Linear Algebra for Computer Science
Homework 2

Read this first:

i You may write your solutions on paper, under a word processing software (MS-word, Libre Office, etc.), or under LaTeX.

ii If writing on paper, you must use a scanner device or a Camera Scanner (CamScanner) software to scan the document and submit a *single* PDF file.

iii Up to 15% extra score will be given to solutions written under LaTeX, provided that you follow either of the following conventions:

(a) Represent scalars with normal (italic) letters (a, A), vectors with bold lower-case letters ($\mathbf{a}$, using `\mathbf{a}`), and matrices with bold upper-case letters ($\mathbf{A}$, using `\mathbf{A}`), or

(b) represent scalars with normal (italic) letters (a, A), vectors with bold letters ($\mathbf{a}$, $\mathbf{A}$), and matrices with typewritter upper-case letters ($\mathtt{A}$, using `\mathtt{A}`).

(c) You latex document must contain a *title*, a *date*, and your name as the author.

(d) In all cases, you must submit a *single* PDF file.

(e) If writing under LaTeX, you must submit the *.tex* source (and other nessesary source files if there are any) in addition to the PDF file.

Here is a short tutorial on LaTeX: `https://www.overleaf.com/learn/latex/Learn_LaTeX_in_30_minutes`

Questions

Basis

1. Let $\mathcal{S} \subset \mathbb{R}^n$ be a *strict* linear subspace of $\mathbb{R}^n$ (*strict* meaning $\mathcal{S} \neq \mathbb{R}^n$ or $\dim(\mathcal{S}) < n$). I argue that the set of standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ form a basis for $\mathcal{S}$ because

   - $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent, and

- evey vector in $\mathcal{S}$ can be written as a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

How is my argument wrong?

Matrix Multiplication

2. Consider the matrices $\mathtt{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, $\mathtt{D} = \text{diag}([d_1, d_2, \ldots, d_n]) \in \mathbb{R}^{n\times n}$, $\mathtt{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] \in \mathbb{R}^{p\times n}$, where $\mathtt{D}$ is a diagonal matrix with diagonal elements d_i. Show that
$$\mathtt{ADB}^T = \sum_{i=1}^{n} d_i\, \mathbf{a}_i \mathbf{b}_i^T$$
.

Row space and Column Space

3. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$. Prove that $\mathcal{C}(\mathtt{AB}) \subseteq \mathcal{C}(\mathtt{A})$, where $\mathcal{C}(\cdot)$ represents the column space.

4. Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$ and a square *invertible* matrix $\mathtt{B} \in \mathbb{R}^{n\times n}$. Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{AB})$. (Hint: to prove that two sets S_1 and S_2 are equation you can show $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$).

5. Consider two matrices $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathtt{B} \in \mathbb{R}^{n\times p}$ where $\mathtt{B}$ has *full row rank* (i.e. $\text{rank}(\mathtt{B}) = n$). Prove that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A}\,\mathtt{B})$.

Linear Equations

To answer the following questions you need to use the fact that the set of solutions to a system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ is in the form of $\{\mathbf{x}_p + \mathbf{x}_n \mid \mathbf{x}_n \in \mathcal{N}(\mathtt{A})\}$, where $\mathbf{x}_p$ is a particular solution.

6. (Bonus) Consider a matrix $\mathtt{A} \in \mathbb{R}^{m\times n}$. Show that $\mathcal{C}(\mathtt{A}) = \mathcal{C}(\mathtt{A}\,\mathtt{A}^T)$. From this conclude that $\text{rank}(\mathtt{A}) = \text{rank}(\mathtt{A}\,\mathtt{A}^T) = \text{rank}(\mathtt{A}^T\,\mathtt{A})$. (You may use the fact that

7. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a fat matrix (i.e. $m < n$) with *full row rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

8. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be a tall matrix (i.e. $m > n$) with *full column rank* and $\mathbf{b} \in \mathbb{R}^m$. Show that $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has either no solution or exactly one solution.

9. Let $\mathtt{A} \in \mathbb{R}^{m\times n}$ be rank-deficient ($\text{rank}(\mathtt{A}) < \min(m, n)$) and $\mathbf{b} \in \mathbb{R}^m$. Show $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ has either no solution or infinitely many solutions.

10. Consider the system of linear equations $\mathtt{A}\,\mathbf{x} = \mathbf{b}$ with $\mathtt{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathcal{S}$ be the set of solutions to it. Show that
    (a) $\mathcal{S}$ is a linear subspace if and only if $\mathbf{b} = \mathbf{0}$.
    (b) If $\mathcal{S}$ is nonempty, then there exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that the set $\{\mathbf{z} - \mathbf{y} \mid \mathbf{z} \in \mathcal{S}\}$ is a linear subspace.

Projections

11. Consider a linear subspace $\mathcal{S}$ and a vector $\mathbf{y} \in \mathcal{S}$. Using the projection formula, show that the projection of $\mathbf{y}$ into $\mathcal{S}$ is itself.

12. For a linear subspace $\mathcal{S} \subseteq \mathbb{R}^n$ its *orthogonal complement* is defined as $\mathcal{S}^\perp = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y}^T\mathbf{x} = 0 \text{ for all } \mathbf{x} \in \mathcal{S}\}$. In other words, $\mathcal{S}^\perp$ comprises all the vectors that are perpendicular to all vectors in $\mathcal{S}$. Show that the orthogonal complement of a linear subspace is a linear subspace.

13. Prove that the *null space* of a matrix is the orthogonal complement of its *row space*.

14. Let $\mathtt{P} \in \mathbb{R}^{n\times n}$ be the projection matrix into a linear subspace $\mathcal{S}$. Show that $\mathtt{I} - \mathtt{P}$ represents the projection into the orthogonal complement of $\mathcal{S}$. Hint: First show that $\mathtt{I} - \mathtt{P}$ is a projection matrix. Then, show that any vector $\mathbf{y} \in \mathcal{S}^\perp$ can be written as $(\mathtt{I} - \mathtt{P})\,\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.

15. Show that $\text{rank}(\mathtt{I} - \mathtt{P}) = n - \text{rank}(\mathtt{P})$ for a projection matrix $\mathtt{P} \in \mathbb{R}^{n\times n}$.

Determinant

16. Prove that the determinant of an orthogonal matrix is either equal to 1 or -1.

17. Show that the determinant of a projection matrix is either equal to 0 or 1. (Hint: remember that projections are *idempotent*.) How do you explain this geometrically?

Eigenvalues and Eigenvectors

18. What is the relation between the eigenvalues and eigenvectors of the square matrix $\mathtt{A}$ and those of $\mathtt{A} - \alpha\mathtt{I}$ where $\mathtt{R}$ and $\mathtt{I}$ is the identity matrix?

19. Prove that any eigenvalue of $\mathtt{A}$ is also an eigenvalue of $\mathtt{A}^T$. (Hint: use the characteristic polynomial).

20. The square matrix $\mathtt{A}$ is called (left) stochastic (or a Markov matrix) if its elements are nonnegative and its columns add up to 1 (programmatically sum(A,axis=0) == ones((1,n))). Prove that $\mathtt{A}$ has at least one unit eigenvalue $\lambda = 1$. (Hint: First prove that $\mathtt{A}^T$ has a unit eigenvalue.)

21. Let $\mathbf{v}$ be an eigenvector of $\mathtt{A}$ with a nonzero corresponding eigenvalue $\lambda \neq 0$. Prove that

    (a) $\mathbf{v}$ is in the column space of $\mathtt{A}$.

    (b) The (orthogonal) projection of $\mathbf{v}$ into the row space of $\mathtt{A}$ is nonzero. (Hint: decompose the vector as $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_n$ where $\mathbf{v}_r$ and $\mathbf{v}_n$ are in the row space and null space of $\mathtt{A}$, respectively. Then show that $\mathbf{v}_r$ is nonzero)

22. Let `A` be a symmetric real matrix with real eigenvalues $1, 2, \dots, n$, and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in \mathbb{R}^n$. Prove that if $\lambda_i \neq \lambda_j$ then $v_i \perp v_j$.