Linear Algebra for Computer Science
Homework 1

Read these first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewritter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other nessesary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

1. A vector space defined on the field of the real numbers $\mathbb{R}$ is a set V, equipped with a vector addition operator (vector + vector) and a scalar multiplication (scalar times vector) with the following properties

(1) *Commutativity:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for $\mathbf{u}, \mathbf{v} \in V$.

(2) *Associativity:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{v} + \mathbf{u}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

(3) *Identity element:* There exists an element $\mathbf{z} \in V$ such that for every $\mathbf{u} \in V$ we have $\mathbf{u} + \mathbf{z} = \mathbf{u}$. ($\mathbf{z}$ is called the identity element and is often denoted by $\mathbf{0}$).

(4) *Inverse:* for every $\mathbf{u} \in V$ there exist $\mathbf{u}' \in V$ such that $\mathbf{u} + \mathbf{u}' = \mathbf{z}$ (where the identity element $\mathbf{z}$ was defined above. The vector $\mathbf{u}'$ is called the (additive) inverse of $\mathbf{u}$ and is usually denoted by $\mathbf{u}^{-1}$ or $-\mathbf{u}$.)

(5) $1\mathbf{u} = \mathbf{u}$. (notice that $1 \in \mathbb{R}$.)

(6) $a(b\mathbf{u}) = (ab)\mathbf{u}$ for all $a, b \in \mathbb{R}$ and $\mathbf{u} \in V$.

(7) $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ for all $a \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.

(8) $(a + b)\,\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ for all $a, b \in \mathbb{R}$ and $\mathbf{u} \in V$.

Prove the following. In each step state which of the above axioms (1-8) you are using. You might also use the previous results, i.e. to prove d you may use parts a,b,c. Notice that there is no notion of subtraction $(-)$ here.

(a) If $\mathbf{z}$ is an identity element, then $\mathbf{z} + \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$. (notice that (3) states $\mathbf{u} + \mathbf{z} = \mathbf{u}$.)

(b) The identity element $\mathbf{z}$ is unique.

(c) The inverse of the identity element is itself.

(d) The inverse of any vector is unique.

(e) $0\,\mathbf{u} = \mathbf{z}$ for the scalar $0 \in \mathbb{R}$.

(f) $a\,\mathbf{z} = \mathbf{z}$ for any scalar $a \in \mathbb{R}$.

(g) $(-1)\,\mathbf{u} = \mathbf{u}'$ where $\mathbf{u}'$ is the inverse element of $\mathbf{u}$.

2. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$, such that $\mathtt{A}\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{R}$. Prove that $\mathtt{A} = \mathtt{0}_{m\times n}$, that is all the entries of $\mathtt{A}$ are zero.

3. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$, such that $\mathtt{A}\,\mathbf{x}_i = 0$ for $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in \mathbb{R}^n$, where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ form a basis for $\mathbb{R}^n$. Prove that $\mathtt{A} = 0_{m\times n}$.

4. Consider a square matrix $\mathtt{A} \in \mathbb{R}^{n\times n}$ for which $\mathtt{A}\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $\mathtt{A} = \mathtt{I}_n$, the n by n identity matrix.

5. Give an example of a matrix $\mathtt{A} \in \mathbb{R}$, such that $\mathtt{A}\mathbf{x} = \mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{R}^n$, $\mathtt{A}$ is not the identity matrix.

6. Assume that the vectors $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$ are linearly independent. Prove that the set of vectors $\mathbf{a}_1', \mathbf{a}_2, ..., \mathbf{a}_n$ are also linearly independent where $\mathbf{a}_1' = \mathbf{a}_1 + \beta\mathbf{a}_2$ for some scalar β.

7. The dot product can also be defined on matrices. Consider two matrices $\mathtt{A}, \mathtt{B} \in \mathbb{R}^{m\times n}$. Their dot product can be defined as $\langle \mathtt{A}, \mathtt{B} \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ij}$.

(a) Prove that $\langle \mathtt{A}, \mathtt{B} \rangle = \text{trace}(\mathtt{A}^T\mathtt{B}) = \text{trace}(\mathtt{B}^T\mathtt{A}) = \text{trace}(\mathtt{A}\mathtt{B}^T)$, where $\text{trace}(\mathtt{S}) = \sum_i S_{ii}$ gives the sum of the diagonal elements of a square matrix $\mathtt{S}$.

(b) Prove that $\langle \mathtt{AB}, \mathtt{C} \rangle = \left\langle \mathtt{B}, \mathtt{A}^T\mathtt{C} \right\rangle = \left\langle \mathtt{A}, \mathtt{C}\mathtt{B}^T \right\rangle$ Hint: $(\mathtt{AB})^T = \mathtt{B}^T\mathtt{A}^T$.

8. Prove that a triangular matrix with at least one zero diagonal element is singular.