Linear Algebra for Computer Science
Fall 1402 (2023)
Instructor: Behrooz Nasihatkon
Final Exam
Dey 1402 (Jan. 2024)

Name: ______________________

ID: ______________________

Note: You can use all the results we proved in the class, but not the solutions to the homework assignments.

1. (24 points) Assume that the matrices $\mathtt{A}, \mathtt{B} \in \mathbb{R}^{n\times n}$ share a common eigenvector $\mathbf{v} \in \mathbb{R}^n$ with *distinct* corresponding eigenvalues for $\mathtt{A}$ and $\mathtt{B}$. Demonstrate that $\mathbf{v}$ is also an eigenvector of the matrix $\mathtt{A}^2 + \mathtt{B}^2 - \mathtt{AB} - \mathtt{BA}$ with a *positive* corresponding eigenvalue. (Note: $\mathtt{A}^2 = \mathtt{AA}$. Note 2: You need to show that the eigenvalue is *positive*, not just *nonnegative*.)

2. (20 points) Show that the determinant of an diagonalizable matrix is equal to the product of its eigenvalues. Hint: Use eigendecomposition.

3. (31 points) Consider a *square* matrix $\mathtt{A} \in \mathbb{R}^{n\times n}$ and let $\mathtt{R} = \mathtt{V}\mathtt{U}^T$ where $\mathtt{U} \in \mathbb{R}^{n\times n}$ is the orthogonal matrix of the left singular vectors of $\mathtt{A}$, and $\mathtt{V} \in \mathbb{R}^{n\times n}$ is that of the right singular vectors of $\mathtt{A}$.

   (a) (5 points) Show that $\mathtt{R}$ is an orthogonal matrix.

(b) (8 points) Show that $\mathtt{RAR} = \mathtt{A}^T$.

(c) (18 points) Show that $\mathtt{RA}$ is *symmetric and positive semi-definite*. Solve without using eigenvalues. What about $\mathtt{AR}$?

4. (25 points) Consider the function $f\colon \mathbb{R}^n_{++} \to \mathbb{R}$ defined as $f(\mathbf{x}) = \mathbf{x}^T \mathbf{log}(\mathbf{x})$ where $\mathbf{log}\colon \mathbb{R}^n \to \mathbb{R}^n$ acts elementwise, that is

$$\mathbf{log}(\mathbf{x}) = \mathbf{log}([x_1, x_2, \ldots, x_n]^T) = [\log(x_1), \log(x_2), \ldots, \log(x_n)]^T,$$

and log is the *natural logarithm*. Show that the gradient of f at $\mathbf{x} \in \mathbb{R}^n_{++}$ is equal to $\mathbf{log}(\mathbf{x}) + \mathbf{1}_n$, where $\mathbf{1}_n \in \mathbb{R}^n$ is the vector of all ones. ($\mathbf{x} \in \mathbb{R}^n_{++}$ means $x_i > 0$ for all i.)