

Linear Algebra for Computer Science

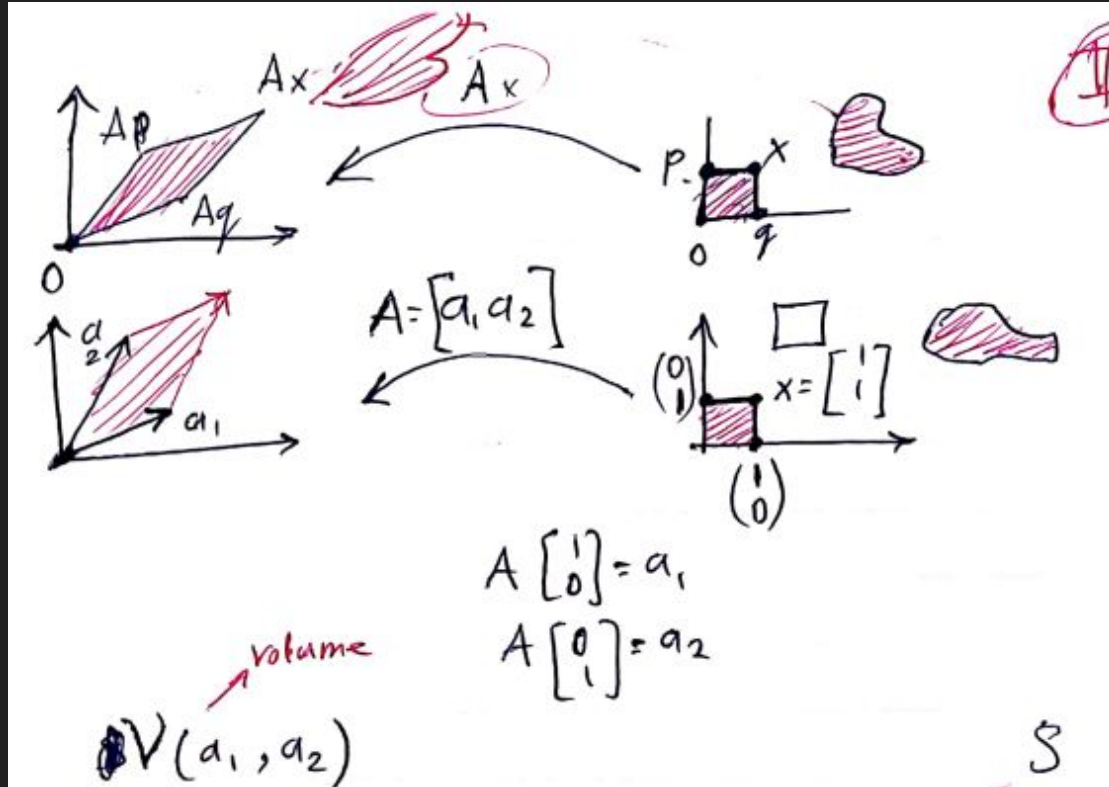
Lecture 19

Signed Volume

Area (2D Volume)



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Area (2D Volume)



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(b)

volume

$V(a_1, a_2)$

$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a_1$

$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_2$

matrix

$[a_1 \ a_2] = A$

set

S

$\text{Area}(AS) = \text{Area}(\{Ax \mid x \in S\})$

$= \text{Area}(S) * \underbrace{V(a_1, a_2)}_{?}$

a_2

a_1

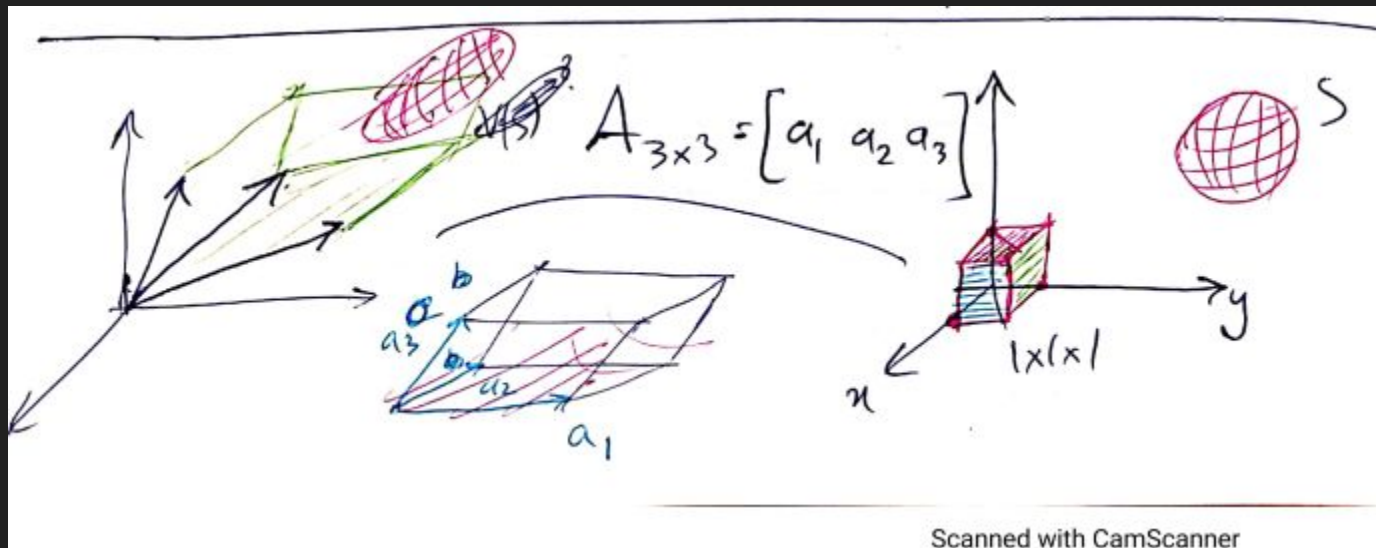
$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

$V(a_1, a_2) = V(A)$

3D Volume



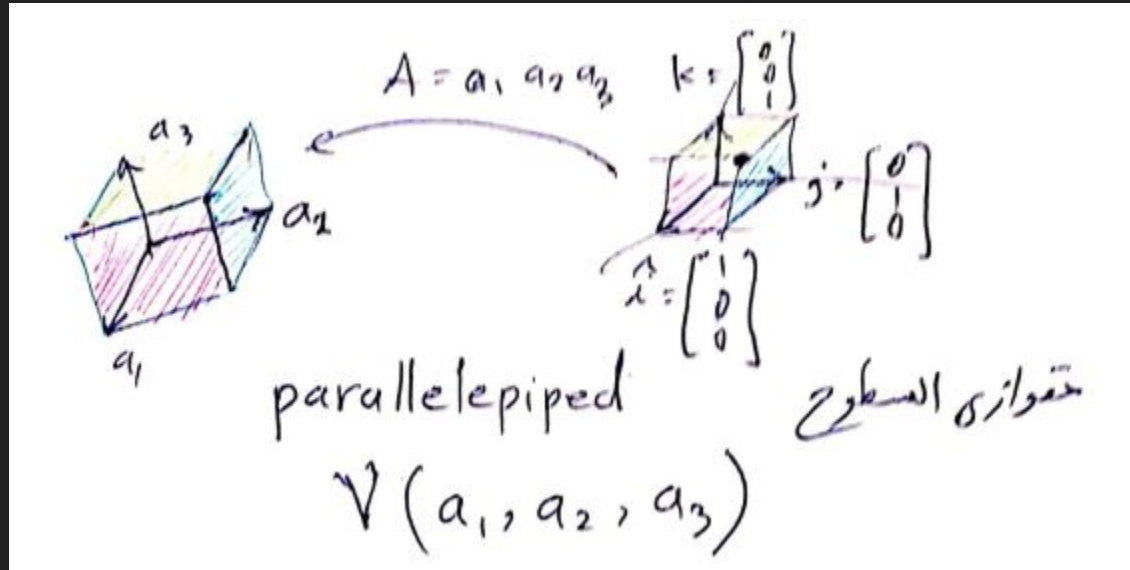
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3D volume



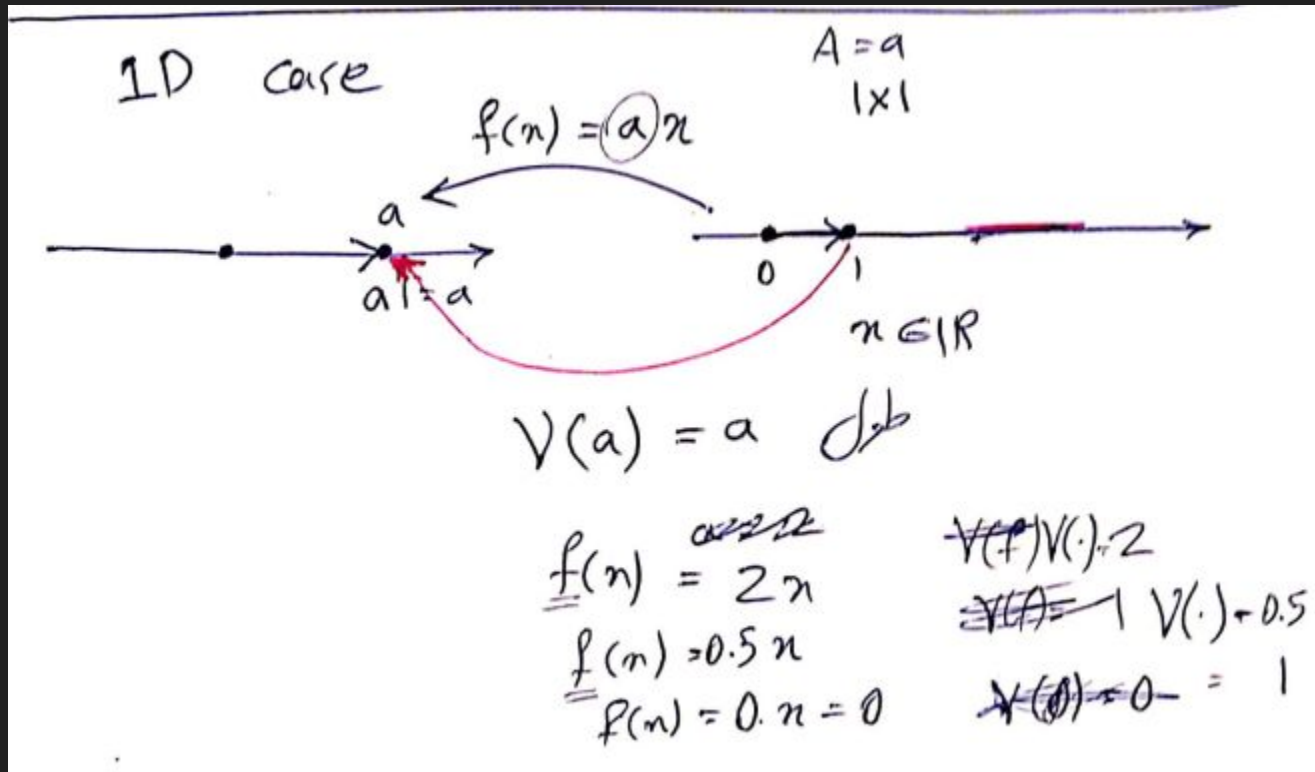
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1D case



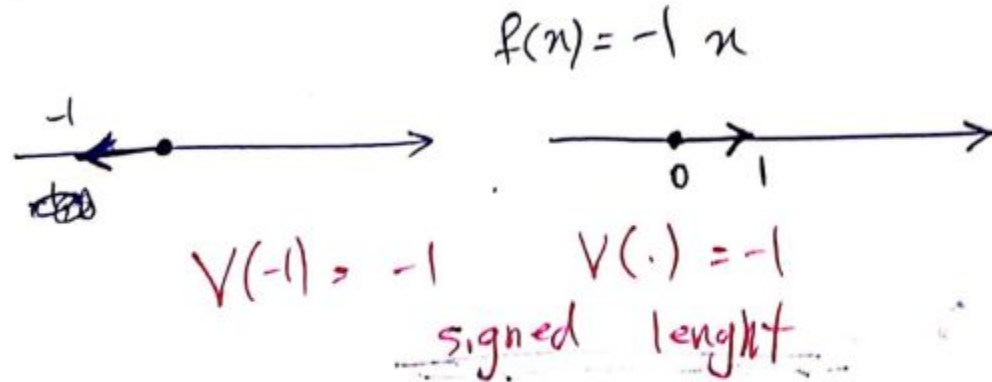
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1D case



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$f(n) = -|n|$

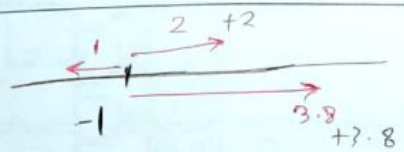
1D-case $A = [a] \quad V(A) = V(a) = \underline{\underline{a}}$

Signed Volume

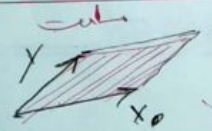


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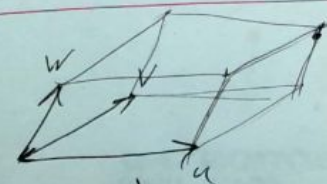
1D $n \in \mathbb{R}$ طول
signed length $|b| = \text{طول}$
 $SL(n) = n$



2D Area مساحت
SA: signed Area
 $SA(x, y) = -SA(y, x)$



3D Volume حجم
SV: signed volume
 $SV(u, v, w) = -SV(u, v, -w)$
 $SV(u, v, w) = -SV(u, w, v)$



Signed Volume



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N -D
 N -Dimensional
Hyper-Volume ~~Volume~~ آبر حجم

$V(a_1, a_2, \dots, a_N)$ $a_i \in \mathbb{R}^N$ $\begin{bmatrix} a_1, a_2, \dots, a_N \end{bmatrix} \in \mathbb{R}^{N \times N}$

Signed Volume



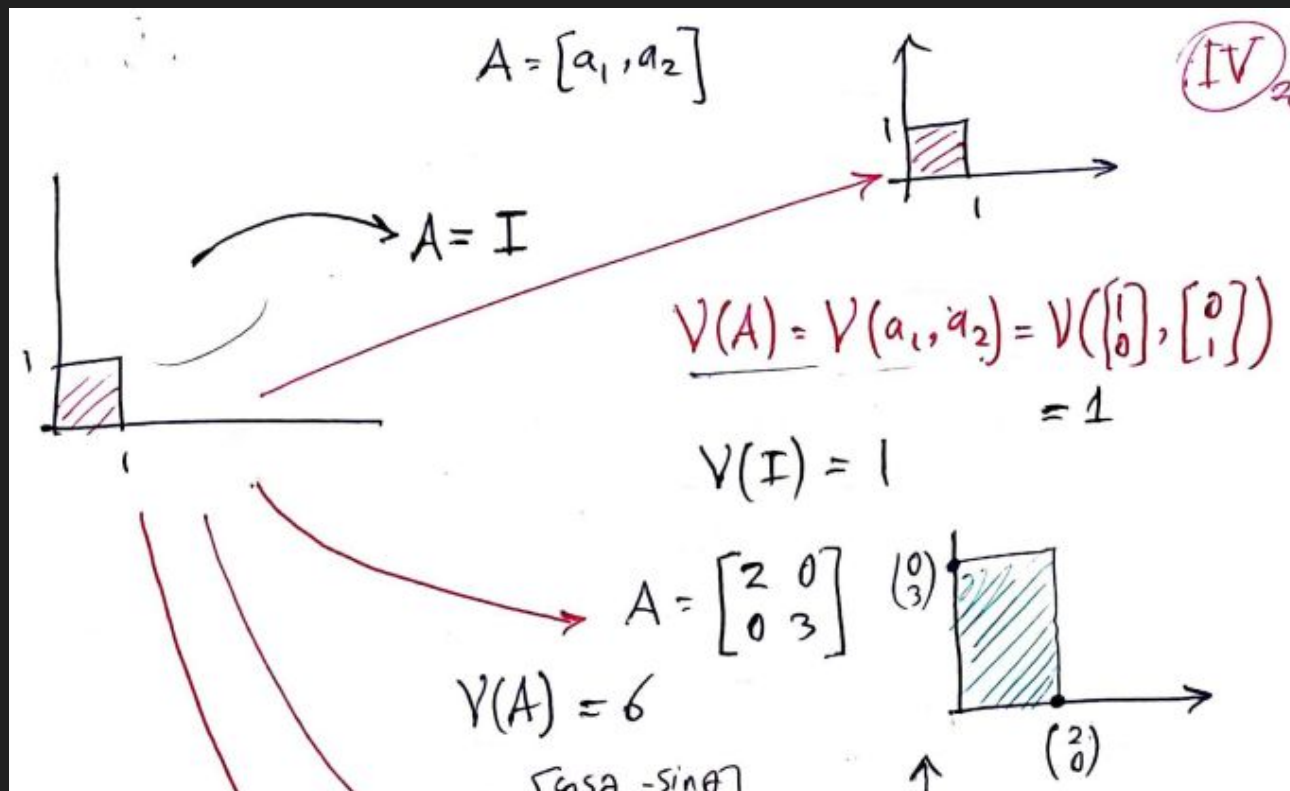
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$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n] \in \mathbb{R}^{n \times n}$$
$$V(a_1, a_2, \dots, a_n) = ?$$

Example: Non-uniform scaling



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Example: diagonal matrices (Non-uniform scaling)



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Handwritten derivation showing the calculation of the volume (determinant) of a 2D matrix A .

Matrix A is given as:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

The diagram illustrates the scaling of a unit square (area 1) into a rectangle with dimensions 2 and 3 (area 6). The unit square is shown with axes i and j . The scaled rectangle is shown with axes 2 and 3 , and its area is shaded.

The volume calculation is shown as:

$$V(A) = V(a_1, a_2) = V\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = 2 \times 3 = 6$$

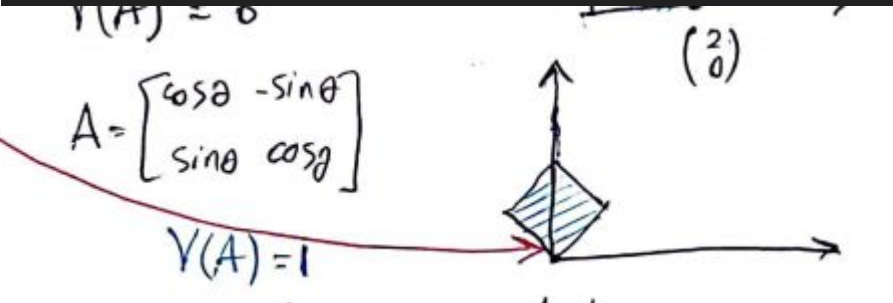
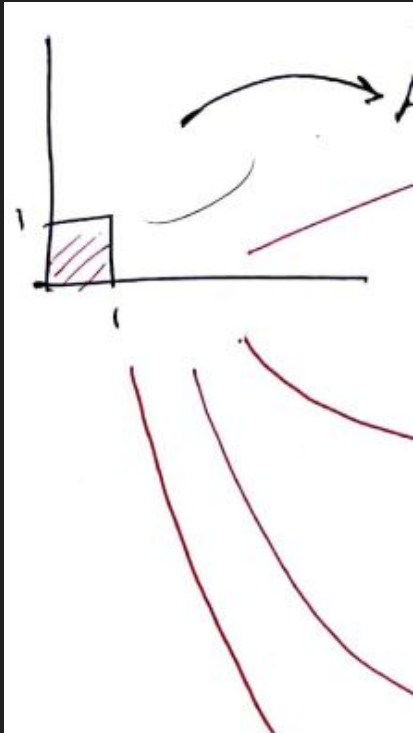
The general formula for the volume of a diagonal matrix is also shown:

$$V\left(\begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{bmatrix}\right) = d_1 \cdot d_2 \cdots d_n = \prod_{i=1}^n d_i$$

Example: Rotation



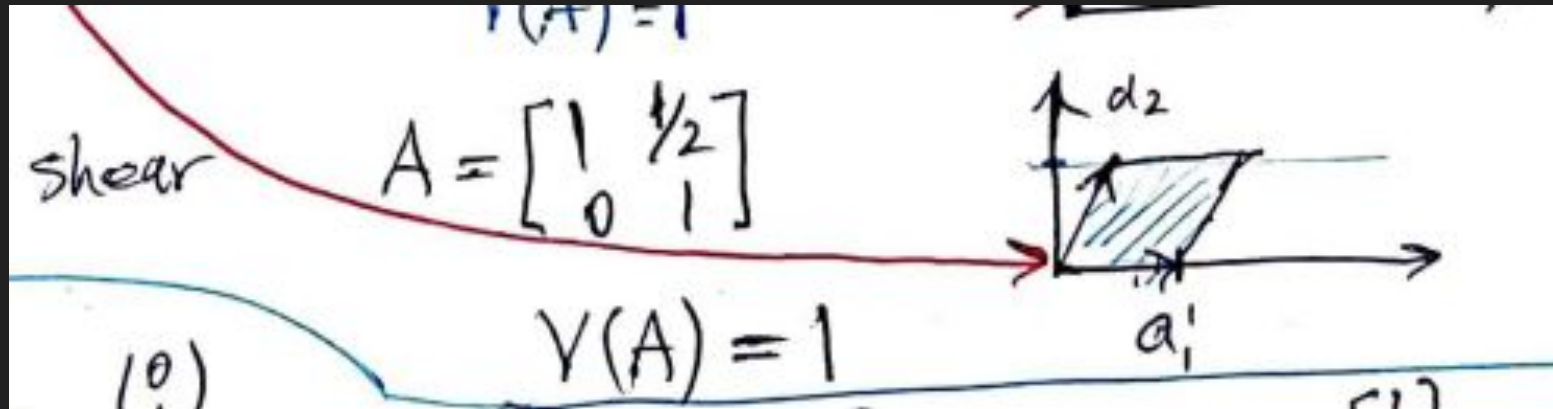
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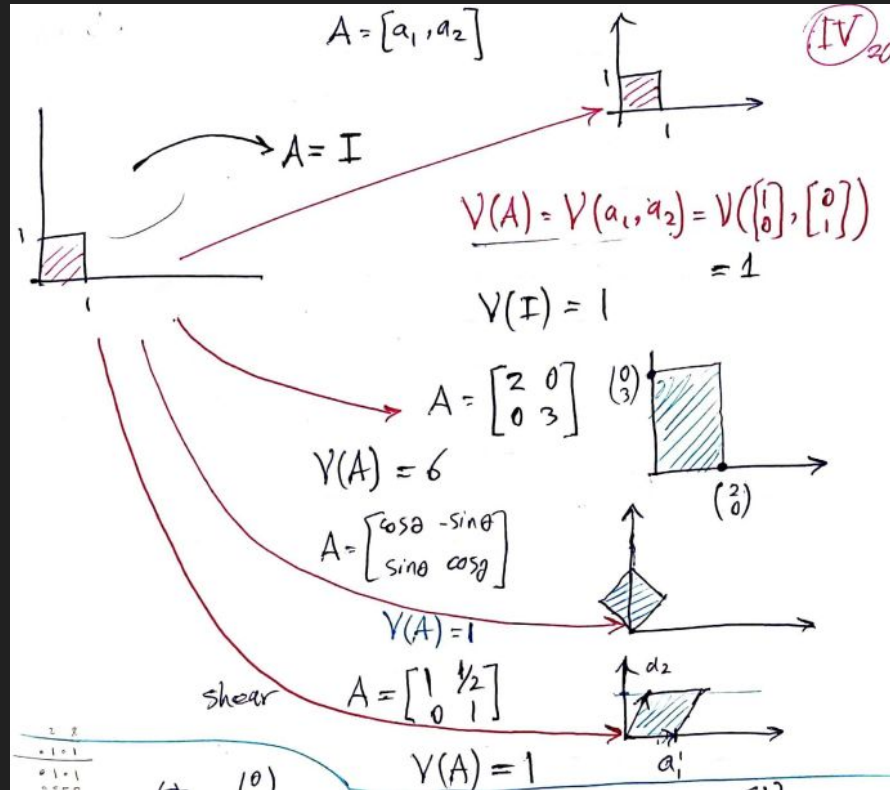
Example: Shear



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Example:



Signed Volume



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Handwritten notes illustrating the signed volume of a parallelepiped defined by vectors a_1 and a_2 .

Left side (Coordinate system):

- Vertical axis: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- Horizontal axis: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- A shaded parallelogram is drawn in the first quadrant.

Center (Matrix and Determinant):

$$A = [a_1 \ a_2]$$
$$V(A) = 1$$
$$V(A) < 0$$

Right side (Vector definitions):

- $a_1 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- $a_2 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

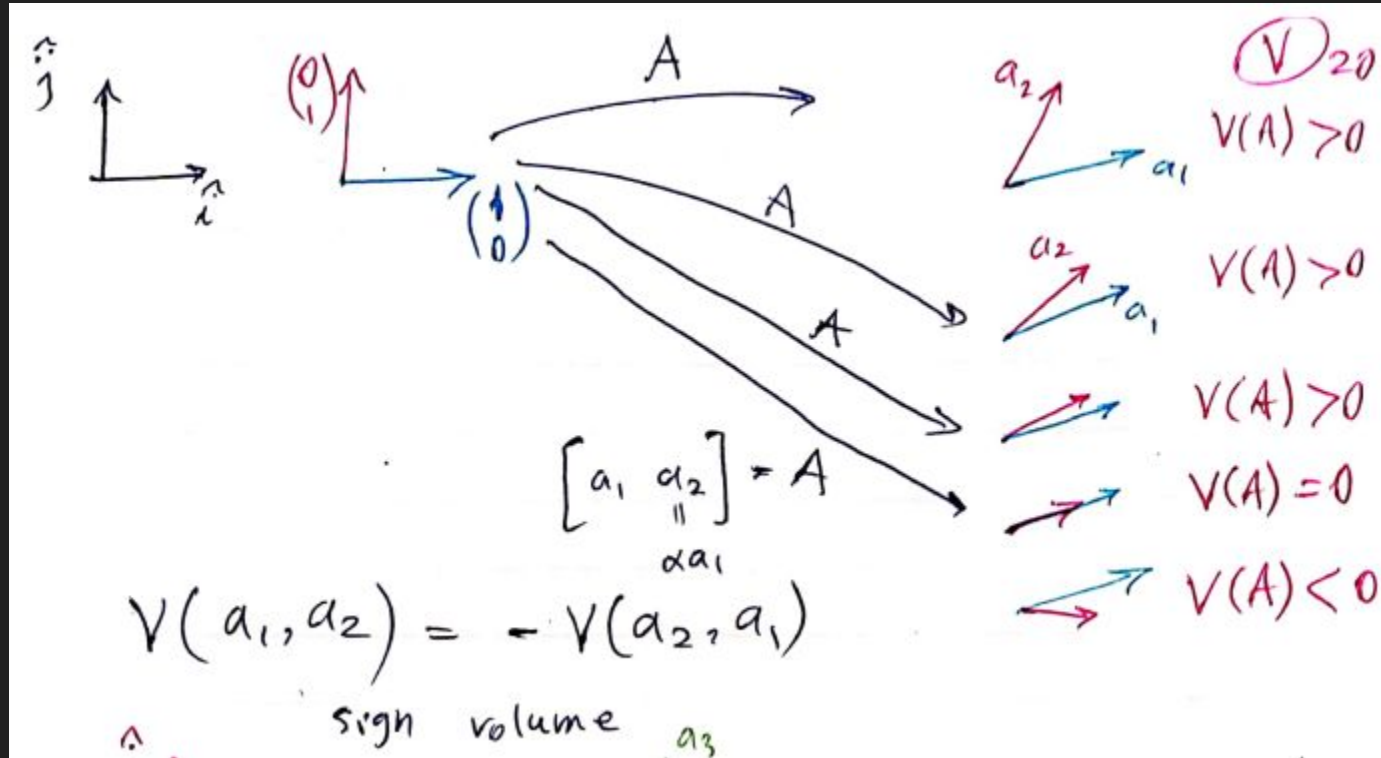
Bottom (Binomial expansion):

$$\sum_{i=1}^n \binom{n}{i} x^i y^{n-i} = (x+y)^n$$

Signed Volume



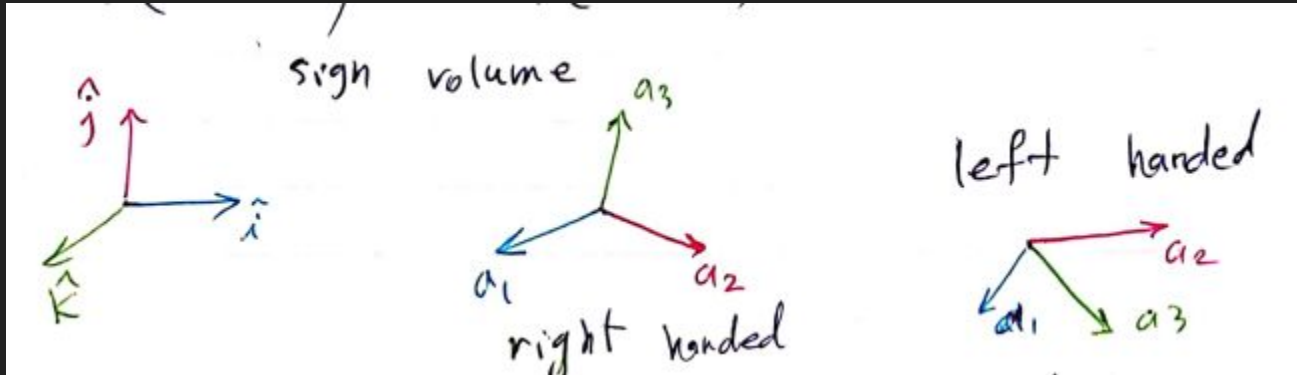
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Signed Volume



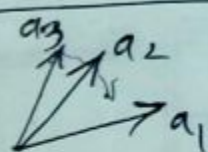
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Singular Case



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$A = [a_1 \ a_2 \ a_3] \in \mathbb{R}^{3 \times 3} \quad a_i \in \mathbb{R}^3$

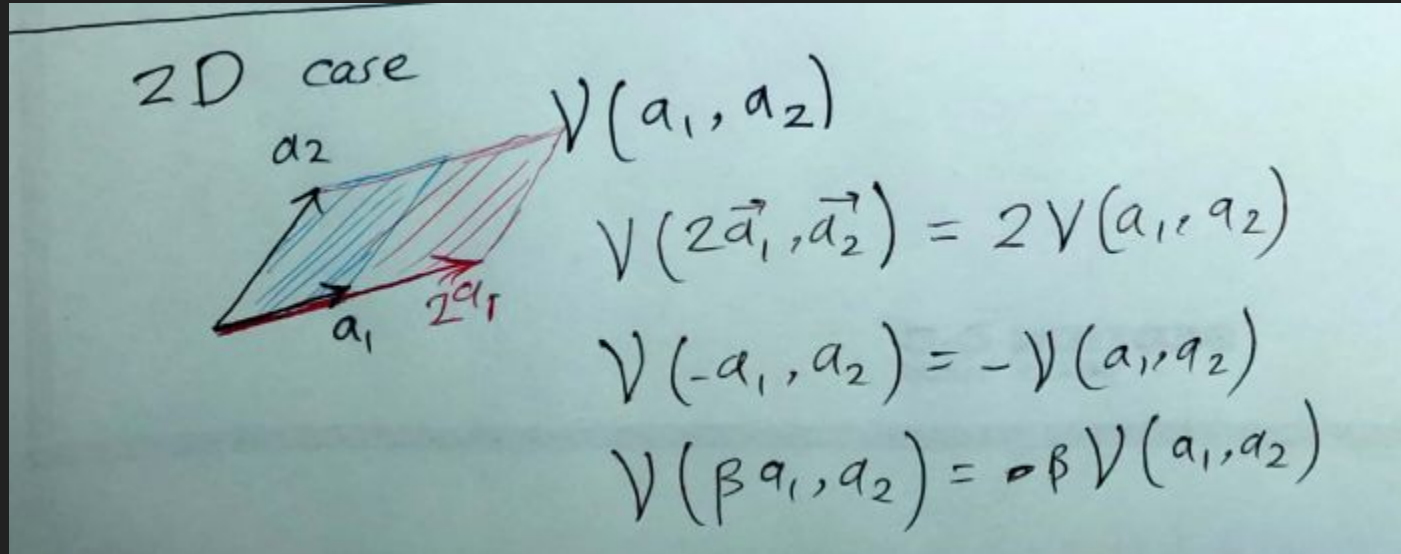
a_1, a_2, a_3 dependent $\Rightarrow V(A) = V(a_1, a_2, a_3) = \emptyset$

A singular

Scaling one vector



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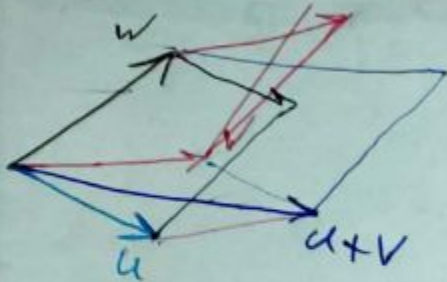


Multilinearity



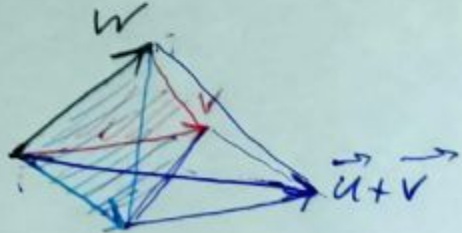
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2D



$$V(u+v, w) = V(\underline{u}, \underline{w}) + V(\underline{v}, \underline{w})$$

$$V(\alpha u + \beta v, w) = \alpha V(u, w) + \beta V(v, w)$$



Multilinearity



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$V(a_1, a_2, \dots, a_n)$ is ^{$a_i \in \mathbb{R}$} multilinear.

↙
linear in each argument

$$V(\underline{\alpha a_1 + \beta a'_1}, a_2, \dots, a_n) = \alpha V(a_1, a_2, \dots, a_n) + \beta V(a'_1, a_2, \dots, a_n)$$

Signed Volume of the Identity matrix



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$$V(I) = 1 = V\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = 1$$

Repeated argument (column)



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$$\vee(a_1, a_1, a_3, a_4, \dots, a_n) = 0$$



Exchanging two arguments (two columns)



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$$V(x, y, z) \quad x, y, z \in \mathbb{R}$$

$$V(y, x, z) = ?$$

$$\begin{aligned} 0 &= V(x+y, \cancel{x+y}, z) = V(x, x+y, z) + V(y, x+y, z) \\ &= V(x, \cancel{x}, z) + V(x, y, z) + V(\cancel{y}, y, z) + V(y, x, z) = 0 \end{aligned}$$

$$\Rightarrow V(x, y, z) = -V(y, x, z)$$

Signed Volume of a 2x2 matrix



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$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = V\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$$V\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$aV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$acV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bdV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$ad - bcV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = ad - bc$$

Signed Volume = Determinant



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$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = V\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$$V\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$aV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$acV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bdV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$ad - bcV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = ad - bc$$

$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$$

determinant