Nonlocal dynamic response of double-nanotube-systems for delivery of lagged-inertial-nanoparticles

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A R T I C L E   I N F O

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A B S T R A C T

This paper deals with transverse vibrations of doubly parallel nanotubes acted upon by doubly lagged-moving nanoparticles accounting for nonlocality. By exploiting the nonlocal Rayleigh and higher-order beam models, equations of motion of the nanosystem are obtained by considering the nonlocal inertial force as well as the lag of moving nanoparticles. For the case of ignorance of lateral inertia of moving nanoparticles, an analytical solution based on the Laplace transform method is suggested and the exact deformation fields of the nanosystem are presented for two patterns. In the case of consideration of the lateral inertia of moving nanoparticles, a numerical approach based on the Galerkin approach and admissible modes is suggested and the elasto-dynamic fields of the nanosystem are appropriately determined at each time. By comparing the predicted results by the proposed nonlocal numerical models with those of another work and the analytical approach, a reasonably good agreement is achieved. Subsequently, the roles of shear deformation, nonlocality, lag effect, geometry of the nanosystem, and kinematic properties of the moving nanoparticles on the dynamic deflections of the constituent tubes are displayed. This scrutiny is aimed to be a solid base for examining vibrations of a membrane of nanotubes for delivery of multiple nanoparticles through their pores.

1. Introduction

Carbon nanotubes (CNTs) have been recognized as highly efficient deliverers of molecules, spherical particles, DNA, drugs, and therapeutics [1–5]. This important function of such tube-like nanostructures is highly indebted to the nearly frictionless nature of their pores [6–8] as well as their high mechanical properties [9–11]. Regarding movement of particles through the pores of the single-walled carbon nanotubes (SWCNTs), Chang [12] reported domino-driven mechanical waves of speed about 1000 m/s due to the collapsing of SWCNTs with diameter greater than 3.5 nm. The conducted molecular dynamics simulations show that this pump of energy could be effectively employed for accelerating the inside nanoparticles. In another study, Chang and Guo [13] investigated the influence of the temperature on the domino process of nanotubes via molecular dynamics. The main obtained result was that the domino motion of the propagated waves does not only rely on the SWCNT’s diameter but also on the temperature of the tube. Actually, the temperature could be used for controlling both speed and direction of the propagated waves resulted from collapsing of the tube. It should be noted that an individual CNT is rarely employed for a specific purpose, and commonly, groups or ensembles of them are fabricated for the considered application. In this view, vibrations of membranes of nanotubes for translocation of nanoparticles are of high interest to the nanotechnologist; however, such a crucial issue has not been investigated yet. Before proceeding on this critical and complicated problem, vibrations of a double-nanotube-system subjected to arbitrarily moving nanoparticles should be realized and discussed carefully. In the present paper, we eagerly study nonlocal transverse vibrations of such a nanosystem acted upon by a set of doubly moving nanoparticles with both lag and lateral inertia using nonlocal beam models.

At the nanoscale, vibrations of an atom could influence on the vibrations of its nearby atoms due to interatomic bonds and forces. This means that the used model for mechanical analysis of the nanostructure should be at such a level of sophistication that could capture at least the nonlocality of elastic fields. One of the most popular models that extensively exploited by researchers is the nonlocal continuum field theory of Eringen [14–16]. This theory could be simply displayed by this fact that the state of each field (for example, strain, stress, temperature, magnetic, or electrical field) at each point of the continuum does not only depend on the state of the field at that point, but also to the states of that field at the adjacent points. This fact is commonly called nonlocality. Mathematically, for instance, the nonlocal stress tensor is related to the local stress one by: \( \sigma_{ij}^{nl}(x, t) = \int_{\Omega} \int_{\Omega} \sigma_{ij}(x', t') dV(x') \frac{1}{(x-x')^q} dV(x) \).
\[ \mathcal{K} = \int V \left( \nabla \mathbf{u}_s \cdot \nabla \mathbf{e}_s - \nabla \mathbf{u}_n \cdot \nabla \mathbf{e}_n \right) dV \]

Fig. 1. A continuum-based representation of DPNTs subjected to doubly moving-inertial nanoparticles with a lag.

Mechanical modeling of CNTs used for particle delivery is initiated by Kiani and Mehr [48], in which transverse vibrations of SWCNTs subjected to moving nanoparticles in the lack of their inertial effects were explicitly evaluated using nonlocal beam theories. In a more advanced research work, Kiani and Wang [49] investigated longitudinal and transverse vibrations of SWCNTs acted upon by a moving nanoparticles accounting for its transverse inertia. Based on the nonlocal beam models, the equations of motion were derived and solved for longitudinal and transverse dynamic displacements. Nonlinear transverse vibrations of SWCNTs for translocation of nanoparticles were also examined in some detail [50,51]. Nikkhoo et al. [52] developed a nonlocal regression-based model to study transverse vibrations of nanotubes due to a moving inertial nanoparticle by employing nonlocal Rayleigh beam model. Simsek [53] researched lateral vibration of a nanotube under a moving pointed load. Without considering the inertia of the moving load, the role of inertial effects on the maximum deflection of the nanotube was displayed. Simsek [54] studied transverse vibration of double-carbon nanotube systems under a moving nanoparticle without considering the nanoparticles’ inertia. This brief survey reveals that vibrations of double-nanotube-systems under doubly moving nanoparticles with inertia and lag effects have not been displayed yet.

In this paper, transverse vibrations of doubly-adjacent-nanotube-systems due to the delivery of nanoparticles with an arbitrary lag are going to be studied in the context of nonlocal elasticity theory. Through modeling of nanotubes by beam-like elements on the basis of the nonlocal Rayleigh beam model (NRBM) and the nonlocal higher-order beam model (NHOBM), accounting for the vdW interactional force, and the lag of moving nanoparticles, useful models are developed. In the case of without considering the inertial effects of R moving nanoparticles, an analytical solution to the obtained nonlocal equations of motion with the lag effect is suggested. For the case of consideration of the inertia of moving nanoparticles, a Galerkin-based assumed mode method is developed for determining the dynamic response of the nanosystem. For low levels of the velocity of moving nanoparticles as well as mass, the suggested numerical methods could reasonably capture the results of the analytical solution. In the remainder of this paper, the roles of intertube distance, velocities and mass weight of doubly moving nanoparticles, lag effect, and slenderness ratio of the nanosystem on the maximum dynamic deflections of constitutive tubes are examined in some details. The importance of shear and nonlocality effects on the dynamic response is displayed and highlighted. The obtained results of this work provide a solid base for better realizing vibrations of vertically aligned membranes made from SWCNTs used for translocation of nanoparticles through their pores.

2. Nanomechanical problem: Description and assumptions

Consider a system of doubly parallel nanotubes (DPNTs) of identical length \( l_b \), radii \( r_{m1} \) and \( r_{m2} \), intertube distance \( d \), which are acted upon by doubly moving nanoparticles as shown in Fig. 1. For structural modeling of constitutive tubes, equivalent continuum structures (ECSs) associated with these tubes have been taken into account. The mass weight and the velocity of the ith moving nanoparticle are, respectively, represented by \( M_i \) and \( v_i \) where \( i = 1, 2 \). The second particle arrives at the nanosystem with the time-delay \( t_d \) and the departure times of the moving nanoparticles inside the nanotubes 1 and 2 are denoted by \( t_{f1} \) and \( t_{f2} \) where \( t_{f1} = \frac{d}{v_1} \) and \( t_{f2} = \frac{d}{v_2} \). The resulted deflections of the first and second nanotubes in order are represented by \( u_{1}^{1}(x,t) \) and \( u_{2}^{1}(x,t) \); \( [\cdot] = R \) or \( H \), and the fields with the superscripts \( R \) and \( H \) are those based on the NRBM and NHOBM, respectively.

The weight of the nanoparticles, the friction forces between the outermost surfaces of the nanoparticles and the innermost surface of the tubes, the vdW forces between the moving nanoparticle and the atoms of the constitutive SWCNTs, and the vdW forces between doubly parallel SWCNTs
Fig. 2. Cross-sectional view of the nanosystem for two possible configurations: (a) pattern #1, (b) pattern #2.

are among the major forces that could influence on the vibrational behavior of the nanosystem. Since only transverse vibrations of the constitutive tubes are of interest, the above-mentioned friction force is not considered. Further, the existing vdW forces between the moving nanoparticle and the constitutive atoms of nanotubes do not affect the transverse vibrations; however, such forces could contribute to the friction force and the caused longitudinal vibrations [49]. It is assumed that the moving nanoparticles are in contact with the inner surfaces of nanotubes during the course of excitation. Therefore, the interactional contact force between the moving nanoparticles and the nanotubes can be stated mathematically by:

$$f_{vi} = M_i \ddot{v}_i \left( 1 - \frac{1}{g} \frac{D^2 w_i}{Dt^2} \right) \delta(x - x_{M_i}); \ i = 1, 2,$$

(1)

where $g$ is the gravitational acceleration, $\frac{D^2 w_i}{Dt^2}$ is the second material derivative, $\delta$ is the Dirac delta function, and $x_{M_i}$ represents the location of the $i$th moving nanoparticle. Actually, $\frac{D^2 w_i}{Dt^2} \delta(x - x_{M_i})$ is the transverse acceleration of the $i$th nanotube at the location of the moving particle. By assuming that the $i$th moving nanoparticle moves axially with the constant velocity $v_i$, the contact force is provided by:

$$f_{vi} = M_i \left( g - \frac{\partial^2 w_i}{\partial t^2} + 2v_i \frac{\partial^2 w_i}{\partial x \partial t} + v_i^2 \frac{\partial^2 w_i}{\partial x^2} \right) \delta(x - x_{M_i}).$$

(2)

The nanosystem exploited for translocation of nanoparticles could be imagined in two patterns as demonstrated in Fig. 2. From continuum mechanics point of view, the intertube vdW force between doubly adjacent tubes is a function of their radii, length, intertube distance, and their lateral motions. Therefore, any discrepancy in transverse displacements of nanotubes (i.e., $\Delta w = w_j(x, t) - w_i(x, t)$) would cause a change in the lateral component of the vdW force (i.e., $\Delta F_{vdW}$). In the light of Taylor’s series of the total transverse vdW force between tubes about $\Delta w = 0$ by excluding the static term, the change in transverse vdW force is related to the discrepancy in transverse displacement fields as follows:

$$\Delta F_{vdW} = C_{vz} \Delta w + \sum_{n=2}^{\infty} C_{vz}^{(n)} (\Delta w)^n,$$

(3)

where the appeared coefficients in this relation are readily calculated by: $C_{vz} = \frac{\partial F_{vdW}}{\partial \Delta w} \big|_{\Delta w=0}$ and $C_{vz}^{(n)} = \frac{1}{n!} \frac{\partial^n F_{vdW}}{\partial (\Delta w)^n} \big|_{\Delta w=0}$. By omitting the nonlinear terms in Eq. (3), the transverse intertube vdW forces could be pictured as an imaginary elastic layer of constant $C_{vz}$ whose value is evaluated by [55]:

- **Pattern 1:**

  $$C_{vz} = -\frac{256 \pi r_{m1} r_{m2}}{9 \pi a^4} \left\{ \begin{array}{l} \frac{\alpha^2 \Lambda^2 (\Lambda - 14 \Lambda^2 (r_{m1} \cos(\phi_2) - r_{m1} \cos(\phi_1))^2) \phi_1 \phi_2 \phi_3} {\alpha^2} \left[ \Lambda^{2} - 8 \Lambda^{2} (r_{m2} \cos(\phi_2) - r_{m2} \cos(\phi_1))^2 \right] \right\} d\phi_1 d\phi_2 d\phi_3 dx_1 dx_2,$$

  (4)

- **Pattern 2:**

  $$C_{vz} = -\frac{256 \pi r_{m1} r_{m2}}{9 \pi a^4} \left\{ \begin{array}{l} \frac{\alpha^2 \Lambda^2 (\Lambda - 14 \Lambda^2 (d + r_{m1} \sin(\phi_2) - r_{m2} \sin(\phi_1))^2) \phi_1 \phi_2 \phi_3} {\alpha^2} \left[ \Lambda^{2} - 8 \Lambda^{2} (d + r_{m1} \sin(\phi_2) - r_{m2} \sin(\phi_1))^2 \right] \right\} d\phi_1 d\phi_2 d\phi_3 dx_1 dx_2,$$

  (5)

where

$$\Lambda = \frac{\pi}{4} \Lambda, \quad a = \text{the length of the carbon-carbon bond}, \quad \epsilon = \text{the well depth}.$$

3. Vibrational analysis without considering inertia of moving nanoparticles

In this context, transverse dynamic deflections of the nanosystem for the case of without consideration of the lateral inertial of the moving nanoparticles (i.e., moving load approach (MLA)) are analytically extracted on the basis of the NRBM and the NOOBM. These models could be also exploited
as a benchmark approach for checking the predicted dynamic response of the nanosystem under moving inertial particles by the Galerkin-based assumed mode method that will be explained in the upcoming section.

3.1. Analytical formulations based on the NRBM

Using the NRBM, in view of the given assumptions in Section 2, the equations of motion that display transverse vibrations of the nanosystem acted upon by moving nanoparticles in terms of the nonlocal bending moment are displayed by:

\[
\begin{align*}
\rho_n & \left(A_{b_1} \frac{\partial^2 w^R}{\partial t^2} - I_{b_1} \frac{\partial^2 u^R}{\partial t \partial x} \right) + C_{\perp z}(u^R_1 - w^R_1) - \frac{\partial^2 (M^R_{1z})}{\partial x^2} = M_1 \delta(x - x_{M_1}) H(t_{f_1} - t), \\
\rho_n & \left(A_{b_2} \frac{\partial^2 w^R}{\partial t^2} - I_{b_2} \frac{\partial^2 u^R}{\partial t \partial x} \right) + C_{\perp z}(u^R_2 - w^R_2) - \frac{\partial^2 (M^R_{2z})}{\partial x^2} = M_2 \delta(x - x_{M_2}) [H(t_{f_2} - t) - H(t_{f_2} - t)],
\end{align*}
\]

(6a)

(6b)

where \(\rho_n\), \(A_{b_i}\), \(I_{b_i}\) are density, cross-sectional area, moment of inertia of the cross section, the nonlocal bending moment of the \(i\)th tube’s ECO, and the Heaviside step function. In the context of a simplified version of the nonlocal elasticity theory of Eringen [14–16], the nonlocal constitutive equations for bending moments within the \(i\)th tube modeled by the NRBM is expressed by: \(\Xi(M^R_{iz}) = (\sigma^R_{iz}) = -E_{b_i} I_{b_i} \frac{\partial^2 w^R}{\partial x^2} \); \(i = 1, 2\) where the nonlocal operator is given by: \(\Xi[.] = [\dot{\Xi} - (\epsilon(\alpha^2) \frac{\partial^2 }{\partial x^2})[.\] Now the recently stated constitutive relation is introduced to Eqs. (6a) and (6b), and we would arrive at the nonlocal governing equations that display transverse vibrations of double-nanotube-system subjected to moving nanoparticles with the lag effect:

\[
\begin{align*}
\Xi(\rho_n) & \left(A_{b_1} \frac{\partial^2 w^R}{\partial t^2} - I_{b_1} \frac{\partial^2 u^R}{\partial t \partial x} \right) + C_{\perp z}(u^R_1 - w^R_1) + E_{b_1} I_{b_1} \frac{\partial^2 u^R}{\partial x^2} = M_1 \delta(x - x_{M_1}) H(t_{f_1} - t), \\
\Xi(\rho_n) & \left(A_{b_2} \frac{\partial^2 w^R}{\partial t^2} - I_{b_2} \frac{\partial^2 u^R}{\partial t \partial x} \right) + C_{\perp z}(u^R_2 - w^R_2) + E_{b_2} I_{b_2} \frac{\partial^2 u^R}{\partial x^2} = M_2 \delta(x - x_{M_2}) [H(t_{f_2} - t) - H(t_{f_2} - t)],
\end{align*}
\]

(7a)

(7b)

for a simply supported nanosystem, the boundary conditions read:

\[
u^R(0, t) = w^R(0, t) = 0, \quad \left( M^R_{1z} \right)(0, t) = \left( M^R_{1z} \right)(l_i, t) = 0,
\]

and the initial conditions are assumed to be:

\[
u^R(x, 0) = 0, \quad \frac{\partial \nu^R}{\partial t}(x, 0) = 0.
\]

(8)

(9)

By considering the following dimensionless quantities:

\[
\begin{align*}
\bar{\xi} &= \frac{l_i}{l_0}, \quad \bar{w}^R = \frac{w^R}{w^0}, \quad \bar{u}^R = \frac{u^R}{u^0}, \quad \bar{R} = \frac{R}{R^0}, \quad \bar{C}_{\perp z} = \frac{C_{\perp z}}{C_{\perp z}^0}, \quad \bar{A}_l = \frac{A_l}{A_l^0}, \quad \bar{I}_l = \frac{I_l}{I_l^0}, \quad \bar{t} = \frac{l_0 t}{R^0}, \quad \bar{\tau} = \frac{l_0 \tau}{R^0}, \\
\bar{\xi}^2_1 &= \frac{\rho_{b_1} I_{b_1}}{E_{b_1} A_{b_1}}, \quad \bar{\xi}^2_2 = \frac{\rho_{b_2} I_{b_2}}{E_{b_2} A_{b_2}}, \quad \bar{\mu} = \frac{\mu}{\rho b l_0}, \quad \Xi[.] = [\dot{\Xi} - \epsilon(\alpha^2) \frac{\partial^2 }{\partial x^2}[.]
\end{align*}
\]

(10)

the dimensionless equations of motion of double-nanotube-system subjected to moving nanoparticles accounting for nonlocality and lag effects are derived:

\[
\begin{align*}
\bar{\Xi} \left( \frac{\partial^2 \bar{u}^R}{\partial t^2} - \bar{\alpha}^2 \frac{\partial^2 \bar{u}^R}{\partial x^2} \right) + \frac{\partial \bar{\nu}^R}{\partial x} &= \bar{M} \delta(\bar{\xi} - \bar{\xi}_M) H(\bar{t} - \bar{\tau}), \\
\bar{\Xi} \left( \frac{\partial^2 \bar{w}^R}{\partial t^2} - \left( \frac{\bar{\xi}^2_1}{\bar{\alpha}^2_1} \right) \frac{\partial^2 \bar{w}^R}{\partial x^2} + \bar{C}_{\perp z}(\bar{u}^R_1 - \bar{w}^R_1) \right) + \bar{\xi}^2_1 \frac{\partial \bar{\nu}^R}{\partial x} &= \bar{M} \delta(\bar{\xi} - \bar{\xi}_M) \left[ H(\bar{t} + \bar{\tau}) - H(\bar{t}) \right],
\end{align*}
\]

(11a)

(11b)

with the following initial and boundary conditions in the dimensionless form:

\[
\begin{align*}
\bar{u}^R(0, \bar{t}) &= \bar{u}^R(1, \bar{t}) = 0, \quad \left( M^R_{1z} \right)(0, \bar{t}) = \left( M^R_{1z} \right)(1, \bar{t}) = 0, \\
\bar{\nu}^R(\bar{\xi}, \bar{t}) &= 0, \quad \frac{\partial \bar{\nu}^R}{\partial \bar{t}}(\bar{\xi}, \bar{t}) = 0.
\end{align*}
\]

(12)

where \((M^R_{1z}) = (\bar{M}^R_{1z})/l_0/\Xi(\bar{E}_{b_1} I_{b_1})\). For the double-nanotube-system at hand, the dimensionless deflection fields of the constitutive tubes in the discretized form are considered as: \(\bar{u}^R(\bar{\xi}, \bar{t}) = \sum_{n=1}^{N_{\bar{\xi}}} \bar{u}^R_n(\bar{\xi}) \Phi_n(\bar{\xi}), \bar{\nu}^R(\bar{\xi}, \bar{t}) = \sum_{n=1}^{N_{\bar{\xi}}} \bar{\nu}^R_n(\bar{\xi}) \Phi_n(\bar{\xi}), \) where \(\Phi_n(\bar{\xi}) = \sin(n\pi \bar{\xi})\). By direct substitution of these expressions into Eqs. (11a) and (11b), we will arrive at the following set of ordinary differential equations (ODEs):

\[
\begin{pmatrix}
\frac{d^2 \bar{u}^R}{d\bar{t}^2} \\
\frac{d^2 \bar{\nu}^R}{d\bar{t}^2}
\end{pmatrix} + \begin{pmatrix}
\bar{\Gamma}_1 & \bar{\Gamma}_2 \\
\bar{\Gamma}_3 & \bar{\Gamma}_4
\end{pmatrix} \begin{pmatrix}
\bar{u}^R \\
\bar{\nu}^R
\end{pmatrix} = \begin{pmatrix}
\bar{M} \\
\bar{M}
\end{pmatrix} \delta(\bar{\xi} - \bar{\xi}_M) H(\bar{t} - \bar{\tau}).
\]
\[
\begin{align*}
\phi_R &= \sin \left( g_R \tau \right) H \left( e_{f_1} + \tau^R - \tau \right) \\
\phi_{R2} &= \sin \left( g_{R2} \tau \right) H \left( e_{f_2} + \tau^R - \tau \right) - H \left( e_{f_2} - \tau \right)
\end{align*}
\]

(13)

with the following initial conditions:

\[
\left\{ \frac{d\phi_R}{dt}, \frac{d^2\phi_R}{dt^2} \right\} = [0, 0].
\]

(14)

where

\[
\Gamma_1 = \frac{C_1 \varepsilon_{11}(\alpha v_z^2 + \omega^2)}{2 \varepsilon_{11}}, \quad \Gamma_2 = \frac{C_2 \varepsilon_{22}(\alpha v_z^2 + \omega^2)}{2 \varepsilon_{22}}, \quad \Gamma_3 = \frac{C_3 \varepsilon_{33}(\alpha v_z^2 + \omega^2)}{2 \varepsilon_{33}}.
\]

(15)

\[
g_m^R = \frac{\nu_m^R \pi}{2 \nu_m^R + \nu_m^R}, \quad i = 1, 2.
\]

Finally, by solving the set of ODEs in Eq. (13) using Laplace transform method (see Appendix A), the dimensionless deflections of the constitutive nanotubes of the nanosystem due to moving nanoparticles without considering the inertial effect but with the lag effect are derived as follows:

\[
\tilde{w}_1^R(t, \tau) = \sum_{n=1}^{\infty} \sum_{i=1}^{2} \left( A_{ni}^R(e_{f_1} + \tau^R - \tau^i) + \frac{\alpha_i v_z^2}{2} \right) \sin(n \pi \tau).
\]

(16a)

\[
\tilde{w}_2^R(t, \tau) = \sum_{n=1}^{\infty} \sum_{i=1}^{2} \left( B_{ni}^R(e_{f_1} + \tau^R - \tau^i) + \frac{\alpha_i v_z^2}{2} \right) \sin(n \pi \tau).
\]

(16b)

3.2. Analytical formulations based on the NHOBM

By employing the Hamilton’s principle in the context of the NHOBM, the equations of motion of the doubly parallel nanotube acted upon by doubly moving nanoparticle with a lag, without considering the inertial effect of the nanoparticles, could be obtained as follows:

\[
\begin{align*}
\frac{\partial^2 \psi^H}{\partial t^2} - (a_1^2 I_{0(1)} - a_1 I_{a(1)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_1 I_{b(1)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(1)}^H}{\partial x} &= 0, \\
\frac{\partial^2 \psi^H}{\partial t^2} - (a_2^2 I_{0(2)} - a_2 I_{a(2)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_2 I_{b(2)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(2)}^H}{\partial x} &= 0,
\end{align*}
\]

(17a)

\[
\begin{align*}
\frac{\partial^2 \psi^H}{\partial t^2} - (a_1^2 I_{0(1)} - a_1 I_{a(1)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_1 I_{b(1)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(1)}^H}{\partial x} &= 0, \\
\frac{\partial^2 \psi^H}{\partial t^2} - (a_2^2 I_{0(2)} - a_2 I_{a(2)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_2 I_{b(2)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(2)}^H}{\partial x} &= 0,
\end{align*}
\]

(17b)

\[
\begin{align*}
\frac{\partial^2 \psi^H}{\partial t^2} - (a_1^2 I_{0(1)} - a_1 I_{a(1)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_1 I_{b(1)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(1)}^H}{\partial x} &= 0, \\
\frac{\partial^2 \psi^H}{\partial t^2} - (a_2^2 I_{0(2)} - a_2 I_{a(2)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_2 I_{b(2)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(2)}^H}{\partial x} &= 0,
\end{align*}
\]

(17c)

\[
\begin{align*}
\frac{\partial^2 \psi^H}{\partial t^2} - (a_1^2 I_{0(1)} - a_1 I_{a(1)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_1 I_{b(1)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(1)}^H}{\partial x} &= 0, \\
\frac{\partial^2 \psi^H}{\partial t^2} - (a_2^2 I_{0(2)} - a_2 I_{a(2)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_2 I_{b(2)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(2)}^H}{\partial x} &= 0,
\end{align*}
\]

(17d)

where \(a_i = \frac{1}{2 \nu_i^H} \) and \(J_{ni} = J_{ni} \rho_h \varepsilon_z^2 \varepsilon A \); \( n = 0, 2, 4, 6 \). The nonlocal resultant shear force (i.e., \( Q_{b(1)}^H + \frac{\partial^2 \psi^H}{\partial x \partial t} \)) and the nonlocal flexural moment (i.e., \( M_{b(1)}^H \)) within the \( i \)th continuum-based tube are linked to the local resultant shear force (i.e., \( Q_{b(1)}^H + \frac{\partial^2 \psi^H}{\partial x \partial t} \)) and the local bending moment (i.e., \( M_{b(1)}^H \)) in the framework of the Eringen’s elasticity theory [51,56,57] as in the following form:

\[
\Xi \left\{ \frac{Q_{b(1)}^H + \frac{\partial^2 \psi^H}{\partial x \partial t}}{\partial x} \right\} = \left( M_{b(1)}^H \right): i = 1, 2.
\]

(18a)

\[
\Xi \left\{ \frac{M_{b(1)}^H}{\partial x} \right\} = \left( M_{b(1)}^H \right) = J_{ni} \frac{\partial^2 \psi^H}{\partial x^2} - a_i J_{ni} \left( \frac{\partial^2 \psi^H}{\partial x^2} + \frac{\partial \psi^H}{\partial x} \right) \cdot i = 1, 2.
\]

(18b)

where \( k_i = J_{ni} \gamma_{6i} \varepsilon_z^2 \varepsilon A \) and \( J_{ni} = J_{ni} \rho_h \varepsilon_z^2 \varepsilon A \); \( n = 2, 4, 6 \). Now by applying the operator \( \Xi \) to both sides of Eqs. (17a)–(17d) and using Eqs. (18a) and (18b), the nonlocal equations of motion that display transverse vibrations of the nanosystem subjected to moving nanoparticles accounting for the lag, nonlocality, and shear effect on the basis of the NHOBM are derived as:

\[
\Xi \left\{ \frac{I_{0(1)} + \frac{\partial^2 \psi^H}{\partial x^2} - (a_1^2 I_{0(1)} - a_1 I_{a(1)}) \frac{\partial^2 \psi^H}{\partial x^2} - a_1 I_{b(1)} \frac{\partial^2 \psi^H}{\partial x \partial t} + \frac{\partial Q_{b(1)}^H}{\partial x}}{\partial x} \right\} = -k_1 \left( \frac{\partial^2 \psi^H}{\partial x^2} + \frac{\partial \psi^H}{\partial x} \right) - a_i J_{ni} \left( \frac{\partial^2 \psi^H}{\partial x^2} + \frac{\partial \psi^H}{\partial x} \right) + \frac{\partial Q_{b(1)}^H}{\partial x},
\]

(19a)
\[ \sum \left( \left( I_{22} + a_2^T I_{66} - 2a_1 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} + \left( a_1^T I_{66} - a_1 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} \right) \]

\[ + k_1 \left( \dot{\psi}_1^H + \frac{\partial \psi_1^H}{\partial t} \right) - \left( J_{22} - 2a_2 J_{44} + a_0^T I_{66} \right) \frac{\partial^2 u^H}{\partial x^2} \]

\[ - \left( a_0^T J_{55} - 2a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} = 0. \quad (19b) \]

\[ \sum \left\{ \left( I_{22} + a_2^T I_{66} - 2a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} + \left( a_1^T I_{66} - a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} \right\} \]

\[ - k_2 \left( \dot{\psi}_2^H + \frac{\partial \psi_2^H}{\partial t} \right) - a_2 J_{44} \frac{\partial^2 u^H}{\partial x^2} + a_0^T I_{66} \frac{\partial^2 u^H}{\partial x^2} \]

\[ = \sum \left\{ M_2 \delta(x - x_M) \left[ H(t_4 + t_f - t) - H(t_2 + t_f - t) \right] \right\}. \quad (19c) \]

\[ \sum \left( I_{22} + a_2^T I_{66} - 2a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} + \left( a_1^T I_{66} - a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} \]

\[ + k_2 \left( \dot{\psi}_2^H + \frac{\partial \psi_2^H}{\partial t} \right) - \left( J_{22} - 2a_2 J_{44} + a_2^T I_{66} \right) \frac{\partial^2 u^H}{\partial x^2} \]

\[ - \left( a_2^T J_{55} - 2a_2 J_{44} \right) \frac{\partial^2 u^H}{\partial x^2} = 0. \quad (19d) \]

In order to examine nonlocal vibrations of doubly parallel nanotube structure under doubly moving nanoparticles in a more general framework, the following dimensionless parameters are taken into account:

\[ \bar{\psi}_1^H = \frac{\psi_1^H}{\psi_0}, \bar{\psi}_2^H = \psi_2^H, \bar{\tau} = \frac{\tau - \tau_0}{\tau_f - \tau_0}, \bar{t}_1 = \frac{t_1 - t_0}{t_f - t_0}, \bar{t}_2 = \frac{t_2 - t_0}{t_f - t_0}, \]

\[ \bar{r}_d = \frac{r_d - r_0}{r_f - r_0}, \bar{\theta} = \frac{\theta - \theta_0}{\theta_f - \theta_0}, \]

\[ \bar{r}_d = \frac{r_d - r_0}{r_f - r_0}, \bar{t}_1 = \frac{t_1 - t_0}{t_f - t_0}, \bar{t}_2 = \frac{t_2 - t_0}{t_f - t_0}, \]

by introducing the given dimensionless factors in Eq. (20) to Eqs. (19a)–(19d), the dimensionless governing equations of the nanosystem used for delivery of nanoparticles are obtained as follows:

\[ \sum \left( \frac{\partial^2 \psi^H}{\partial \theta^2} + \frac{\partial^2 \psi^H}{\partial \tau^2} + \frac{\partial^2 \psi^H}{\partial r_1^2} \frac{\partial^2 \psi^H}{\partial r_2^2} + \frac{\partial^2 \left( \dot{\psi}_1^H - \ddot{\psi}_1^H \right)}{\partial \theta^2} \right) \]

\[ - \sum \left( \frac{\partial^2 \psi^H}{\partial \tau^2} + \frac{\partial^2 \psi^H}{\partial \theta^2} + \frac{\partial^2 \psi^H}{\partial r_1^2} \frac{\partial^2 \psi^H}{\partial r_2^2} + \frac{\partial^2 \left( \dot{\psi}_2^H - \ddot{\psi}_2^H \right)}{\partial \theta^2} \right) \]

\[ = \sum \left( \bar{J} \bar{\psi} \delta(x - x_M) \left[ H(t_4 + t_f - t) - H(t_2 + t_f - t) \right] \right). \quad (21a) \]

\[ \sum \left( \frac{\partial^2 \psi^H}{\partial \tau^2} + \frac{\partial^2 \psi^H}{\partial \theta^2} + \frac{\partial^2 \psi^H}{\partial r_1^2} \frac{\partial^2 \psi^H}{\partial r_2^2} \right) + \sum \left( \dot{\psi}_1^H + \frac{\partial \psi_1^H}{\partial t} \right) \]

\[ - \sum \left( \bar{J} \bar{\psi} \frac{\partial \phi^H}{\partial \tau} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial \theta} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial r_1} \frac{\partial \phi^H}{\partial r_2} \right) \]

\[ = \sum \left\{ \bar{J} \bar{\psi} \delta(x - x_M) \left[ H(t_4 + t_f - t) - H(t_2 + t_f - t) \right] \right\}. \quad (21b) \]

\[ \sum \left( \frac{\partial^2 \psi^H}{\partial \tau^2} + \frac{\partial^2 \psi^H}{\partial \theta^2} + \frac{\partial^2 \psi^H}{\partial r_1^2} \frac{\partial^2 \psi^H}{\partial r_2^2} \right) + \sum \left( \dot{\psi}_2^H + \frac{\partial \psi_2^H}{\partial t} \right) \]

\[ - \sum \left( \bar{J} \bar{\psi} \frac{\partial \phi^H}{\partial \tau} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial \theta} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial r_1} \frac{\partial \phi^H}{\partial r_2} \right) \]

\[ = \sum \left\{ \bar{J} \bar{\psi} \delta(x - x_M) \left[ H(t_4 + t_f - t) - H(t_2 + t_f - t) \right] \right\}. \quad (21c) \]

\[ \sum \left( \frac{\partial^2 \psi^H}{\partial \tau^2} + \frac{\partial^2 \psi^H}{\partial \theta^2} + \frac{\partial^2 \psi^H}{\partial r_1^2} \frac{\partial^2 \psi^H}{\partial r_2^2} \right) + \sum \left( \dot{\psi}_2^H + \frac{\partial \psi_2^H}{\partial t} \right) \]

\[ - \sum \left( \bar{J} \bar{\psi} \frac{\partial \phi^H}{\partial \tau} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial \theta} + \frac{\bar{J} \bar{\psi}}{\partial \tau} \frac{\partial \phi^H}{\partial r_1} \frac{\partial \phi^H}{\partial r_2} \right) \]

\[ = \sum \left\{ \bar{J} \bar{\psi} \delta(x - x_M) \left[ H(t_4 + t_f - t) - H(t_2 + t_f - t) \right] \right\}. \quad (21d) \]

For the nanosystem with simple ends, the boundary conditions read: \( \bar{\psi}_1^H(0, \tau) = \bar{\psi}_1^H(1, \tau) = 0 \) and \( M^H \right|_{1=H}(1, \tau) = 0 \). It is assumed that the nanosystem is at rest just before entrance of moving nanoparticles; therefore, the initial conditions are expressed by: \( \bar{\psi}_1^H(\xi, 0) = 0, \psi_1^H(\xi, 0) = 0, \)

\( \frac{\partial \psi_1^H}{\partial \xi}(\xi, 0) = 0, \text{ and } \frac{\partial \psi_1^H}{\partial \xi}(\xi, 0) = 0 \). For dynamic analysis of the problem, the dimensionless deformation fields are discretized in terms of admissible modes as:

\[ \bar{\psi}_1^H(\xi, \tau) = \sum_{m=1}^{M} \bar{a}_m \phi_m^H(\xi), \quad \bar{\psi}_1^H(\xi, \tau) = \sum_{m=1}^{M} \bar{b}_m \phi_m^H(\xi). \]
\[ \ddot{w}^H(\xi, t) = \sum_{n=1}^{\infty} \ddot{c}^H_n(t) \phi_n^H(\xi), \quad \ddot{\psi}_1^H(\xi, t) = \sum_{n=1}^{\infty} d_n^H(t) \phi_{n+1}^H(\xi), \]  

(22b)

where \( \phi_n^H \) and \( \psi_{n+1}^H \) are denoting the \( n \)th and \( n+1 \)th admissible mode shapes associated with the deflection and angle of rotation of the \( n \)th nanotube modeled on the NHOBM, \( \ddot{c}^H_n(t) \) and \( \ddot{d}_n^H(t) \) are the time-dependent parameters. For simply supported nanosystems, we consider the following appropriate modes: \( \phi_n^H(\xi) = \sin(n \pi \xi), \psi_{n+1}^H(\xi) = \cos(n \pi \xi) \). Now by direct substitution of Eqs. (22a) and (22b) into Eqs. (21a)–(21d), the following set of second-order ODEs is derived:

\[
\begin{bmatrix}
\ddot{Z}_1 \\
\ddot{Z}_2 \\
\ddot{Z}_3 \\
\ddot{Z}_4
\end{bmatrix} = 
\begin{bmatrix}
\dddot{c}_1 & c_2 & c_3 & \ddot{c}_4 \\
0 & \ddot{c}_2 & c_3 & c_4 \\
0 & c_3 & \dddot{c}_4 & c_4 \\
0 & 0 & c_4 & \dddot{c}_4
\end{bmatrix} + 
\begin{bmatrix}
e_1 & e_2 & e_3 & 0 \\
e_2 & 0 & e_3 & 0 \\
e_3 & 0 & e_4 & 0 \\
e_4 & 0 & e_4 & 0
\end{bmatrix} 
\begin{bmatrix}
\ddot{c}_1 \\
\ddot{c}_2 \\
\ddot{c}_3 \\
\ddot{c}_4
\end{bmatrix}
\]

(23)

with the following initial conditions:

\[
\begin{bmatrix}
c_1^H(0) \\
c_2^H(0) \\
c_3^H(0) \\
c_4^H(0)
\end{bmatrix} = \begin{bmatrix}
\frac{d^2c_1}{dt^2}(0) \\
\frac{d^2c_2}{dt^2}(0) \\
\frac{d^2c_3}{dt^2}(0) \\
\frac{d^2c_4}{dt^2}(0)
\end{bmatrix} = \{0, 0, 0, 0\},
\]

(24)

where

\[
\begin{align*}
e_1 &= \gamma_1^2(1 + (n \sigma \mu)^2) + \gamma_2^2(n \sigma)^2 + (n \sigma)^2, \\
e_2 &= \gamma_2^2(1 + (n \sigma \mu)^2), \\
e_3 &= \gamma_2^2(1 + (n \sigma \mu)^2), \\
e_4 &= \gamma_2^2(1 + (n \sigma \mu)^2) + \gamma_2^2(n \sigma)^2 + \gamma_2^2(n \sigma)^2, \\
e_5 &= \gamma_2^2(1 + (n \sigma \mu)^2) + \gamma_2^2(n \sigma)^2, \\
e_6 &= \gamma_2^2(1 + (n \sigma \mu)^2), \\
e_7 &= \gamma_2^2(1 + (n \sigma \mu)^2)
\end{align*}
\]

(25)

Through solving the set of four coupled 2nd-order ODEs in Eq. (23) using Laplace transform method (see Appendix B), it is obtainable:

\[
\begin{align*}
\ddot{w}_1^H(\xi, t) &= \sum_{n=1}^{\infty} L_1^{\gamma_s} \left[ A_{n_1}^H \left( A_{n_1}^H(t) + B_{n_1}^H(t) \right) + A_{n_1}^H \left( C_{n_1}^H(t) + D_{n_1}^H(t) \right) \right] \sin(n \pi \xi), \\
\ddot{w}_2^H(\xi, t) &= \sum_{n=1}^{\infty} L_2^{\gamma_s} \left[ B_{n_2}^H \left( A_{n_1}^H(t) + B_{n_1}^H(t) \right) + B_{n_2}^H \left( C_{n_1}^H(t) + D_{n_1}^H(t) \right) \right] \cos(n \pi \xi), \\
\ddot{w}_3^H(\xi, t) &= \sum_{n=1}^{\infty} L_3^{\gamma_s} \left[ C_{n_2}^H \left( A_{n_2}^H(t) + B_{n_2}^H(t) \right) + C_{n_2}^H \left( C_{n_1}^H(t) + D_{n_1}^H(t) \right) \right] \sin(n \pi \xi), \\
\ddot{w}_4^H(\xi, t) &= \sum_{n=1}^{\infty} L_4^{\gamma_s} \left[ D_{n_2}^H \left( A_{n_1}^H(t) + B_{n_1}^H(t) \right) + D_{n_2}^H \left( C_{n_1}^H(t) + D_{n_1}^H(t) \right) \right] \cos(n \pi \xi).
\end{align*}
\]

(26a–26d)

in which the procedure to evaluate the unknown constants as well as the time-dependent coefficients in Eqs. (26a)–(26d) has been stated in Appendix B.

It should be emphasized that the provided formulations in this part are based on the nonlocal differential elasticity whose fundamental frequencies for simply supported nanoscaled beams are fairly close to those predicted by the nonlocal integral elasticity \( [25, 58, 59] \). However, for statics and vibrational problems problems rod-like nanostructures \([24, 60, 62]\), the above-mentioned differences are so obvious particularly for the fixed-fixed and fixed-free ends. With regard to some complexities in evaluating the stiffness matrices of the nanosystem acted upon by moving lagged-inertial-nanoparticles in the context of the nonlocal integral elasticity, the work on this interesting subject could be considered for future works.

4. Vibrational analysis by considering inertia of moving nanoparticles

In order to generalize the presented equations of motion in the previous sections, the lateral inertia of moving nanoparticles passes through the pores of the nanotubes should be taken into account in the formulations (i.e., moving mass approach (MMAA)). In this section, these nonlocal governing equations for the nanosystem at hand are displayed and then solved for the unknown fields using Galerkin approach. In the upcoming section, the predicted results by this numerical method is verified with those of the analytical methods provided in the previous section. Additionally, the limitations of the presented analytical methods for predicting dynamic response of the nanosystem are revealed and explained in some details.
4.1. Numerical formulations based on the NRBM

Using Eqs. (6a) and (6b) in conjunction with Eq. (1), the transverse vibrations of doubly parallel nanotubes subjected to doubly moving nanoparticles with the lag and the inertial effects are expressed by:

\[
\rho_b (A_b \frac{\partial^2 u_b}{\partial t^2} - \frac{\partial u_b}{\partial x} a_x) + C_{iv} (u_b^R - u_b^L) - \frac{\partial (M_{by})}{\partial x} = M_{y}(g - \frac{D_{y} u_b}{\partial t}) \delta(x - x_{M_y}) H(t_f - t).
\]

(27a)

\[
\rho_b (A_b \frac{\partial^2 u_b}{\partial t^2} - \frac{\partial u_b}{\partial x} a_x) + C_{iv} (u_b^R - u_b^L) - \frac{\partial (M_{by})}{\partial x} = M_{y}(g - \frac{D_{y} u_b}{\partial t}) \delta(x - x_{M_y}) [H(t_f + t_y - t) - H(t_y - t)].
\]

(27b)

By introducing the dimensionless parameters in Eq. (10) to Eqs. (27a) and (27b), the dimensionless equations of motion that display transverse vibrations of the nanosystem acted by moving nanoparticles by consideration of both lag and inertia effects according to the NRBM would be:

\[
\begin{align*}
\sum_{i=1}^{2} \left\{ \frac{\partial^2 \bar{u}_i}{\partial t^2} - \zeta \bar{u}_i + \frac{\partial}{\partial z} \left[ \bar{C}_{iv} \bar{u}_i (\bar{u}_i^R - \bar{u}_i^L) \right] \right\} + \frac{\partial^2 \bar{u}_i}{\partial z^2} &= \frac{1}{f_R} \left( 1 - \frac{D_{y} \bar{u}_i}{\partial t} \right) \delta(z - \zeta M_y) H(t_f - t) , \quad \text{(28a)}
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{2} \left\{ \left( 1 - \left( \frac{\partial \bar{u}_i}{\partial x} a_x \right) \zeta \right) + \frac{\partial}{\partial z} \left[ \bar{C}_{iv} \bar{u}_i (\bar{u}_i^R - \bar{u}_i^L) \right] \right\} + \frac{\partial^2 \bar{u}_i}{\partial z^2} &= \frac{1}{f_R} \left( 1 - \frac{D_{y} \bar{u}_i}{\partial t} \right) \delta(z - \zeta M_y) [H(t_f + t_y - t) - H(t_y - t)] , \quad \text{(28b)}
\end{align*}
\]

where the dimensionless second-order material derivative of \( \bar{u}_i \) is given by:

\[
\begin{align*}
\frac{D_{y} \bar{u}_i}{\partial t} &= \frac{\partial^2 \bar{u}_i}{\partial t^2} + 2a_x \left( \rho_b A_{2b} \frac{\partial \bar{u}_i}{\partial x} + (v_J) \right) \left( \rho_b A_{2b} \frac{\partial \bar{u}_i}{\partial x} + (v_J) \right) \zeta ,
\end{align*}
\]

(29)

To solve Eqs. (28a) and (28b) for the unknown dimensionless deflections, Galerkin approach in conjunction with the assumed mode method (AMM) is employed. For this purpose, we premultiply both sides of Eqs. (28a) and (28b) by \( \delta \bar{u}_i^R \) and \( \delta \bar{u}_i^L \), respectively, and then we take integral from the resulted relations over \([0,1]\). Hence, we will arrive at the following relations by exploiting integration by part technique:

\[
\begin{align*}
\int_{0}^{1} \left\{ \frac{\partial^2 \bar{u}_i^R}{\partial z^2} - \frac{\partial}{\partial z} \left[ \bar{C}_{iv} \bar{u}_i^R (\bar{u}_i^R - \bar{u}_i^L) \right] \right\} dx - \int_{0}^{1} \frac{\partial \bar{u}_i^R}{\partial z} \delta(z - \zeta M_y) H(t_f - t) dz &= \frac{1}{f_R} \left( 1 - \frac{D_{y} \bar{u}_i^R}{\partial t} \right) \delta(z - \zeta M_y) H(t_f - t) dz , \quad \text{(30a)}
\end{align*}
\]

\[
\begin{align*}
\int_{0}^{1} \left\{ \frac{\partial^2 \bar{u}_i^L}{\partial z^2} - \frac{\partial}{\partial z} \left[ \bar{C}_{iv} \bar{u}_i^L (\bar{u}_i^R - \bar{u}_i^L) \right] \right\} dx - \int_{0}^{1} \frac{\partial \bar{u}_i^L}{\partial z} \delta(z - \zeta M_y) [H(t_f + t_y - t) - H(t_y - t)] dz &= \frac{1}{f_R} \left( 1 - \frac{D_{y} \bar{u}_i^L}{\partial t} \right) \delta(z - \zeta M_y) [H(t_f + t_y - t) - H(t_y - t)] dz , \quad \text{(30b)}
\end{align*}
\]

Now we discretize the unknown deflection fields of the problem in terms of admissible mode shapes as: \( \bar{u}_i ^R(z, t) = \sum_{n=1}^{NM} \bar{\phi}_{i}^{R}(\zeta) \bar{w}_{in}^{R}(\zeta) \) and \( \bar{u}_i ^L(z, t) = \sum_{n=1}^{NM} \bar{\phi}_{i}^{L}(\zeta) \bar{w}_{in}^{L}(\zeta) \) where \( NM \) denotes the number of considered modes. By substituting these relations into Eqs. (30a) and (30b), it is derived:

\[
\begin{bmatrix}
\tilde{M}_{by}^{R} & 0 \\
0 & \tilde{M}_{by}^{L}
\end{bmatrix}
\begin{bmatrix}
\bar{w}_{in}^{R} \\
\bar{w}_{in}^{L}
\end{bmatrix}
+ \begin{bmatrix}
\tilde{C}_{iv}^{R} & 0 \\
0 & \tilde{C}_{iv}^{L}
\end{bmatrix}
\begin{bmatrix}
\frac{d^{2} \bar{u}_{i}^{R}}{d z^{2}} \\
\frac{d^{2} \bar{u}_{i}^{L}}{d z^{2}}
\end{bmatrix}
= \begin{bmatrix}
\frac{d \bar{u}_{i}^{R}}{d t} \\
\frac{d \bar{u}_{i}^{L}}{d t}
\end{bmatrix}
\]

(31)

where the elements of the dimensionless mass, damping, and stiffness matrices are given in Appendix C. For the rest condition (i.e., zero initial deflections as well as zero initial lateral velocity), we consider the following initial conditions:

\[
\begin{bmatrix}
\bar{\phi}_{i}^{R}(0) \\
\bar{\phi}_{i}^{L}(0)
\end{bmatrix} = \begin{bmatrix}
\frac{d \bar{u}_{i}^{R}}{d t}(0) \\
\frac{d \bar{u}_{i}^{L}}{d t}(0)
\end{bmatrix} = \{0,0\} .
\]

(32)

In order to calculate 2NM time-dependent parameters in Eq. (31) at each time with the given initial conditions in Eq. (32), generalized Newmark-\( \beta \) approach is exploited.
4.2. Numerical formulations based on the NOHBM

In this part, we are interested in dynamic analysis of doubly parallel nanotubes subjected to doubly moving nanoparticles accounting for lag, nonlocality, and shear effects. To this end, we employ Eqs. (17a)–(17d) by virtue of Eq. (1) to construct the equations of motion using the NOHBM:

\[ I_{0(1)} \frac{\partial^2 w_H}{\partial t^2} - (a_1^2 I_{6(1)} - \alpha_1 I_{4(1)}) \frac{\partial^2 w_H}{\partial x^2} - a_1^2 I_{6(1)} \frac{\partial^2 w_H}{\partial t^2} - \frac{\partial Q_H}{\partial x} = \frac{\partial^2 I_{0(1)}}{\partial t^2}, \]  
\[ -a_1 \frac{\partial^2 w_H}{\partial x^2} + C_{12}(w_H - \frac{\partial^2 w_H}{\partial t^2}) = M_f(g - \frac{D_{12}^H}{D_{22}^H}) \delta(x - s_M) H(t_1 - t), \]  
\[ (I_{2(1)} + a_2^2 I_{6(2)} - 2a_2 I_{4(2)}) \frac{\partial^2 w_H}{\partial t^2} + (a_2^2 I_{6(2)} - a_1 I_{4(2)}) \frac{\partial^2 w_H}{\partial x^2} + \frac{\partial Q_H}{\partial x} = \frac{\partial I_{0(2)}}{\partial t^2}, \]  
\[ +a_2 \frac{\partial^2 w_H}{\partial x^2} - \frac{\partial M_{0(2)}}{\partial x} = 0, \]

by combining Eqs. (33a)–(33d) with Eqs. (18a) and (18b) through using Eq. (20), the dimensionless governing equations of the nanosystem conveying nanoparticles could be derived in the following form:

\[ \Xi \left\{ \frac{\partial^2 \psi_H}{\partial t^2} + \frac{\partial \psi_H}{\partial x^2} - \frac{\partial^2 \psi_H}{\partial t^2} + \frac{\partial^2 \psi_H}{\partial x^2} \right\} = \frac{\partial^2 \psi_H}{\partial t^2} - \frac{\partial^2 \psi_H}{\partial x^2}, \]  
\[ -\int_0^L \left\{ \Theta \left( f^H_0 \left( g - \frac{D_{12}^H}{D_{22}^H} \right) \delta(x - s_M) H(t_1 - t) \right) \right\} \]  
\[ \Xi \left\{ \frac{\partial^2 \psi_H}{\partial t^2} - \frac{\partial^2 \psi_H}{\partial x^2} \right\} = \left( f^H_0 \left( 1 - \frac{D_{12}^H}{D_{22}^H} \right) \delta(x - s_M) H(t_1 - t) \right), \]  
\[ \Xi \left\{ \frac{\partial^2 \psi_H}{\partial t^2} - \frac{\partial^2 \psi_H}{\partial x^2} \right\} = \left( f^H_0 \left( 1 - \frac{D_{12}^H}{D_{22}^H} \right) \delta(x - s_M) H(t_1 - t) \right), \]  
\[ \Xi \left\{ \frac{\partial^2 \psi_H}{\partial t^2} - \frac{\partial^2 \psi_H}{\partial x^2} \right\} = \left( f^H_0 \left( 1 - \frac{D_{12}^H}{D_{22}^H} \right) \delta(x - s_M) H(t_1 - t) \right), \]

where \( \frac{D_{12}^H}{D_{22}^H} = \frac{\partial^2 \psi_H}{\partial t^2} - \frac{\partial^2 \psi_H}{\partial x^2} \). To study vibrations of the nanosystem, both sides of Eqs. (34a)–(34d) in order are multiplied by \( \delta \frac{\partial^2 \psi_H}{\partial t^2} \), \( \delta \frac{\partial \psi_H}{\partial x^2} \), \( \delta \frac{\partial \psi_H}{\partial t^2} \), and \( \delta \frac{\partial^2 \psi_H}{\partial x^2} \). Subsequently, the resulting expressions are integrated over the dimensionless spatial domains of the constitutive nanotubes, and then the integration by part technique is employed to reduce the order of derivatives of unknown dynamic deformation fields:
5. Results and discussion

In this section, parametric studies associated with the transverse vibration of nanosystems under the influence of moving nanoparticles with and without consideration of the lateral inertial effect based on the NRBM and NHOBM are presented. For this purpose, the constitutive nanotubes have the following continuum-based specifications: $t_0 = 0.34$ nm, $r_{nm} = 1$ nm, $p_b = 2300$ kg/m$^3$, $E_b = 1$ TPa, $v_b = 0.2$. For demonstrating the results, normalized weight, normalized speed, normalized deflections, normalized lag time, and normalized free space between nanotubes have been used frequently, and these are defined by:

$$W_N = \frac{\omega_N x(t)}{(M_N + m_N) E_{tx}}, \quad V_N = \frac{v_{tx}}{v_{tx}}, \quad d_N = \frac{d}{d_{min}}, \quad d_{max} = \frac{\rho_v}{\rho_b},$$

$$d_{min} = r_{nm} + t_0 + t, \quad d_{max} = \frac{r_{nm} + r_{nm} + t_0}{r_{nm} + t_0} + \frac{t}{x_B} \left( I_{tx} \Gamma_4 + \Gamma_2 \Gamma_1 \right).$$

### 5.1. A comparison study

In order to verify the validity of the calculations, the results of the proposed model are compared with the results of another work. Using nonlocal Euler–Bernoulli beam theory, Lu et al. [63] studied transverse vibrations of a nanosystem consisting of two SWCNTs under moving nanoparticles by considering their lag but without considering their lateral inertia. To provide a fair comparison between the results of Lu et al. [63] and those of the proposed model, the rotational inertia of the nanotubes in the Rayleigh beam theory as well as the lateral inertial of the travelling nanoparticles has been neglected. The results of this comparison, the plots of maximum normalized deflections as a function of normalized velocity, are shown in Fig. 3. As it is obvious, for different values of the lag, a fairly good match between the results of the proposed model and those of the suggested model by Lu et al. [63] is observed.

### 5.2. Roles of influential factors on vibrations of the nanosystem

#### 5.2.1. Influence of the intertube distance

Maximum normalized deflections in terms of the normalized free space of the nanotubes are demonstrated in Fig. 4. As shown in Fig. 4a, when the mechanical and geometric properties of the nanotubes are the same and similar nanoparticles have the same speed and time of arrival, the maximum...
transverse displacements of the nanotubes are the same. In this case, the existing vdW force does not play a critical role in transverse vibrations of the nanotubes, and therefore, the maximum transverse displacements would not vary with the free space between nanotubes. By changing the arrival time of one of the nanoparticles (Fig. 4b), the dynamic deflection of each tube is affected by that of its neighboring one, particularly for low levels of the intertube free space. A close comparison of the plotted results in Fig. 4(a) and (b) evidently shows that for high levels of the intertube distances, the existence of lag would not obvious influence on the resulted maximum deflections of nanotubes. This is mainly attributed to decreasing of the coefficient of vdW forces due to an increasing of the intertube distance. A more detailed of the plotted results in Fig. 4(a) indicates that the inertial and shear effects do not change with increasing of the intertube distance when there is no lag between moving nanoparticles. In such a case, the relative differences between the results of the MLA and those of the MMA based on the NRBM and the NHOBM are about 20% and 25%, respectively, such that the predicted results based on the MMA are generally higher than those predicted by the MLA. Furthermore, the predicted results by the NHOBM are commonly greater than those of the NRBM. The maximum relative discrepancies between the results of these two models for the cases of the MLA and the MMA in order are 13.5% and 7.5%. According to Fig. 4(b), by growing the free intertube distance, the inertial and shear effects would initially change oscillatory; then, the relative differences between two approaches and the two theories approach to the corresponding values of these differences in Fig. 4(a). The maximum relative differences between the two approaches according to the NHOBM(NRBM) for the first and second nanotubes are 26(32)% and 27(19.5)%, respectively.

5.2.2. Influence of the mass weight of the moving nanoparticles

In Fig. 5, the maximum transverse displacements of the nanotubes as a function of the mass weight of the nanoparticles are plotted for two velocities of the moving nanoparticles. In contrast to the MMA, the maximum values of the normalized dynamic deflections would not alter in terms of the mass weight of moving nanoparticles in the case of the MLA. At the velocity level $V_{in} = 0.3$ (see Fig. 5(a)), increase in the mass of moving nanoparticles has a small effect on the maximum deflections predicted by the MMA. In such a case, the relative discrepancies between the results of the MLA and those of the MMA based on the developed-nonlocal models in order are 7% and 3.5% for the first and second nanotubes. This is while at the velocity $V_{in} = 0.9$ (see Fig. 5(b)), the inertial force of the nanoparticles increases with their mass weight, and thereby, the maximum

---

**Fig. 3.** The normalized maximum deflections of the midspan points of the DPNTs as a function of normalized velocity for different lags: (△) $\tau_d = 0$, (□) $\tau_d = 0.2\tau_p$, (○) $\tau_d = 0.4\tau_p$, (◊) $\tau_d = 0.6\tau_p$, (●) $\tau_d = 0.8\tau_p$. $M_{iN} = 0.1$, $e_i a = 1$ nm, $r_n = 0.33$ nm, $l_p/D = 10$, $D = 2r_n + t_b$, $C_{nr} = 1000$; (—) Lu et al. [63], (—) present study.

**Fig. 4.** Normalized maximum deflections of DPNTs as a function of normalized intertube distance for two lags in pattern#1: (a) $\tau_d = 0$, (b) $\tau_d = 6.5$; (□) $\tau_d = 0.1$ nm, $\lambda_i = 20$, $V_{in} = 0.9$, $M_{iN} = 0.3$; (○) without inertial effects, (●) with inertia effects, (…) NRBM, (—) NHOBM.
Fig. 5. Normalized maximum deflections of DPNTs as a function of normalized mass weight of the moving nanoparticles for two levels of the velocity: (a) \( V_N = 0.3 \), (b) \( V_N = 0.9 \); \( d_N = 0, \lambda = 20, e_N = 1 \) nm, \( \tau_{in} = 6.5 \); (□) without inertial effects, (○) with inertia effects; (--) NRBM, (—) NHOBM.

Fig. 6. Normalized maximum deflections of DPNTs as a function of the normalized lag for different velocities of moving nanoparticles based on the MMA: (a) \( V_N = 0.3 \), (b) \( V_N = 0.6 \), (c) \( V_N = 0.9 \); \( d_N = 0, \lambda = 20, e_N = 1 \) nm, \( M_{in} = 0.3 \); (□) pattern#1, (△) pattern#2; (--) NRBM, (—) NHOBM.

Dynamic deflections according to the MMA would almost vary linearly in terms of the mass weight of the moving nanoparticles. These plots display that the maximum relative difference between two approaches for the first and second nanotubes are about 21% and 24% for the first and the second nanotubes, respectively.

For the carried out calculations based on the MLA, it should be noticed that the shear effect does not affect by the mass weight of the moving nanoparticles. For velocity levels of \( V_N = 0.3 \) and 0.9, the relative discrepancies between the results of the NRBM and those of the NHOBM for the first(second) nanotube in order are about 9.9(8.2)% and 11.8(9.1)%. A close verifications of the results of the NRBM on the basis of the MMA and those of the NHOBM in the case of \( V_N = 0.3 \) reveals that the shear effect would increase initially by increasing of the mass weight of nanoparticles, and thereafter, it would decrease. The maximum relative discrepancies between the results of these two models are about 10.1% and 9.8% for the first and the second tubes, respectively. Concerning the case of \( V_N = 0.6 \) (see Fig. 5(b)), the above-mentioned discrepancies or shear effect for the first tube is generally reduced and for the second tube would be fluctuating. In such a case, the maximum relative discrepancies between the results of the NRBM and those of the NHOBM for the first and second nanotubes are 12% and 9.7%, respectively.

### 5.2.3. Influence of the lag of the moving nanoparticles

In Fig. 6(a)–(c), the graphs of the maximum normalized deflections in terms of the lag for two patterns of nanotubes and three values of the velocity have been shown. At low levels of velocities (Fig. 6(a)), the maximum displacements commonly occur in the forced vibration phase. In such a case, the maximum deflections associated with \( \tau_{in} = 0 \) are generally greater than those pertinent to \( \tau_{in} \neq 0 \). For high levels of velocities (i.e., \( V_N \geq 0.6 \)), (see Fig. 6(b) and (c)) maximum values of deflections take place in the free vibration phase, so, with increasing the lag, the maximum displacements of both patterns would vary in an oscillatory manner. In the first pattern, contrary to the second one, maximum displacements with the lag are greater than those without the lag. On the other hand, the maximum displacements of the first pattern are always higher than those of the second pattern. This issue is mainly raised from difference between the vdW force coefficients of two patterns. In the case of \( V_N = 0.3 \), a detailed survey of the plotted results for the first nanotube indicates that the shear effect fluctuates for low levels of the lag, and then, it remains constant (see
Further by nanoparticle displacements first relative and up approaches and the arrival 5.2.4. forces (see Fig. 6(a)). The maximum differences between the two theories for the first and second patterns are, respectively, 10.1% and 10.9%. In other plots (see Fig. 6(b) and (c)), variation of the shear effect due to the variation of the lag commonly presents a fluctuated curve such that the amplitudes of these fluctuations would grow with an increase of the lag.

According to Figs. 6(a)-6(c), irrespective of the velocity of the moving nanoparticles, the discrepancies between the predicted deflections of the considered patterns in the absence of lag vanish exactly. In fact, the dynamic deflection of each nanotube is not influenced by the vdW interactional forces for the zero-lag circumstance. In the cases of $SV$ (IN)$S = 0.3, 0.6$, and 0.9, the maximum of such discrepancies in the presence of lag for the first(second) nanotube are 39(44)%, 59(53)%, and 61(56)%, respectively.

5.2.4. Influence of the velocities of the moving nanoparticles

In Fig. 7, the maximum transverse displacements of the nanotubes in terms of the speed of the nanoparticles for two patterns are presented for two levels of the lag: $\tau_{dN} = 0$ and 6.5. In Fig. 7(a), the coefficient of the vdW forces does play a trivial role in the vibration of the nanosystem. Therefore, the maximum dynamic displacements of the nanotubes for patterns are almost identical. As shown in Fig. 7(b), if there exists a lag in arrival of the nanoparticles into the nanosystem, the difference between vibrational behavior of two patterns would be obvious. This fact reveals the impact of both lag and coefficient of vdW forces on the dynamic response of the nanosystem. On the other hand, with increasing the speed, the transverse inertia of the nanoparticles in both patterns increases and this issue corresponds to the more discrepancies between the results of the MLA and those of the MMA. In Fig. 7(a) (i.e., the case of without a lag), the maximum difference between the plots of the MLA and those of the MMA for both patterns, irrespective of the considered model, reaches almost 32%. In the case of $\tau_{dN} = 6.5$ and the first pattern of the nanotubes, such differences for the first and second tubes are about 44% and 27.5%, respectively, while in the second pattern of nanotubes, these differences reach 27% for both nanotubes.

Regarding the role of the shear deformation on the vibrational behavior of the nanosystem (see Fig. 7(a) and (b)), for both nanotubes, both approaches and patterns, the relative discrepancies between the results of the NRBM and those of the NHOBM would fluctuate in terms of velocity up to $V_{N} = 0.5$. Subsequently, these discrepancies would slightly increase in Fig. 7(a) while they remain relatively constant in Fig. 7(b). According to the demonstrated results in Fig. 7(a), the maximum relative discrepancy between the proposed theories for both nanotubes is approximately equal to 24%, while the above-mentioned discrepancies for the first and second nanotubes in Fig. 7(b) are 34.5% and 36.5%, respectively.

In Fig. 7(a), the maximum relative differences between the predicted results of two patterns are less than 1.5%, irrespective of the used model and the nanotube. According to Fig. 7(b), this relative difference for both nanotubes in two patterns is generally growing such that the maximum relative difference of two patterns reaches 44%.

5.2.5. Influence of a combination of the lag and velocity of the moving nanoparticles

The combined effects of the lag and speed of moving nanoparticles on transverse dynamic responses of the constitutive tubes of the nanosystem have been demonstrated in Fig. 8. According to the plotted results, at low levels of speeds (i.e., $V_{N} < 0.3$), maximum displacements of nanotubes acted upon by lagged-moving nanoparticles occur at $\tau_{d} = 0$. This issue is chiefly related to the occurrence of the maximum displacement in the first phase of vibration (i.e., when the moving nanoparticles have not left the nanosystem). At high levels of speeds (i.e., $V_{N} > 0.3$), maximum displacements take place in the free vibration phase, and so, the deformed shape of the second nanotube at the moment of entrance of the last nanoparticle plays a key role in general trend of vibrations of the nanosystem. Therefore, in this range of speeds, the plots of maximum deflections in terms of the lag are generally oscillating. With increasing the speed, the effect of inertia of the moving nanoparticles generally increases, but by increasing of the lag such effect would commonly vary oscillatory. A close scrutiny of the obtained results also indicates that the maximum relative discrepancies between the results of the MLA and those of the MMA based on the NHOBM for the first and second tubes in order are 52.4% and 48.5%; however, these discrepancies on the basis of the NRBM are, respectively, reduce to 51.2% and 45.1%. A careful investigation of the predicted results shows that the shear effect generally increases by simultaneous increasing of the lag and the speed of the moving nanoparticles. Further examinations of the obtained results by both NRBM and NHOBM on the basis of the MMA, in which the results of the NRBM have not been presented for the sake of brevity, indicates that the maximum relative discrepancies between the results of the NRBM and those of the NHOBM for both nanotubes are about 24%, while such discrepancies based on the MLA for the first and second nanotubes are 17% and 20.8%, respectively.
The first order of effect (see Fig. 8) of the moving nanoparticles velocities as well as the lag for pattern#1 using NHOBM; \(\eta = 0, \epsilon_0, a = 1 \text{ nm}, \lambda_1 = 20, M_{\text{NRBM}} = 0.3\).

**Fig. 8.** Normalized maximum deflections of DPNTs as a function of the moving nanoparticles velocities as well as the lag for pattern#1 using NHOBM; \(\eta = 0, \epsilon_0, a = 1 \text{ nm}, \lambda_1 = 20, M_{\text{NRBM}} = 0.3\).

\[
W_{N_{\text{max}}} = \begin{cases} 1 & \text{if } \lambda_1 = 0.3, \\ 1.5 & \text{if } \lambda_1 = 0.9; \end{cases}
\]

\[\tau_{dN} = 0, \epsilon_0, a = 1 \text{ nm}, \lambda_1 = 20, V_{dN}(\lambda_1) = 0.3, \tau_{dN} = 0; (\bigcirc) \text{ pattern#1}, (\bigtriangleup) \text{ pattern#2}; (...) \text{ NRBM}, (-) \text{ NHOBM}.

5.2.6. **Influence of the slenderness ratio of the constitutive tubes**

In Fig. 9, the maximum normalized displacements in terms of the slenderness ratio for two suggested patterns and two levels of the velocity of moving nanoparticles have been plotted. Irrespective of the velocity of the moving nanoparticles, with the increase in the slenderness ratio, the shear effect reduces for both patterns of the nanosystem. For example, the discrepancy between the plots of the NRBM and those of the NHOBM for both MLA and MMA in the case of \(\lambda_1 = 10\) is about 37%, and at \(\lambda_1 = 120\), it reaches under 0.5%. On the other hand, by increasing the slenderness ratio, the plotted results of two patterns become closer. For example, in the case of \(V_{N}(\lambda_1 = 50) = 0.3\) (see Fig. 9(a)), the discrepancies between the results of two patterns are negligible at high levels of the slenderness ratio such that their graphs are indistinguishable, while in the case of \(V_{N}(\lambda_1 = 50) = 0.9\) (see Fig. 9(b)), the above-mentioned discrepancies for the first and second nanotubes are roughly equal to 19% and 26%, respectively.

Regarding the inertia effect of the moving nanoparticles, further calculations and complementary plots associated with Fig. 9(a) display that the effect of inertia increases with the increase of the slenderness ratio such that the relative discrepancies between the results of the MLA and those of the MMA reach the maximum of 18%. Additionally, complementary studies pertinent to Fig. 9(b) reveal that changes in the inertia effect for the first pattern and both nanotubes would be oscillatory as a function of \(\lambda_1\) for low levels of the slenderness ratio. For the second pattern, variation of the inertia effect is initially incremental for the first nanotube, but in the second nanotube, we observe oscillation in the inertia effect by growing of the slenderness ratio, and then, it becomes almost constant.

6. **Conclusions**

Using the NRBM and NHOBM, transverse vibrations of double-tube-nanosystems subjected to doubly moving nanoparticles are investigated. The nanotubes are both straight and horizontal while the nanoparticles move through their pores with constant velocities, however, their lag in arrival into the nanosystem has been taken into account in the modeling of the problem. When the lateral inertia of the moving nanoparticles is ignored (i.e., MLA), the exact deformation fields of the nanosystem are derived carefully based on the NRBM and the NHOBM by exploiting Laplace transform methodology. When the lateral inertia of both particles is considered (i.e., MMA), finding an analytical approach is a cumbersome job. To
overcome this difficulty, a Galerkin-based assumed mode method is developed and the equations of motion of the nanosystem used for translocation of nanoparticles are transformed into the ordinary differential equations whose mass, damping, and stiffness submatrices are time-dependent. We employ generalized Newmark-\(\beta\) method to solve the resulted set of equations numerically at each time. The proposed models for these two approaches (i.e., MLA and MMA) are verified with each other and the predicted results by the proposed models on the basis of the MLA are also successfully checked with those of another work. The roles of the slenderness ratio, lag, mass and velocity of the nanoparticles on the vibrations of the nanosystem are explained in some details.

Appendix A. Evaluation of the time-dependent parameters of the NRBM-based model

By taking the Laplace transform from both sides of Eq. (13) and using the identity: \(L\left[\frac{d^s f(t)}{dt^s}\right] = s^s f(s) - s^{s-1} f(0) - \frac{d^{s-1} f(0)}{dt^{s-1}}\), the Laplace transform of the dimensionless time-dependent factors are readily derived:

\[
\mathcal{L}\{a_{n1}^R\} = \begin{cases} \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} & \frac{\Gamma_1 n s}{2} < \pi n \frac{\Gamma_2 n s}{2} \\ \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} & \pi n \frac{\Gamma_2 n s}{2} < \frac{\Gamma_1 n s}{2} \end{cases},
\]

(A.1a)

\[
\mathcal{L}\{b_{n1}^R\} = \begin{cases} \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} & -\frac{\Gamma_1 n s}{2} < \pi n \frac{\Gamma_2 n s}{2} \\ \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} & \pi n \frac{\Gamma_2 n s}{2} < -\frac{\Gamma_1 n s}{2} \end{cases}.
\]

(A.1b)

where \(\Delta^R(s) = (s^2 + \Gamma_1 n s)(s^2 + \Gamma_2 n s) - \Gamma_3 n \Gamma_4 n\). To evaluate the time-dependent parameters from Eqs. (A.1a) and (A.1b), we should express the right hand side terms in the form of the appropriate fractional statements as follows:

\[
L\{a_{n1}^R\} = \sum_{i=1}^{3} \left[ (1 - \cos(n \pi s))e^{-\frac{\Gamma_1 n s}{2}} \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right] + \left( e^{-\frac{\Gamma_1 n s}{2}} - e^{-\frac{\Gamma_2 n s}{2}} \right) \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \cos(n \pi s) \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})},
\]

(A.2a)

\[
L\{b_{n1}^R\} = \sum_{i=1}^{3} \left[ (1 - \cos(n \pi s))e^{-\frac{\Gamma_1 n s}{2}} \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right] + \left( e^{-\frac{\Gamma_1 n s}{2}} - e^{-\frac{\Gamma_2 n s}{2}} \right) \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \cos(n \pi s) \frac{\rho_n^R r^R}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})},
\]

(A.2b)

where

\[
\Delta^R(s) = (s^2 + \Gamma_1 n s)(s^2 + \Gamma_2 n s) = \pi n \Gamma_3 n \Gamma_4 n.
\]

\[
r_{n1} = \left( \Gamma_1 n + \Gamma_4 n \right) / 2 + \frac{\sqrt{\left( \Gamma_1 n - \Gamma_4 n \right)^2 / 4 + \Gamma_3 n \Gamma_4 n}},
\]

\[
r_{n2} = \left( \Gamma_1 n + \Gamma_4 n \right) / 2 - \frac{\sqrt{\left( \Gamma_1 n - \Gamma_4 n \right)^2 / 4 + \Gamma_3 n \Gamma_4 n}},
\]

\[
A_{n1}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right),
\]

\[
B_{n1}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} \right),
\]

\[
A_{n2}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right),
\]

\[
B_{n2}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} \right),
\]

\[
A_{n12}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right),
\]

\[
B_{n12}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} \right),
\]

\[
A_{n22}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Delta^R_n (s^2 - \frac{\Gamma_2 n s}{2})} \right),
\]

\[
B_{n22}^R = \left( \frac{\rho_n^R r^R \cosh(\Gamma_1 n s - \frac{\Gamma_2 n s}{2})}{\Gamma_2 n s} \right),
\]

(A.4)

By taking the inverse Laplace transform from Eqs. (A.2a) and (A.2b), it is obtainable:

\[
\hat{a}_{n1}^R(t) = \sum_{i=1}^{3} (A_{n1}^R(t) + B_{n1}^R(t)) + A_{n12}^R(C_{n1}^R(t) + D_{n1}^R(t)),
\]

\[
\hat{b}_{n1}^R(t) = \sum_{i=1}^{3} (B_{n1}^R(t) + B_{n12}^R(t)) + B_{n12}^R(C_{n1}^R(t) + D_{n1}^R(t)),
\]

(A.5)
where
\[
A^R_n(t) = \frac{\sin^R_n(\theta)}{r^R_{n1}},
\]
\[
B^R_n(t) = -\frac{\cos^R_n(\theta - \gamma^R_n)H(t - \tau^R_n)}{r^R_{n1}},
\]
\[
C^R_n(t) = \frac{\sin^R_{n1}(\theta)}{r^R_{n1}},
\]
\[
D^R_n(t) = -\frac{\cos^R_{n1}(\theta - \gamma^R_n)H(t - \tau^R_n)}{r^R_{n1}}.
\]

(A.6)

Appendix B. Evaluation of the time-dependent parameters of the NHOBM-based model

By application of the Laplace transform to both sides of Eq. (23) in view of the given initial conditions in Eq. (24), one can arrive at:

\[
\mathcal{L}\{\beta^R_n\} = \left\{ \begin{array}{cl}
\frac{\rho^R_n H(t - \tau^R_n) e^{-s \tau^R_n}}{s} & \left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\

\left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\
\end{array} \right)
\end{array} \right)
\end{array} \right. 
\]

(B.1a)

\[
\mathcal{L}\{\delta^R_n\} = \left\{ \begin{array}{cl}
\frac{\rho^R_n H(t - \tau^R_n) e^{-s \tau^R_n}}{s} & \left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\

\left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\
\end{array} \right)
\end{array} \right)
\end{array} \right. 
\]

(B.1b)

\[
\mathcal{L}\{\gamma^R_n\} = \left\{ \begin{array}{cl}
\frac{\rho^R_n H(t - \tau^R_n) e^{-s \tau^R_n}}{s} & \left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\

\left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\
\end{array} \right)
\end{array} \right)
\end{array} \right. 
\]

(B.1c)

\[
\mathcal{L}\{\delta^R_n\} = \left\{ \begin{array}{cl}
\frac{\rho^R_n H(t - \tau^R_n) e^{-s \tau^R_n}}{s} & \left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\

\left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\
\end{array} \right)
\end{array} \right)
\end{array} \right. 
\]

(B.1d)

where

\[
\Delta^R_n(s) = \left\{ \begin{array}{l}
\frac{\rho^R_n H(t - \tau^R_n) e^{-s \tau^R_n}}{s} \left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\

\left( \begin{array}{c}
\left( e_n \hat{\epsilon}_n + e_n \hat{\chi}_n \right)^2 (x_n^2 + s_n^2) \\
\end{array} \right)
\end{array} \right)
\end{array} \right.
\]

(B.2)

The statement of \(\Delta^R_n(s)\) could be rewritten in a more compact form as:

\[
\Delta^R_n(s) = \psi (s^2 + (r^R_{n1})^2)(s^2 + (r^R_{n1})^2)(s^2 + (r^R_{n1})^2),
\]

(B.3)

\[
\psi = x_{n1} x_{n2} (x_{n2} x_{n3} - x_{n2} x_{n4}) + x_{n2} x_{n3} (x_{n3} x_{n4} - x_{n4} x_{n2}).
\]

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where $\pm ir^H_{ij}$, $i = 1, 2, 3, 4$ represent the roots of $\Delta^H(s) = 0$ ($i = \sqrt{-1}$). To evaluate the inverse Laplace of the expressions in Eqs. (B.1a)–(B.1d), it is required that these are presented in terms of appropriate fractional statements:

\[ \mathcal{L}[\delta^H_1] = \sum_{i=1}^{5} \left\{ \frac{(1 - \cos(nx)e^{-r^H_1v})A^H_{ij}}{(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))} \right\} \]

\[ \mathcal{L}[\delta^H_2] = \sum_{i=1}^{5} \left\{ \frac{(1 - \cos(nx)e^{-r^H_1v})B^H_{ij}}{(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))} \right\} \]

\[ \mathcal{L}[\delta^H_3] = \sum_{i=1}^{5} \left\{ \frac{(1 - \cos(nx)e^{-r^H_1v})C^H_{ij}}{(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))} \right\} \]

\[ \mathcal{L}[\delta^H_4] = \sum_{i=1}^{5} \left\{ \frac{(1 - \cos(nx)e^{-r^H_1v})D^H_{ij}}{(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))(e^{-x^H_1} - e^{-\pi(\xi^H_1+\theta^H_1)}\cos(nx))} \right\} \]

in which $r^H_{ij} = r^H_{kj}$, and the values of the parameters $A^H_{ij}$, $B^H_{ij}$, $C^H_{ij}$, and $D^H_{ij}$; $j = 1, 2$ are given by:

\[ A^H_{i1} = -\Theta_{i1}\left\{ (-x_{k1}(r^H_{i1})^2 + \epsilon_{x_{k1}})(x_{k1}x_{k2} - x_{k1}x_{k4}) - (r^H_{i1})^2(e_{x_{k1}}x_{k2} + e_{x_{k1}}x_{k4} - e_{x_{k1}}x_{k1} - e_{x_{k1}}x_{k4}) \right\} \]

\[ A^H_{i2} = -\Theta_{i2}\left\{ (e_{x_{k1}}x_{k1} + e_{x_{k1}}x_{k2})(-x_{k1}(r^H_{i1})^2 + \epsilon_{x_{k1}}) \right\} \]

\[ B^H_{i1} = \Theta_{i1}\left\{ (r^H_{i1})^2(e_{x_{k1}}x_{k2} + e_{x_{k1}}x_{k4} - e_{x_{k1}}x_{k1} - e_{x_{k1}}x_{k4}) \right\} \]

\[ B^H_{i2} = \Theta_{i2}\left\{ (e_{x_{k1}}x_{k1} + e_{x_{k1}}x_{k2})(-x_{k1}(r^H_{i1})^2 + \epsilon_{x_{k1}}) \right\} \]

\[ C^H_{i1} = -\Theta_{i1}\left\{ (e_{x_{k1}}x_{k1} + e_{x_{k1}}x_{k2})(x_{k1}x_{k4} - x_{k1}x_{k4}) \right\} \]

\[ C^H_{i2} = -\Theta_{i2}\left\{ (r^H_{i1})^2(e_{x_{k1}}x_{k2} + e_{x_{k1}}x_{k4} - e_{x_{k1}}x_{k1} - e_{x_{k1}}x_{k4}) \right\} \]

\[ D^H_{i1} = \Theta_{i1}\left\{ (e_{x_{k1}}x_{k1} + e_{x_{k1}}x_{k2})(x_{k1}x_{k4} - x_{k1}x_{k4}) \right\} \]

\[ D^H_{i2} = \Theta_{i2}\left\{ (r^H_{i1})^2(e_{x_{k1}}x_{k2} + e_{x_{k1}}x_{k4} - e_{x_{k1}}x_{k1} - e_{x_{k1}}x_{k4}) \right\} \]

and the values of $\Theta_{kj}$; $k = 1, 2, \ldots, 5$ are as follows:

\[ \Theta_{1j} = \frac{\rho^H_{ij}h^H_{ij}}{\rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij}} \]

\[ \Theta_{2j} = \frac{\rho^H_{ij}h^H_{ij}}{\rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij}} \]

\[ \Theta_{3j} = \frac{\rho^H_{ij}h^H_{ij}}{\rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij}} \]

\[ \Theta_{4j} = \frac{\rho^H_{ij}h^H_{ij}}{\rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij}} \]

\[ \Theta_{5j} = \frac{\rho^H_{ij}h^H_{ij}}{\rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij} + \rho^H_{ij}} \]

Finally, by taking the inverse Laplace from the given expressions in Eqs. (B.4a)–(B.4d), the dimensionless time-dependent parameters are evaluated in the following form:

\[ \bar{a}^H_n(t) = \sum_{i=1}^{5} \left( A^H_{ij}(A^H_n(t) + B^H_n(t)) + A^H_{ij}(C^H_n(t) + D^H_n(t)) \right) \]
\[\begin{align*}
\dot{H}_n^H (\tau) &= \sum_{k=1}^{n} (B_{n_k}^H (A_{n_k}^H (\tau) + B_{n_k}^H (\tau)) + B_{n_k+1}^H (C_{n_k}^H (\tau) + D_{n_k}^H (\tau))), \\
\dot{\beta}_n^H (\tau) &= \sum_{k=1}^{n} (C_{n_k}^H (A_{n_k}^H (\tau) + B_{n_k}^H (\tau)) + C_{n_k+1}^H (C_{n_k}^H (\tau) + D_{n_k}^H (\tau))), \\
\ddot{H}_n^H (\tau) &= \sum_{k=1}^{n} (D_{n_k}^H (A_{n_k}^H (\tau) + B_{n_k}^H (\tau)) + D_{n_k+1}^H (C_{n_k}^H (\tau) + D_{n_k}^H (\tau))),
\end{align*}\]

where
\[\begin{align*}
A_{n_k}^H (\tau) &= \frac{\sinh(\eta_{n_k}^H (\tau))}{\eta_{n_k}^H}, \\
B_{n_k}^H (\tau) &= -\frac{\cosh(\eta_{n_k}^H (\tau)) \sin(\eta_{n_k}^H (\tau))}{\eta_{n_k}^H} H(\tau - \eta_{n_k}^H), \\
C_{n_k}^H (\tau) &= \frac{\sin(\eta_{n_k}^H (\tau))}{\eta_{n_k}^H} H(\tau - \eta_{n_k}^H), \\
D_{n_k}^H (\tau) &= -\frac{\cosh(\eta_{n_k}^H (\tau)) \sin(\eta_{n_k}^H (\tau))}{\eta_{n_k}^H} H(\tau - \eta_{n_k}^H).
\end{align*}\]

Appendix C. Expressions of the mass, damping, and stiffness matrices pertinent to the NRBM-inertial-based model

\[\begin{align*}
\begin{bmatrix} M_b^R \end{bmatrix}_{ij} &= 0.5(\tau_{ij} + \Delta_{ij}^2)^2 + \gamma_{ij}/\mu + \gamma_{ij}^2) + \tau_{ij}^2 H(\tau_{ij} - \tau), \\
+ & M_{1N}(1 + \gamma_{ij}^2) \sin(j \pi \xi M_2) \sin(\Gamma_2) H(\tau_{ij} - \tau), \\
\begin{bmatrix} B_b^R \end{bmatrix}_{ij} &= 0.5(\Delta_{ij}^2 + \tau_{ij}^2)^2 + \gamma_{ij}/\mu + \gamma_{ij}^2) + \tau_{ij}^2 H(\tau_{ij} - \tau) - H(\tau_{ij} - \tau), \\
+ & M_{2N}(1 + \gamma_{ij}^2) \sin(j \pi \xi M_2) \sin(\Gamma_2) H(\tau_{ij} - \tau) - H(\tau_{ij} - \tau), \\
\begin{bmatrix} C_b^R \end{bmatrix}_{ij} &= 2\lambda \sqrt{\beta_R^2} M_{1N} (j \pi \xi M_2) \cos(j \pi \xi M_2) \cos(j \pi \xi M_2) H(\tau_{ij} - \tau), \\
+ & \begin{bmatrix} D_b^R \end{bmatrix}_{ij} = 2\lambda \sqrt{\beta_R^2} M_{2N} (j \pi \xi M_2) \cos(j \pi \xi M_2) \cos(j \pi \xi M_2) H(\tau_{ij} - \tau) - H(\tau_{ij} - \tau), \\
\begin{bmatrix} K_b^R \end{bmatrix}_{ij} &= 0.5(\tau_{ij} + \Delta_{ij}^2)^2 + \gamma_{ij}/\mu + \gamma_{ij}^2) + \tau_{ij}^2 H(\tau_{ij} - \tau) - H(\tau_{ij} - \tau), \\
+ & M_{1N}(1 + \gamma_{ij}^2) \sin(j \pi \xi M_2) \sin(\Gamma_2) H(\tau_{ij} - \tau), \\
\begin{bmatrix} f_b^R \end{bmatrix}_{ij} &= 2\lambda \sqrt{\beta_R^2} M_{2N} (j \pi \xi M_2) \cos(j \pi \xi M_2) \cos(j \pi \xi M_2) H(\tau_{ij} - \tau) - H(\tau_{ij} - \tau), \\
\end{align*}\]

where
\[\begin{align*}
\gamma_{ij} &= \left\{ \begin{array}{ll}
1 & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{array} \right.
\end{align*}\]

Appendix D. Expressions of the mass, damping, and stiffness matrices pertinent to the NHOBM-inertial-based model

\[\begin{align*}
\begin{bmatrix} M_b^H \end{bmatrix}_{ij} &= (\tau_{ij} + \Delta_{ij}^2)^2 + \gamma_{ij}/\mu + \gamma_{ij}^2 \tau_{ij}^2 H(\tau_{ij} - \tau), \\
+ & M_{1N}(1 + \gamma_{ij}^2) \sin(j \pi \xi M_2) \sin(\Gamma_2) H(\tau_{ij} - \tau), \\
\begin{bmatrix} B_b^H \end{bmatrix}_{ij} &= -\gamma_{ij}^2 (\tau_{ij} + \Delta_{ij}^2)/2, \\
\begin{bmatrix} C_b^H \end{bmatrix}_{ij} &= -\gamma_{ij}^2 (\tau_{ij} + \Delta_{ij}^2)/2, \\
\begin{bmatrix} D_b^H \end{bmatrix}_{ij} &= (\tau_{ij} + \Delta_{ij}^2)/2.
\end{align*}\]
\[ [M^H]^\mu_{ij}^{\text{w2}} = (\delta_1 \gamma_{ij} + (\delta_1 \mu)^2 + \beta_1^2 \Gamma_{ij} + r_1^2 (\delta_3 \mu \Gamma_{ij})^2) / 2 + M_{N2} (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \cos(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.5)

\[ [M^H]^\mu_{ij}^{\text{w2}} = \delta_2^2 (Y_{ij} + \Gamma_{ij} \mu^2) / 2, \]
(D.6)

\[ [M^H]^\mu_{ij}^{\text{w2}} = -\delta_2^2 \gamma_1 (\Xi_{ij} + \Pi_{ij} \mu^3) / 2, \]
(D.7)

\[ [M^H]^\mu_{ij}^{\text{w2}} = -\delta_2^2 \gamma_1 (\Xi_{ij} + \Pi_{ij} \mu^3) / 2, \]
(D.8)

\[ C^H_{ij}^{\text{w1}} = 2 \beta_1^H M_{N2} \phi (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \cos(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.9)

\[ C^H_{ij}^{\text{w1}} = 2 \beta_2^H M_{N2} \phi (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \cos(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.10)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 (Y_{ij} + \Gamma_{ij} \mu^2) + r_1^2 \Gamma_{ij} + \Gamma_2^2) / 2 - M_{N2} (j \phi \pi \mu)^2 (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \cos(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.11)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 (Y_{ij} + \Gamma_{ij} \mu^2) + r_1^2 \Gamma_{ij} + \Gamma_2^2) / 2 - M_{N2} (j \phi \pi \mu)^2 (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \cos(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.12)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 \Xi_{ij} - \gamma_1^2 \Pi_{ij}) / 2, \]
(D.13)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 \Xi_{ij} - \gamma_1^2 \Pi_{ij}) / 2, \]
(D.14)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 (Y_{ij} + \gamma_2^2 \Gamma_{ij}) / 2, \]
(D.15)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 (Y_{ij} + \Gamma_{ij} \mu^2) + \delta_2^2 \gamma_1 \Gamma_{ij} + \beta_1^2 \Gamma_{ij}) / 2 - M_{N2} (j \phi \pi \mu)^2 (1 + (i \pm \mu)^2 \sin(i \pi \xi_{M2}) \sin(j \pi \xi_{M1}) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.16)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 \Xi_{ij} - \delta_2^2 \gamma_1 \Pi_{ij}) / 2, \]
(D.17)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 \Xi_{ij} - \delta_2^2 \gamma_1 \Pi_{ij}) / 2, \]
(D.18)

\[ K^H_{ij}^{\text{w1}} = (r_1^2 \Xi_{ij} - \delta_2^2 \gamma_1 \Pi_{ij}) / 2, \]
(D.19)

\[ \{H^H_{ij}^{\mu2} = M_{N1} (r_1^2 (Y_{ij} + \Gamma_{ij} \mu^2) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.20)

\[ \{H^H_{ij}^{\mu2} = M_{N1} (r_1^2 (Y_{ij} + \Gamma_{ij} \mu^2) H([r_{fj}^H + \tau_d - \tau] - H(\tau_d - \tau)), \]
(D.21)

where
\[
\Pi_{ij} = \left\{ \begin{array}{ll}
\{ i = j & \Xi_{ij} = \{ i = j \\
0; & \{ i \neq j
\end{array} \}
\right.
\]
(D.22)

\[ C^H = \frac{\alpha_1^H}{\alpha_2^H} \frac{\Gamma_{11} \Gamma_{12}}{\Gamma_{21} \Gamma_{22}} \gamma^H = \frac{\Gamma_{11} \Gamma_{12}}{\Gamma_{21} \Gamma_{22}} \]

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