Introduction to 8086 Assembly

Lecture 19

Introduction to Floating Point

How to represent rational/real numbers

- Decimal
  - $78.173 = 7 * 10^1 + 8 * 10^0 + 1 * 10^{-1} + 7 * 10^{-2} + 3 * 10^{-3}$
- Binary
  - $1001.1011 = ?$

How to represent rational/real numbers

- Decimal
  - $78.173 = 7 * 10^1 + 8 * 10^0 + 1 * 10^{-1} + 7 * 10^{-2} + 3 * 10^{-3}$
- Binary
  - $1001.1011 = 1 * 2^3 + 0 * 2^2 + 0 * 2^1 + 1 * 2^0 + 1*2^{-1} + 0 * 2^{-2} + 1 * 2^{-3} + 1 * 2^{-4}$

Convert decimal to binary

float1.c

```c
#include <stdio.h>


int main() {

  double x = 12.625;
  double y = 12.35;

  printf("x= %f\n", x);
  printf("y= %f\n", y);


  return 0;
}
```

K. N. Toosi
University of Technology

Convert decimal to binary

float1.c

```c
#include <stdio.h>


int main() {

  double x = 12.625;
  double y = 12.35;

  printf("x= %f\n", x);
  printf("y= %f\n", y);


  return 0;
}
```

```
CS@kntu:lecture19$ gcc float1.c && ./a.out
x= 12.625000
y= 12.350000
```

Convert decimal to binary

float2.c

```c
#include <stdio.h>


int main() {

  double x = 12.625;
  double y = 12.35;

  printf("x= %.16f\n", x);
  printf("y= %.16f\n", y);


  return 0;
}
```

K. N. Toosi
University of Technology

Convert decimal to binary

float2.c

```
#include <stdio.h>


int main() {

  double x = 12.625;
  double y = 12.35;

  printf("x= %.16f\n", x);
  printf("y= %.16f\n", y);


  return 0;
}
```

```
CS@kntu:lecture19$ gcc float2.c && ./a.out
x= 12.6250000000000000
y= 12.3499999999999996
```

K. N. Toosi
University of Technology

Fixed point representation

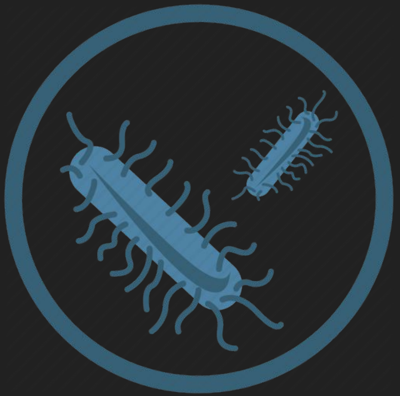
K. N. Toosi
University of Technology

1	2	3	1	2	7	8	4

1231.2784

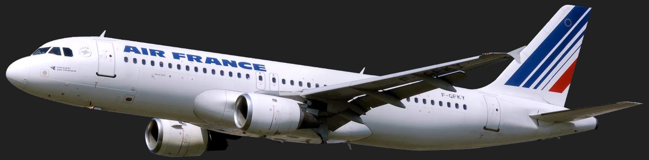
Fixed point representation

https://goo.gl/EGnmXc



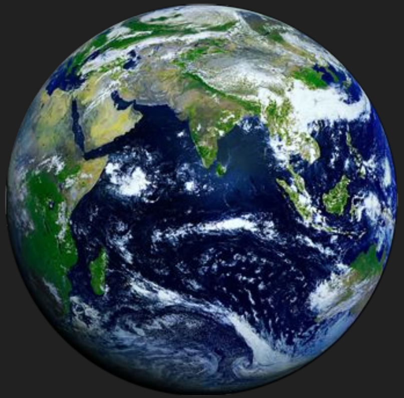
0.0000023718 m

https://goo.gl/yjPBnm



53.2843453 m

https://goo.gl/NSBgxj

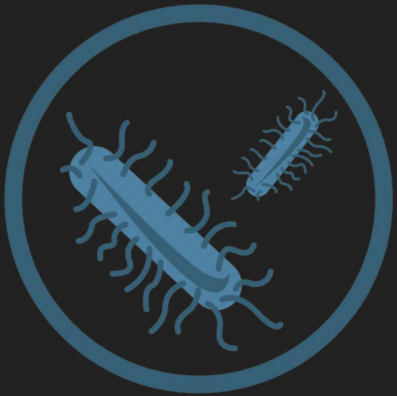


12742345.23 m

Floating point representation

K. N. Toosi
University of Technology

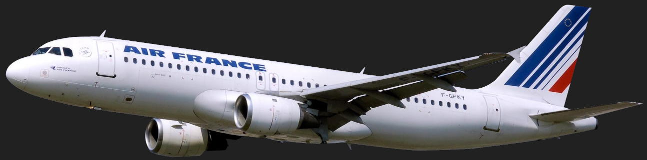
https://goo.gl/EGnmXc



0.0000023718 m

2.3718 * 10^{-6}

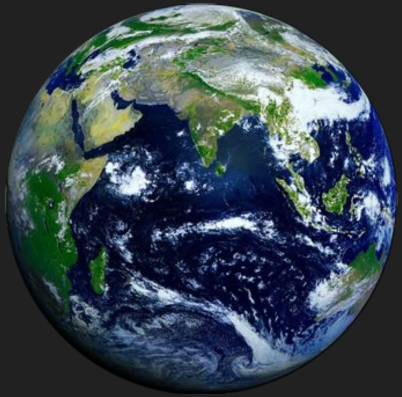
https://goo.gl/yjPBnm



53.2843453 m

5.3284 * 10^{1}

https://goo.gl/NSBgxj



12742345.23 m

1.2742 * 10^{7}

IEEE 754 standard floating point representation

K. N. Toosi
University of Technology

- Why floating point standards?

- single precision (32 bits)
- double precision (64 bits)

IEEE 754 - Normalization

- 0.0000023718
- 53.2843453
- 12742345.23

IEEE 754 - Normalization

- 0.0000023718 => ?
- 53.2843453 => ?
- 12742345.23 => ?

$x * 10^p \quad 1 <= x < 10$

IEEE 754 - Normalization

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => ?
- 12742345.23 => ?

$x * 10^{p}$ $\quad 1 <= x < 10$

IEEE 754 - Normalization

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => $5.3284 * 10^{1}$
- 12742345.23 => ?

$x * 10^{p}$ $1 <= x < 10$

IEEE 754 - Normalization

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => $5.3284 * 10^{1}$
- 12742345.23 => $1.2742 * 10^{7}$

$x * 10^{p}$ $\quad 1 <= x < 10$

K. N. Toosi
University of Technology

IEEE 754 - Normalization

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => $5.3284 * 10^{1}$
- 12742345.23 => $1.2742 * 10^{7}$

$x * 10^{p}$ $1 <= x < 10$

- 1010.1010101
- 0.00010101
- 0

K. N. Toosi
University of Technology

IEEE 754 - Normalization

K. N. Toosi
University of Technology

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => $5.3284 * 10^{1}$
- 12742345.23 => $1.2742 * 10^{7}$

$x * 10^{p} \quad 1 <= x < 10$

- 1010.1010101 => ?
- 0.000010101 => ?
- 0

$x * 2^{p} \quad 1 <= x < 2$

IEEE 754 - Normalization

K. N. Toosi
University of Technology

- 0.0000023718 => $2.3718 * 10^{-6}$
- 53.2843453 => $5.3284 * 10^{1}$
- 12742345.23 => $1.2742 * 10^{7}$

$x * 10^{p}$ $1 <= x < 10$

- 1010.1010101 => $1.0101010101 * 2^{3}$
- 0.000010101 => ?
- 0

$x * 2^{p}$ $1 <= x < 2$

IEEE 754 - Normalization

K. N. Toosi
University of Technology

- $0.0000023718 \Rightarrow 2.3718 * 10^{-6}$
- $53.2843453 \Rightarrow 5.3284 * 10^{1}$
- $12742345.23 \Rightarrow 1.2742 * 10^{7}$

$x * 10^{p} \quad 1 <= x < 10$

- $1010.1010101 \Rightarrow 1.0101010101 * 2^{3}$
- $0.000010101 \Rightarrow 1.0101 * 2^{-5}$
- 0

$x * 2^{p} \quad 1 <= x < 2$

IEEE 754 - Normalization

- $0.0000023718 \Rightarrow 2.3718 * 10^{-6}$
- $53.2843453 \Rightarrow 5.3284 * 10^{1}$
- $12742345.23 \Rightarrow 1.2742 * 10^{7}$

$x * 10^{p} \quad 1 <= x < 10$

- $1010.1010101 \Rightarrow 1.0101010101 * 2^{3}$
- $0.000010101 \Rightarrow 1.0101 * 2^{-5}$
- $0 \Rightarrow ?$

$x * 2^{p} \quad 1 <= x < 2$

IEEE 754 - Representation

$$x * 2^p \qquad 1 <= x < 2$$

$$x = ?$$

IEEE 754 - Normalization

- $x * 2^p$
  - x : significand (mantissa) $1 <= x < 2$
  - p : exponent

IEEE 754 - Normalization

- $x * 2^p$
  - x : significand (mantissa) $1 <= x < 2$
  - p : exponent
- x = 1.??????

IEEE 754 - Normalization

K. N. Toosi
University of Technology

- $x * 2^p$
  - x : significand (mantissa) $1 <= x < 2$
  - p : exponent
- $x = 1.abcdefg$

±	exponent	abcdefg
	8 bits	23 bits

IEEE 754 - Normalization

- $x * 2^p$
  - x : significand (mantissa) $1 <= x < 2$
  - p : exponent
- $x = 1.abcdefg$

±	exponent	abcdefg
	8 bits	23 bits

IEEE 754 - Normalization

- $x * 2^p$
  - x : significand (mantissa) $\quad 1 <= x < 2$
  - p : exponent
- x = 1.abcdefg

±	exponent	abcdefg	
	8 bits	23 bits	single precision
	11 bits	52 bits	double precision

IEEE 754 - Representation

S	exponent + 127	fraction = significand - 1
	8 bits	23 bits

IEEE 754 - Representation

S	exponent + 127	fraction = significand - 1
	8 bits	23 bits

S	exponent + 1023	fraction = significand - 1
	11 bits	52 bits

IEEE 754 - Representation

K. N. Toosi
University of Technology

S	biased-exponent	fraction
	8 bits	23 bits

- significand: 1 + fraction
- exponent = biased-exponent - 127
- sign = (-1)^S
- x = sign * significand * 2^exponent

IEEE 754 - Representation

K. N. Toosi
University of Technology

S	biased-exponent	fraction
	11 bits	52 bits

- significand: 1 + fraction
- exponent = biased-exponent - 1023
- sign = (-1)^S
- x = sign * significand * 2^exponent

IEEE 754 - Representation

S	biased-exponent	fraction

- Exponents 00000000 and 11111111 are reserved for special cases
  - +0
  - -0
  - +Inf
  - -Inf
  - NaN
  - very small (denormalized) numbers

IEEE 754 - Representation

S	biased-exponent	fraction

- Single precision range:
  - Minimum abs: $2^{-126} \approx 1.18 \times 10^{-38}$ (disregarding denormalized values)
  - Maximum abs: $2 \times 2^{127} \approx 3.4 \times 10^{38}$
- Double precision range:
  - Minimum abs: $\approx 2.2 \times 10^{-308}$ (disregarding denormalized values)
  - Maximum abs: $\approx 1.8 \times 10^{308}$

IEEE 754 - Representation

K. N. Toosi
University of Technology

S	biased-exponent	fraction

- Relative Precision (after decimal point of significand):
  - Single Precision: 2^{-23} (6 digits after decimal point)
  - Double Precision: 2^{-52} (16 digits after decimal point)

Floating point arithmetic

K. N. Toosi
University of Technology

- add
- subtract
- multiply
- divide

floating point range

32 bit int: up to $2.1*10^9$

32 bit float: up to $+3.4*10^{38}$

?

Example

float3.c

```c
#include <stdio.h>

int main() {

  float x = 1e12;
  float y = 1e-12;
  float z = x+y;

  printf("x= %e\n", x);
  printf("y= %e\n", y);
  printf("z= %e\n", z);
  printf("%d\n", x==z);

  return 0;
}
```

Example

float3.c

```c
#include <stdio.h>

int main() {

  float x = 1e12;
  float y = 1e-12;
  float z = x+y;

  printf("x= %e\n", x);
  printf("y= %e\n", y);
  printf("z= %e\n", z);
  printf("%d\n", x==z);

  return 0;
}
```

```
b.nasihatkon@kntu:lecture19$ gcc float3.c && ./a.out
x= 1.000000e+12
y= 1.000000e-12
z= 1.000000e+12
1
```

Example

float4.c

```c
#include <stdio.h>

int main() {

  int a = 88888888;
  int b = 88888889;

  float x = a;
  float y = b;

  printf("x=%.10f\n", x);
  printf("y=%.10f\n", y);
  printf("%d\n", x==y);

  return 0;
}
```

Example

float4.c

```c
#include <stdio.h>

int main() {

  int a = 88888888;
  int b = 88888889;

  float x = a;
  float y = b;

  printf("x=%.10f\n", x);
  printf("y=%.10f\n", y);
  printf("%d\n", x==y);

  return 0;
}
```

```
b.nasihatkon@kntu:lecture19$ gcc float4.c && ./a.out
x=88888888.0000000000
y=88888888.0000000000
1
```

K. N. Toosi
University of Technology

Example

float5.c

```c
#include <stdio.h>

int main() {

  float x = 88888888;
  printf("x=%f\n", x);

  for (int i = 0; i < 100000; i++)
    x++;

  printf("x=%f\n", x);

  return 0;
}
```

K. N. Toosi University of Technology

Example

float5.c

```c
#include <stdio.h>

int main() {

  float x = 88888888;
  printf("x=%f\n", x);

  for (int i = 0; i < 100000; i++)
    x++;

  printf("x=%f\n", x);

  return 0;
}
```

```
b.nasihatkon@kntu:lecture19$ gcc float5.c && ./a.out
x=88888888.000000
x=88888888.000000
```

Example: Newton's method

test_root1.c

```c
double h(double x) {
  return x*x*x*x*x - 1.25;

}

int main() {
  double x = newton(h, 2);

  if (fabs(h(x)) == 0)
    printf("root= %e\n", x);
  else
    printf("root not found!\n");

  return 0;
}
```

$$x_{t+1} = x_t - f(x_t) / f'(x_t)$$

test_root1.c

```c
double newton(double (*f)(double x), double x0) {
  double x = x0;
  const double delta = 1e-7;

  while (f(x) != 0 ) {
    double df_dx = (f(x+delta)-f(x))/delta;

    x = x - f(x) / df_dx;

    printf("%.10f, %e\n", x, f(x));
  }
  return x;
}
```

K. N. Toosi
University of Technology

Example: Newton's method

test_root1.c

```
double newton(double (*f)(double x), double x0) {
  double x = x0;
  const double delta = 1e-7;

  while ( f(x) != 0 ) {
    double df_dx = (f(x+delta)-f(x))/delta;

    x = x - f(x) / df_dx;

    printf("%.10f, %e\n", x, f(x));
  }
  return x;
}
```

test_root2.c

```
double newton(double (*f)(double x), double x0) {
  double x = x0;
  const double delta = 1e-7;

  while ( fabs(f(x)) >= 1e-10 ) {
    double df_dx = (f(x+delta)-f(x))/delta;

    x = x - f(x) / df_dx;

    printf("%.10f, %e\n", x, f(x));
  }
  return x;
}
```

Example: Newton's method

test_root1.c

```c
int main() {
  double x = newton(h, 2);

  if (fabs(h(x)) == 0)
    printf("root= %e\n", x);
  else
    printf("root not found!\n");

  return 0;
}
```

test_root2.c

```c
int main() {
  double x = newton(h, 2);

  if ( fabs(h(x)) < 1e-10 )
    printf("root= %e\n", x);
  else
    printf("root not found!\n");

  return 0;
}
```

Example: Computing Convergent Series

K. N. Toosi
University of Technology

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Example: Convergent Series - forward summation

float6.c

```c
#include <stdio.h>
#include <math.h>

int main() {
  double sum = 1;
  double p_factorial = 1;
  int max_itr = 20;

  for (int p = 1; p <= max_itr; p++) {
    p_factorial *= p;
    sum += 1/p_factorial;
  }

  printf("sum= %.20f\n", sum);
  printf(" e = %.20f\n", M_E);
  printf("|sum-e|=%e\n", fabs(sum-M_E));

  return 0;
}
```

K. N. Toosi
University of Technology

Example: Convergent Series - forward summation

K. N. Toosi
University of Technology

float6.c

```c
#include <stdio.h>
#include <math.h>

int main() {
  double sum = 1;
  double p_factorial = 1;
  int max_itr = 20;

  for (int p = 1; p <= max_itr; p++) {
    p_factorial *= p;
    sum += 1/p_factorial;
  }

  printf("sum= %.20f\n", sum);
  printf(" e = %.20f\n", M_E);
  printf("|sum-e|=%e\n", fabs(sum-M_E));

  return 0;
}
```

```
b.nasihatkon@kntu:lecture19$ gcc float6.c && ./a.out
sum= 2.71828182845904553488
 e = 2.71828182845904509080
|sum-e|=4.440892e-16
```

Example: Convergent Series - backward summation

K. N. Toosi
University of Technology

float7.c

```
#include <stdio.h>
#include <math.h>

int main() {
  double p_factorial = 1;
  int max_itr = 21;

  for (int p = 1; p <= max_itr; p++)
    p_factorial *= p;

  double sum = 0;
  for (int p = max_itr; p >= 0; p--) {
    sum += 1/p_factorial;
    p_factorial /= p;
  }

  printf("sum= %.20f\n", sum);
  printf(" e = %.20f\n", M_E);
  printf("|sum-e|=%e\n", fabs(sum-M_E));

  return 0; }
```

Example: Convergent Series - backward summation

K. N. Toosi
University of Technology

float7.c

```c
#include <stdio.h>
#include <math.h>

int main() {
  double p_factorial = 1;
  int max_itr = 21;

  for (int p = 1; p <= max_itr; p++)
    p_factorial *= p;

  double sum = 0;
  for (int p = max_itr; p >= 0; p--) {
    sum += 1/p_factorial;
    p_factorial /= p;
  }

  printf("sum= %.20f\n", sum);
  printf(" e = %.20f\n", M_E);
  printf("|sum-e|=%e\n", fabs(sum-M_E));

  return 0; }
```

```
b.nasihatkon@kntu:lecture19$ gcc float7.c && ./a.out
sum= 2.71828182845904509080
 e = 2.71828182845904509080
|sum-e|=0.000000e+00
```

References

- Alark Joshi, IEEE 754 F LOATING POINT REPRESENTATION
  - http://cs.boisestate.edu/~alark/cs354/lectures/ieee754.pdf
- Carter, Paul A. PC Assembly Language, 2007.
- Wikipedia https://en.wikipedia.org/wiki/IEEE_754-1985

K. N. Toosi
University of Technology