Linear Algebra for Computer Science
Fall 2023
Instructor: Behrooz Nasihatkon
Midterm Exam
Azar 1402 (Nov. 2023)

Name: ______________________

ID: ______________________

Note: You can use all the results we proved in the class, but not the solutions to the homework assignments.

1. (20 points) Consider the matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$. Show that if $\mathtt{B}$ has full row rank then $\text{rank}(\mathtt{AB}) = \text{rank}(\mathtt{A})$.

2. (15 points) Consider three *nonzero* vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^n$, such that $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\mathbf{z} = \mathbf{z}^T\mathbf{x} = 0$. Prove that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are independent. Hint: If a vector $\mathbf{u}$ is nonzero then $\mathbf{u}^T\mathbf{u} > 0$.

3. (30 points) Consider two matrices $\mathtt{A}, \mathtt{B} \in \mathbb{R}^{m \times n}$ such that $\mathcal{C}(\mathtt{B}) \subseteq \mathcal{C}(\mathtt{A})$, where $\mathcal{C}(\cdot)$ denotes the column space.

   (a) (15 points) Show that there exists a matrix $\mathtt{H} \in \mathbb{R}^{n \times n}$ such that $\mathtt{B} = \mathtt{AH}$. (Hint: Look at the columns of $\mathtt{B}$ and $\mathtt{H}$. First, show that $\mathbf{b}_i \in \mathcal{C}(\mathtt{A})$ where $\mathbf{b}_i$ is the i-th column of $\mathtt{B}$.)

   (b) (15 points) Show that if $\mathtt{A}$ is of full column rank then $\mathtt{H}$ is unique. (Hint: Assume that $\mathtt{B} = \mathtt{AH} = \mathtt{AH}'$. Then show that $\mathtt{H} = \mathtt{H}'$).

4. (35 points) Consider the following system of linear equations.

$$\begin{bmatrix} 1 & a & 0 & b & 0 \\ 0 & 0 & 1 & a & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ a+1 \end{pmatrix}$$

(a) (5 points) If the system has at least one solution, then what is the value of a? Why?

(b) (10 points) Find a basis for the null space of the coefficient matrix (the matrix on the left hand side) in terms of b.

(c) (10 points) If the vector $[-8, 4, -2, -2, 0]^T$ is in the null space of the coefficient matrix, then what is the value of b? Why?

(d) (10 points) Find the complete set of solutions to the system of equations above. (If you could not solve the previous questions, write the solution in terms of b).