2300$ kg/m$^3$, $E_p = 1$ TPa, $\beta_p = 0.35$ nm, $d_1 = 1.4$ nm, and $d_1 = 0.7$ nm. Let us define the ratio: $\gamma_1 = \frac{\omega_1}{(\omega_0)^{1/2}}$ where $\omega_0$ denotes the NEBM-based lowest frequency of the nth vibration mode with(out) consideration of the uniform temperature change. In Fig. 3, the predicted results by the proposed nonlocal model of Tounsi et al. [39] and those of our suggested discrete models based on the NRBM and the NHOMB for three levels of the uniform temperature change have been demonstrated in the case of environment with a low temperature. As it is seen, in the case of $n = 2$, there exists a reasonably good agreement between the predicted frequency ratios by the NRBM-based discrete model of this work and those of the NEBM-based model of Tounsi et al. [39]. Furthermore, the predicted results by the NHOMB are slightly lower than those of the NEBM. Such discrepancies in the results of these models are related to the effect of shear deformation.
Fig. 3. Comparison of the predicted frequency ratio of the second vibration mode by the proposed NRBM-based and NHOBM-based discrete models and those of Tounsi et al. [39] for different values of the uniform temperature change; (□) \( \Delta T = 20 \), (○) \( \Delta T = 40 \), (△) \( \Delta T = 60 \, ^\circ C \); (--) Tounsi et al. [39], (---) present study based on the NRBM, (--) present study based on the NHOBM; \( \varepsilon_0a = 2 \) nm.

Fig. 4. Influence of the temperature change in high temperatures on the out-of-plane fundamental frequency for different slenderness ratios: (a) \( \lambda_1 = 20 \), (b) \( \lambda_1 = 50 \), (c) \( \lambda_1 = 100 \); (---) NRBM, (--) NHOBM; (○) \( \varepsilon_0a = 0 \), (□) \( \varepsilon_0a = 1 \), ( Δ) \( \varepsilon_0a = 2 \) nm; \( N = 100 \), \( T_1 = T_2 = \Delta T \).

7.3. Role of the temperature gradient and a discussion on potential instability of the nanosystem

In this part, the effect of the temperature change on the free transverse vibrations of the nanosystem is evaluated. For this purpose, in Fig. 4(a)–4(c), the effect of temperature variation on the fundamental out-of-plane frequencies is plotted for thermally influenced nanosystems in the presence of high temperatures. The plotted results are based on the nonlinear continuous models for three slenderness ratios of 20, 50 and 100, as well as three values of the small-scale parameter 0, 1 and 2. The results of various theories show that the fundamental frequency of the nanosystem would reduce by increasing the thermal change. This is because of the reduction of the nanosystem’s transverse stiffness due to the increase in temperature. Further, the influence of the temperature changes on the variation of fundamental frequency of slenderer nanosystems is more obvious. As shown in Fig. 4(b) and 4(c), with the increase of the slenderness ratio, not only the fundamental frequency lessens, but also the predicted results by the NRBM become closer to those of the NHOBM. For instance, in the case of \( \lambda_1 = 20 \) and \( \varepsilon_0a = 2 \) nm, the NRBM predicts the NHOBM results with a relative error lower than 7 percent for the considered range of the thermal change, while in the case of \( \lambda_1 = 50 \) and \( \varepsilon_0a = 2 \) nm, the NRBM predicts the NHOBM’s results with a relative error lesser than 2 percent. In all proposed models, the fundamental frequencies reduce by increasing the small-scale parameter. This behavior is due to the reduction of the ratio of the transverse stiffness to the mass. The effect of the small-scale parameter is lowered by increasing the slenderness ratio such that the results of different models approach each other, which is clearly seen in Fig. 4(c). The plotted results confirm that the NHOBM’s graphs are commonly lower than those of the NRBM. As shown in Fig. 4(c), for a nanosystem with \( \lambda_1 = 100 \) subjected to a temperature change greater than a specific value, the out-of-plane fundamental frequency is zero, and thereby, the nanosystem would be no longer stable according to the linear nonlocal elasticity. The lowest value of the temperature change with zero frequency is called critical temperature change. Actually, for temperature changes greater than the critical value, the nonlinear deformation would be generated and the
nonlinear strain terms should be incorporated into the nonlocal stress-strain relations. The large deformation/postbuckling behavior of thermally affected membranes made from CNTs is also an interesting topic that should be pursued by the nanomechanics community in the near future.

According to the NRBM, the critical values of the uniform temperature change of a nanosystem with slenderness ratio of 100 and the small-scale parameters 0, 1, and 2 nm in order are 1535, 1529, and 1511°C, while the critical temperature changes on the basis of the NHOBM are 1526, 1521, and 1504°C, respectively. The comparison of these results shows that the predicted critical temperature change by the NRBM are greater than those obtained based on the NHOBM. Additionally, the critical temperature change would grow with the small-scale parameter. This brief discussion encourages us to evaluate the critical temperature changes analytically. Using the proposed nonlocal-continuous models by setting the fundamental frequency equal to zero, the explicit expression of the critical temperature of the nanosystem based on the NRBM and NHOBM are derived as:

\[
\Delta T_{ci}^R = - \frac{(\phi^R_{i1})^2(\phi^R_{i2})^2(\phi^R_{i3})^2 + 2(\phi^R_{i1})(\phi^R_{i2})(\phi^R_{i3})^2 + (\phi^R_{i3})^4 + (\phi^R_{i2})^2(\phi^R_{i3})^2 + (\phi^R_{i1})^2(\phi^R_{i3})^2 + (\phi^R_{i1})(\phi^R_{i2})(\phi^R_{i3})^2 + (\phi^R_{i1})^2(\phi^R_{i2})^2 + 2(\phi^R_{i1})^2(\phi^R_{i2})(\phi^R_{i3})^2 + (\phi^R_{i1})^4)}{2(\phi^R_{i1})(\phi^R_{i2})(\phi^R_{i3})^2}, \quad (46a)
\]

\[
\Delta T_{ci}^H = - \frac{(\phi^H_{i1})^2(\phi^H_{i2})^2(\phi^H_{i3})^2 + 2(\phi^H_{i1})(\phi^H_{i2})(\phi^H_{i3})^2 + (\phi^H_{i3})^4 + (\phi^H_{i2})^2(\phi^H_{i3})^2 + (\phi^H_{i1})^2(\phi^H_{i3})^2 + (\phi^H_{i1})(\phi^H_{i2})(\phi^H_{i3})^2 + (\phi^H_{i1})^2(\phi^H_{i2})^2 + 2(\phi^H_{i1})^2(\phi^H_{i2})(\phi^H_{i3})^2 + (\phi^H_{i1})^4)}{2(\phi^H_{i1})(\phi^H_{i2})(\phi^H_{i3})^2}, \quad (46b)
\]

where \(\phi^R_{i} \), \(i = 1, 2, \ldots, 9\) and \(\phi^H_{i} \), \(j = 1, 2, \ldots, 12\) are given in supplementary material, part F. Eq. (46a) and (46b) display that the critical values of temperature change for membranes made from DWCNTs rely on geometrical data of the membrane as well as mechanical properties of both membrane and elastic matrix. In fact, each factor that assists transverse stiffness of the nanosystem would result in a higher critical temperature change.

In Fig. 5(a)-(c), the effect of temperature change at high temperatures on the in-plane fundamental frequency has been demonstrated. A close verification of the plotted results in Figs. 4 and 5 indicate that the in-plane frequencies are generally higher than those of the out-of-plane ones, especially in slenderer nanosystems.

In Fig. 6, the diagrams of the variations of both in-plane and out-of-plane fundamental frequencies as a function of temperature change have been demonstrated for nanosystems at low temperatures. Unlike the case of temperature changes at high temperature situation, in this case, both models predict that both in-plane and out-of-plane frequencies would increase as the temperature change grows. By comparing the frequencies associated with the in-plane and out-of-plane vibrations, it is realized that the rate of change of the out-of-plane frequency in terms of temperature change is greater than that of the in-plane frequency. In addition, the predicted in-plane fundamental frequencies are commonly larger than those of the out-of-plane ones. The results show that by increasing the slenderness ratio, in addition to the reduction of the fundamental frequencies, the results of the two proposed models approach each other. In most of the cases, the NHOBM’s results are lower than the NRBM’s results which is mainly related to the incorporation of shear stresses into the elastic strain energy of the nanosystem modeled by the NHOBM.

7.4. Role of the slenderness ratio

One of the most important parameters affecting free dynamic response of ensembles of CNTs is the slenderness ratio. The graphs of the out-of-plane fundamental frequency as a function of slenderness ratio have been shown in Fig. 7(a)–7(c) for three values of the temperature change and three
populations. The obtained results based on the proposed nonlocal models indicate that the fundamental frequency of the nanosystem decreases with the increase of slenderness ratio. The rate of reduction is more obvious in nanosystems with more DWCNTs. On the other hand, with the increase in the number of DWCNTs, the fundamental frequency would lessen, particularly in nanosystems with higher slenderness ratio. As shown in Fig. 7(a), the results predicted by the NRBM, especially in lower slenderness ratios, are higher than the NHOBM’s results. By increasing the slenderness ratio, the results of these models approach each other. This reflects the great effect of the shear deformation on the free vibration of nanosystems with low slenderness ratio. By comparing the provided graphs in Fig. 7(a)–7(c), it is understood that by increasing the temperature of the nanosystem, the fundamental frequency decreases. In addition, the relative difference between the results of the NRBM and those of the NHOBM increases at high temperatures.

In Fig. 7(b), in the case of $\Delta T = 2100$ and $N = 20$, the frequency of the nanosystem becomes zero in the slenderness ratio of 105. The lowest slenderness ratio in which the nanosystem frequency vanishes is called the critical slenderness ratio. Actually, the nanosystem with slenderness ratio great than the critical one would be dynamically unstable. Therefore, identifying the critical slenderness ratio is important in optimal design of the membranes from DWCNTs affected by the temperature gradients. As it is seen in the demonstrated graphs, the critical slenderness ratio would reduce as the temperature change increases for those nanosystems in environments with high temperatures. The critical slenderness ratio in the cases of $N = 30$ and $N = 100$ reaches 90 and 85, respectively. This means that the instability of the thermally affected nanosystems with higher population occurs at smaller slenderness ratio. In the case of $\Delta T = 700$, the relative difference between the results of the NRBM and NHOBM would reduced by increasing the slenderness ratio; for example, in the cases of $\lambda_1 = 10$ and 110, the relative differences between the results of these models are, respectively, about 20 and 0.5 percent. In the case of $\Delta T = 700$ and 2100, as the slenderness ratio grows the relative discrepancies between the results of the proposed models decrease up to a certain slenderness ratio. For the slenderness ratios greater than this particular level, the aforementioned discrepancies increase with growing of the slenderness ratio such that reaches to its maximum value at the zero frequency. This issue is more apparent for the thermally affected nanosystems with higher populations.

With regard to the crucial role of the slenderness ratio on the critical temperature change of the nanosystem, this fact has been illustrated in Fig. 8. The results are given for a membrane consists of 4000 DWCNTs in the cases of $e_0\pi = 0$, 1 and 2 nm. The results of both nonlocal-continuous-based theories indicate that the critical temperature of the nanosystem decreases with increasing the slenderness ratio. In addition, the critical temperature of the nanosystem decreases with the increase of the small-scale parameter. By comparing the results of the two proposed models, the results of the NHOBM are generally lower than those of the NRBM. By increasing the slenderness ratio, the difference between the predicted critical temperatures by suggested models would reduce for all considered levels of the small-scale parameter. As discussed in previous parts, this issue is mainly related to the reduction of the shear effect. The predicted results show that for $\lambda_1 > 30$, the NRBM could reproduce the results of the NHOBM model with relative error lesser than 7 percent, irrespective of the small-scale parameter.

7.5. Role of the radius of constitutive tubes

In this part, the aim is to investigate the effect of the radius variation of constitutive DWCNTs on the out-of-plane fundamental frequency. For this purpose, three nanosystems with nanotube’s lengths of 20, 30 and 50 nm are considered in a high temperature medium. According to Fig. 9, the results of proposed models indicate that the fundamental frequency of the nanosystem would increase by growing of the radius of DWCNTs. This fact is more obvious at high temperatures; such vibrational behavior of the nanosystem can be attributed to the increase of vdW forces due to the growth of radius. In all plots in Fig. 9(a)–(c), with increasing radius of DWCNTs, the relative difference between the fundamental frequencies predicted by the NRBM and those of the NHOBM generally increases. For example, for nanosystems of length 20 nm subjected to $\Delta T = 700$ °C, NRBM can estimate the results of the NHOBM with relative error in the range of 1–5 percent. However, at $\Delta T = 2100$ °C, the relative error of these models ranges from 2 to 5.5 percent, which reveals an increase in discrepancies between the results of the proposed models due to an increasing of the temperature. At different temperatures, it is also clearly observed that the variations of relative discrepancies of two models’ results in terms of the radius would reduce by increasing the length of nanotubes. In all proposed models, by growing the temperature change from 700 to 2100°C, and also through increasing the length of the nanotubes from 20 to 50 nm, the fundamental frequency has dropped. Further studies show that the variation of fundamental frequency is higher in slenderer nanosystems. According to the demonstrated results in Fig. 9(b) and 9(c), for a specific value of the radius of the nanotubes, the fundamental frequency becomes zero in the case of $l_n = 50$ nm. The radius in which the fundamental frequency of the nanosystem would be zero is called the critical radius. Based on
the suggested nonlocal theories, the critical radii for a nanosystem of length 50 nm at \( \Delta T = 1400 \) and 2100°C are about 0.66 and 0.86 nm, respectively. In other words, the critical radius of the nanosystem at high temperatures would grow by an increase of the temperature change.

7.6. Role of the adjacent elastic environment

Using nonlocal-continuous models based on the NRBM and NHOBM, the effect of transverse stiffness of the elastic environment on the in-plane and out-of-plane fundamental frequencies of the nanosystem is shown in Fig. 10. The results are provided for a nanosystem with 300 vertically aligned DWCNTs under temperature changes of 0, 800, and 1600°C. For both proposed models, the fundamental frequencies associated with the in-plane and out-of-plane vibrations increase with increasing the transverse stiffness. In addition, the fundamental frequencies of the elastically embedded nanosystem at high temperatures reduce by increasing the temperature. The influence of the temperature increase on the estimated results by both models is approximately the same. Also, the fundamental frequencies of the in-plane vibration and those of the out-of-plane vibration are nearly equal in most of the cases, which can be attributed to the high level of the transverse stiffness of surrounding environment in both \( y \) and \( z \) directions. The presented plots show that NHOBM’s results are always lower than the NRBM’s results. Also, the relative discrepancies between the plotted results based on the NRBM and those of the NHOBM would decrease as the transverse stiffness grows.

The role of the rotational stiffness of the neighboring environment on the fundamental frequencies of the thermally affected nanosystem has been shown in Fig. 11. The results of the proposed models indicate that the fundamental frequencies of the nanosystem increase by growing the rotational stiffness. In addition, an increase of the temperature change from zero to 1600°C, the fundamental frequencies of the nanosystem would decrease. The plotted diagrams show that the relative discrepancies between the results of the NRBM and those of the

Fig. 7. Influence of the slenderness ratio in high temperatures on the out-of-plane fundamental frequency for different levels of the temperature change and number of DWCNTs: (a) \( T_1 = T_2 = \Delta T = 700 \), (b) \( T_1 = T_2 = \Delta T = 1400 \), (c) \( T_1 = T_2 = \Delta T = 2100 \) °C; (□) NRBM, (○) NHOBM; (□) \( N = 20 \), (○) \( N = 30 \), (\( \Delta \)) \( \Delta N = 100 \), \( e_0, h = 2 \) nm.
NHOBM grow by an increase of the rotational stiffness of the adjacent environment. For instance, in the case of $\Delta T = 800$ and $K^R = 0$, the relative difference between the two models is about 3 percent, which reaches 20 percent at $K^R = 100$. The main reason of this fact is the incorporation of the rotational stiffness into the shear strain energy of the thermally affected nanosystem modeled by the NHOBM.

7.7. Role of the intertube distance

Herein the influence of the intertube distance on the thermally affected nanosystem’s vibration behavior is considered. For this purpose, the effect of gap between DWCNTs at high temperatures on both in-plane and out-of-plane fundamental frequencies is studied for three nanosystems of populations $N = 10$, 20 and 80 (see Fig. 12). By increasing the temperature change, the fundamental frequencies associated with the in-plane and out-of-plane vibrations would reduce according to the proposed nonlocal-continuous models. In a low-population nanosystem, the in-plane frequency decreases with growing the intertube distance such that it reaches to its locally minimum value at $d = 2.49$ nm. As the distance increases, in contrast to the previous branch, the frequency increases, and for intertube distances greater than 2.65 nm, the variation of the frequency as a function of the intertube distance becomes almost negligible. Further investigations show that this trend is very similar to the variation of vdW force coefficient in terms of the intertube distance. As the nanosystem’s population increases, the rate of change of in-plane frequencies in terms of intertube distance decreases such that in largely populated nanosystems, the fundamental frequency almost does not alter with the intertube distance. In contrast to the results obtained for the in-plane vibration, the out-of-plane frequency has fairly no sensitive to the intertube distance. Generally, this behavior is due to the low variation of the vdW forces in terms of the intertube distance. Additionally, the relative discrepancy between the predicted in-plane frequencies based on the NRBM and those of the NHOBM reaches to its maximum value at the locally minimum points of the plots. This issue is more apparent for thermally affected nanosystems with low number of nanotubes.
8. Concluding remarks

Transverse vibrations of elastically confined monolayers membranes from vertically aligned DWCNTs in the presence of longitudinal temperature field are examined using nonlocal elasticity theory. To this end, a continuum-based model is developed for evaluating vdW interactional forces within the nanosystem. This model explains simply that how variation of such forces could be linearly linked to the variations of the in-plane and out-of-plane deflections of nanotubes. By deploying a nonlocal Fourier law model, the nonlocal temperature fields and the resulted thermal forces within nanotubes because of the longitudinal temperature gradients are calculated accounting for heat dissipation through the surface of constitutive DWCNTs. Then, by implementing NRBM and NHOBM and the Hamilton’s principle, both nonlocal-discrete and nonlocal-continuous models are developed and solved for fundamental frequencies using assumed mode approach. The recently novel continuous models enable us to study free dynamic response of thermally affected membranes of DWCNTs irrespective of their populations. The explicit expressions of the in-plane and the out-of-plane frequencies based on the continuous-based models are obtained on the basis of the NRBM and NHOBM, and thereafter, the critical values of temperature change are evaluated analytically. The influences of temperature change, slenderness ratio, radius of nanotubes, intertube distance, transverse and rotational stiffness of the elastic foundation on both in-plane and out-of-plane fundamental frequencies of the nanosystem acted upon by uniform temperature changes as well as longitudinal temperature gradients are examined comprehensively.

The proposed nonlocal continuum-based models and the suggested methodologies in this work could be extended to vibrational analysis of thermally influenced membranes made from multi-walled carbon nanotubes. Further, vibrational examination of three-dimensional jungles of vertically aligned DWCNTs in the presence of longitudinal and transverse temperature gradients could be considered as a hot topic for the future explorations. Through advancements in continuum-based modeling of doubly orthogonal DWCNTs subjected to temperature gradients, we could also explore free vibrations of thermally affected...
double-layered or even multi-layered orthogonal-membranes made from DWCNTs by some efforts. These important issues could be paid attention to by the applied mechanics community in the near future.

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Supplementary material

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