Graph Connectivity in Sparse Subspace Clustering

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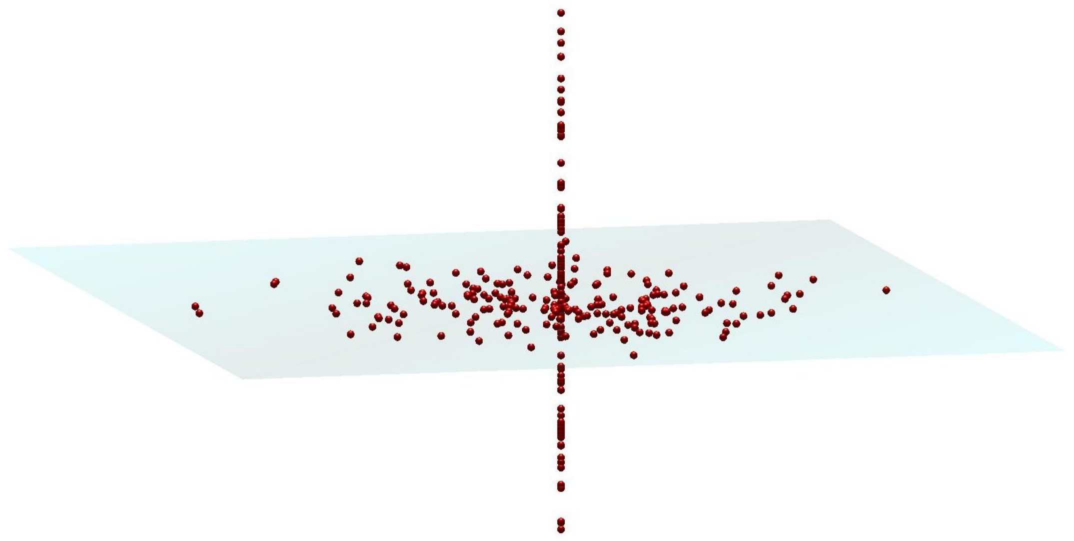
1 Subspace Clustering

- Clustering data from different subspaces



1.1 Application to Motion Segmentation

A set of points $\mathbf{X}_1, X_2, \ldots, \mathbf{X}_P$ in 3D of a rigid body.
At each frame f:

$$\mathbf{x}_{fp} = A_f \begin{bmatrix} \mathbf{X}_p \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}_{11} & \cdots & \mathbf{x}_{1P} \\ & \vdots & \\ \mathbf{x}_{F1} & \cdots & \mathbf{x}_{FP} \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_F \end{bmatrix}_{2F\times 4} \begin{bmatrix} \mathbf{X}_1 & \cdots & \mathbf{X}_P \\ 1 & \cdots & 1 \end{bmatrix}_{4\times P}$$

- Data lie on a 4D linear subspace.
- Having several rigid moving objects, they lie on several affine subspaces.

2 Sparse Representation

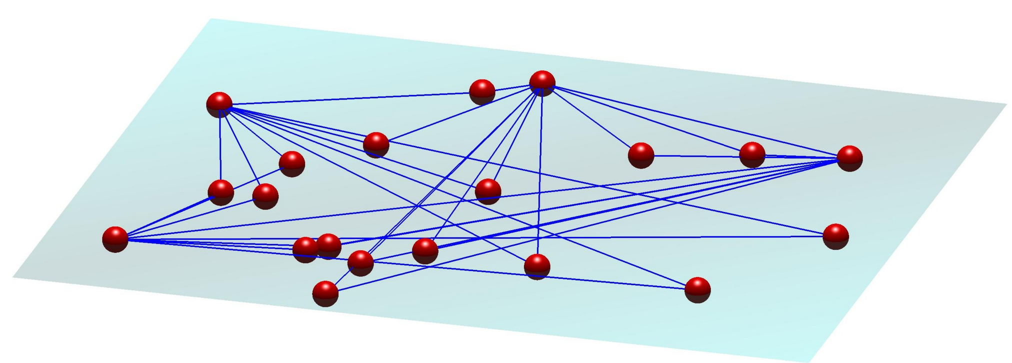
Data: $\mathbf{X} = [\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_n]$
Sparse Representation: $\mathbf{x} = \mathbf{X}\mathbf{s}$, with $\mathbf{s}$ sparse.

$$\min \|\mathbf{s}\|_0 \quad s.t. \quad \mathbf{x} = \mathbf{X}\,\mathbf{s}$$ NP-hard!

$$\min \|\mathbf{s}\|_1 \quad s.t. \quad \mathbf{x} = \mathbf{X}\,\mathbf{s}$$

3 Sparse Subspace Clustering

$$\min \|\mathbf{s}\|_1 \quad s.t. \quad \mathbf{x}_i = \mathbf{X}_{-i}\,\mathbf{s}$$
$$\mathcal{N}_i = \{j \,|\, s_j \neq 0\}$$



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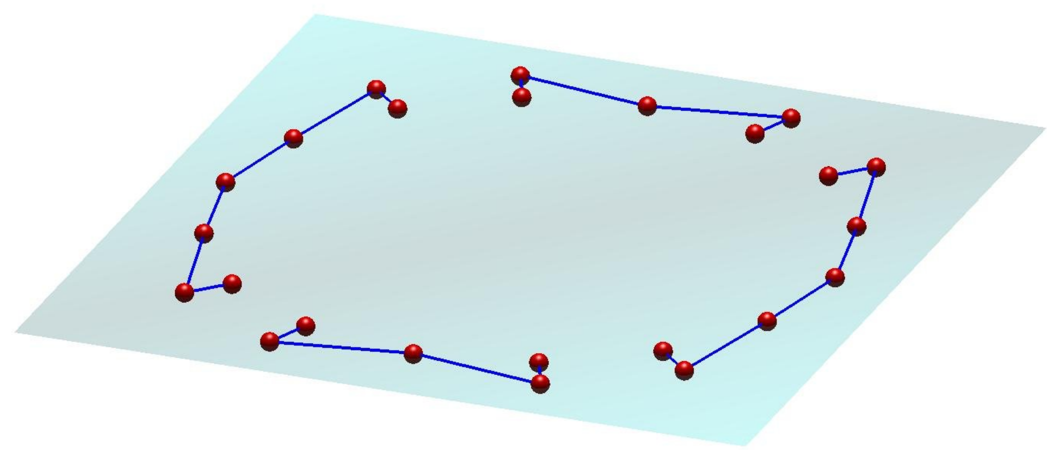
Question: Is the corresponding graph of each subspace connected?

Non-generic Cases:



Connectedness for Generic Cases?

4 Basics

- Add the negative of each point:

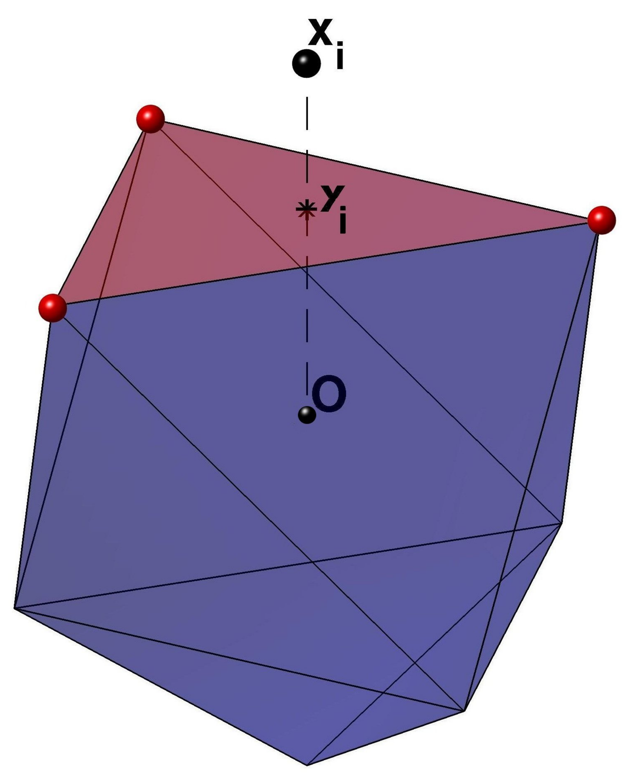
$$X_{\pm} = X \cup \{-\mathbf{x} \,|\, \mathbf{x} \in \mathbf{X}\}$$
$$X_{-i} = X_{\pm} - \{\mathbf{x}_i\}$$

Then we have:

$$\mathbf{x}_i = \sum_{j \notin i} a_j \mathbf{x}_j \qquad a_j \geq 0$$
$$= \mathbf{X}_{-i}\mathbf{a} \qquad a_j \succ 0$$

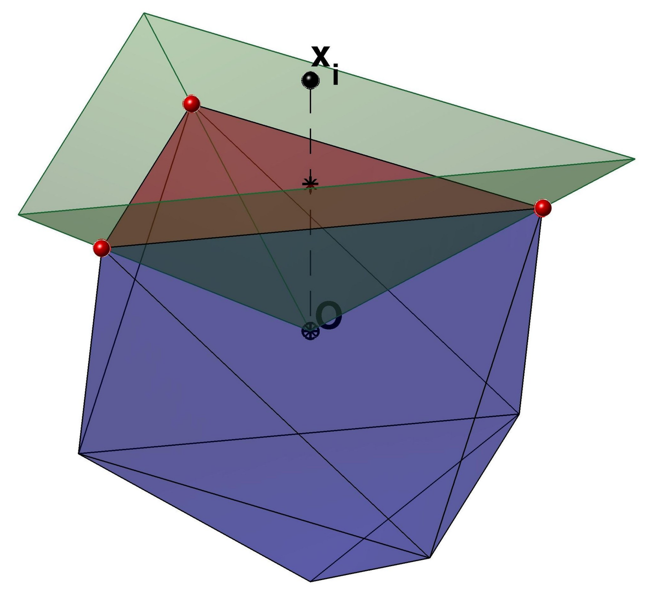
The problem turns into:

minimize α
s.t.
$\mathbf{x}_i = \alpha \mathbf{y}$
$\mathbf{y} \in H_{\text{convex}}\{X_{-i}\}$



5 Neighbourhood Cones

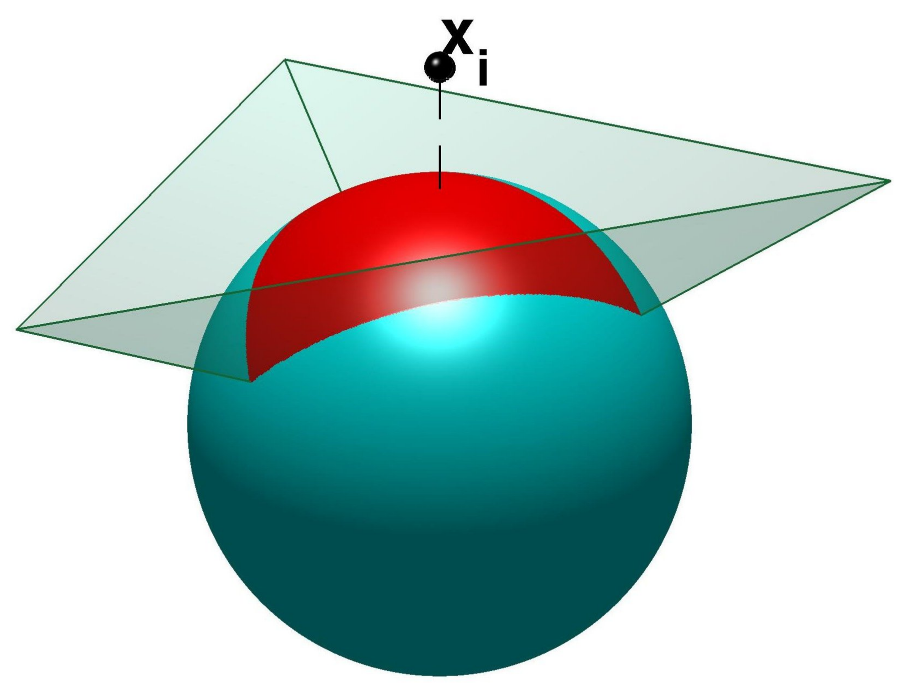
$\stackrel{\text{def}}{=} C_{\text{convex}}\{X_{\mathcal{N}_i}\}$



Theorem. Two points are neighbours if and only if their neighbourhood cones intersect.

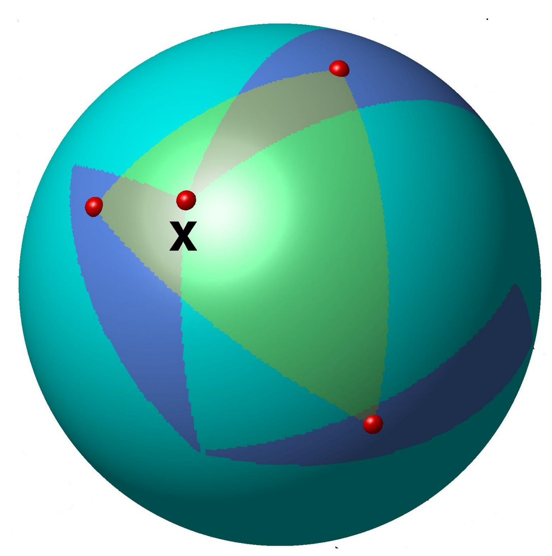
6 Projecting onto $\mathbb{S}^{d-1}$

- Cones are reduced to Hyper-spherical Simplices
- Reduce dimensionality by 1



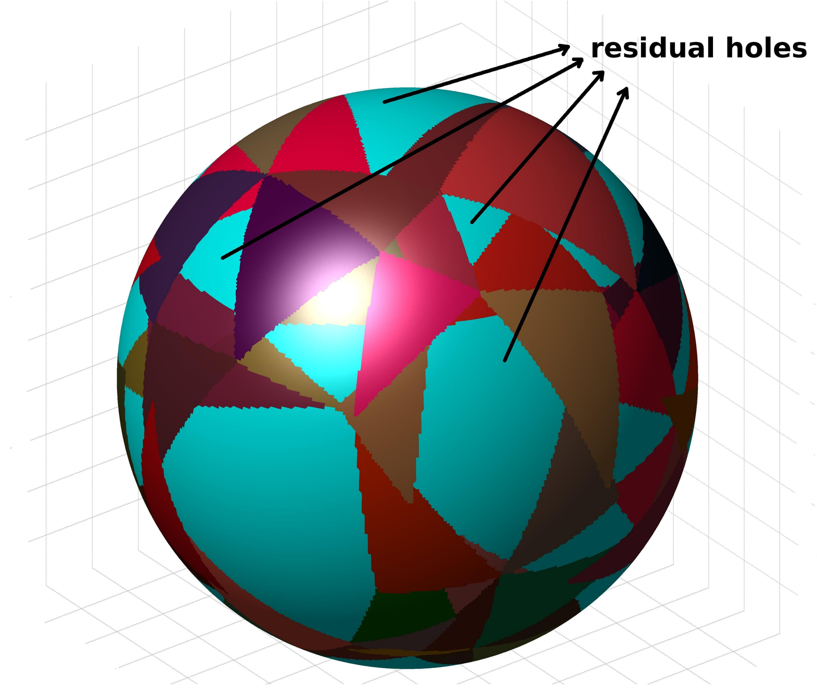
7 A Proof for 3D

7.1 Spherical Triangles



7.2 Residual Holes

Each connected component leaves residual holes on the sphere.



7.3 Gauss Bonnet Theorem

$$\int_M K\,dA + \int_{\partial M} \kappa_g ds = 2\pi\chi(M)$$

Applying to a residual hole:

$$A + \sum_i \alpha_i = 2\pi \Rightarrow A < 2\pi$$

- All triangles of one connected component must lie inside one residual hole of the other.
- Area of each residual hole is less than a half-sphere (2π).

8 Generic Counterexamples for $\geq$ 4D

- Data around two non-intersecting great circles:

$$[\cos\theta_k,\ \sin\theta_k,\ \pm\delta,\ \pm\delta]^T$$
$$[\pm\delta,\ \pm\delta,\ \cos\theta_k,\ \sin\theta_k]^T, \qquad (\theta_k = k\pi/m)$$

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