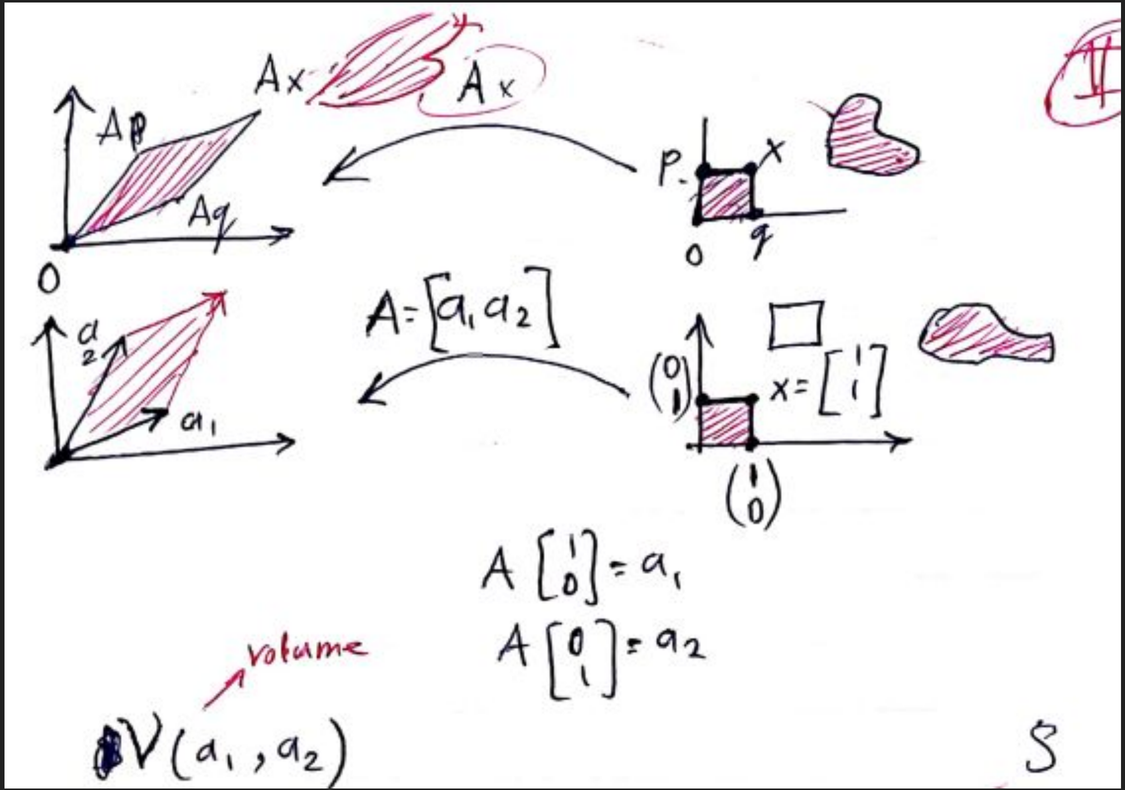
Linear Algebra for Computer Science

Lecture 19

Signed Volume

Area (2D Volume)



Area (2D Volume)

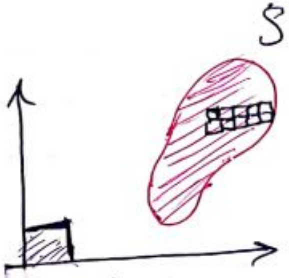
$$A\begin{bmatrix}1\\0\end{bmatrix} = a_1$$

$$A\begin{bmatrix}0\\1\end{bmatrix} = a_2$$

volume

$$V(a_1, a_2)$$

$$[a_1\ a_2] = A$$

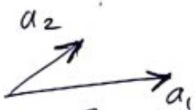


matrix

set

$$\text{Area}(AS) = \text{Area}(\{Ax \mid x \in S\})$$

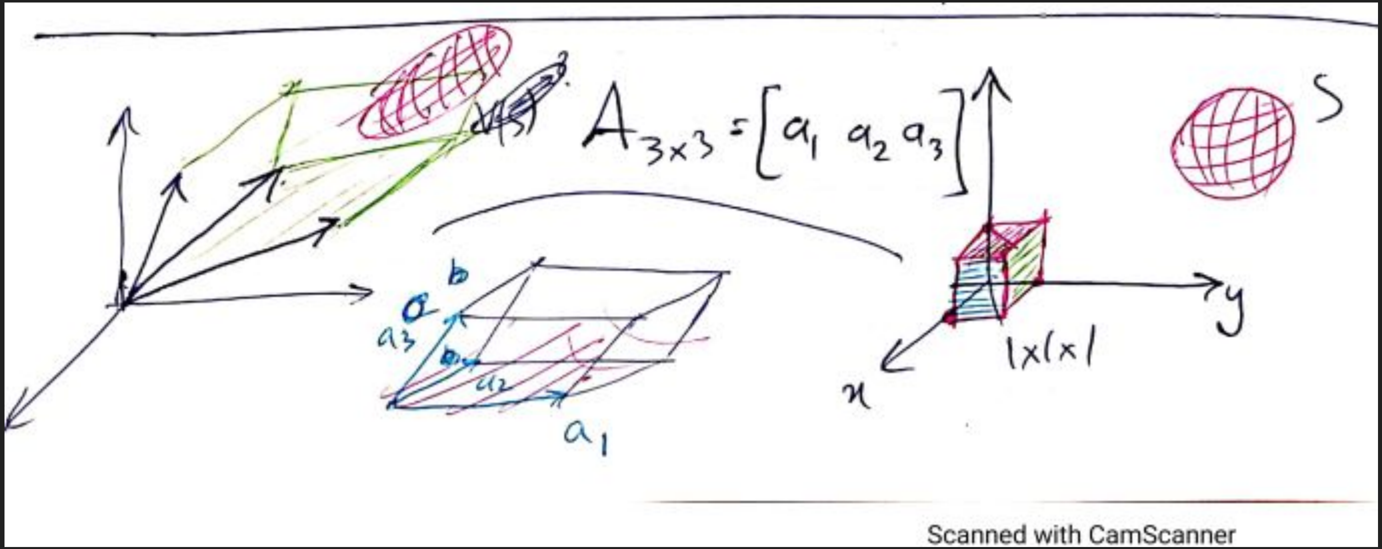
$$= \text{Area}(S) * \underbrace{V(a_1, a_2)}_{?}$$



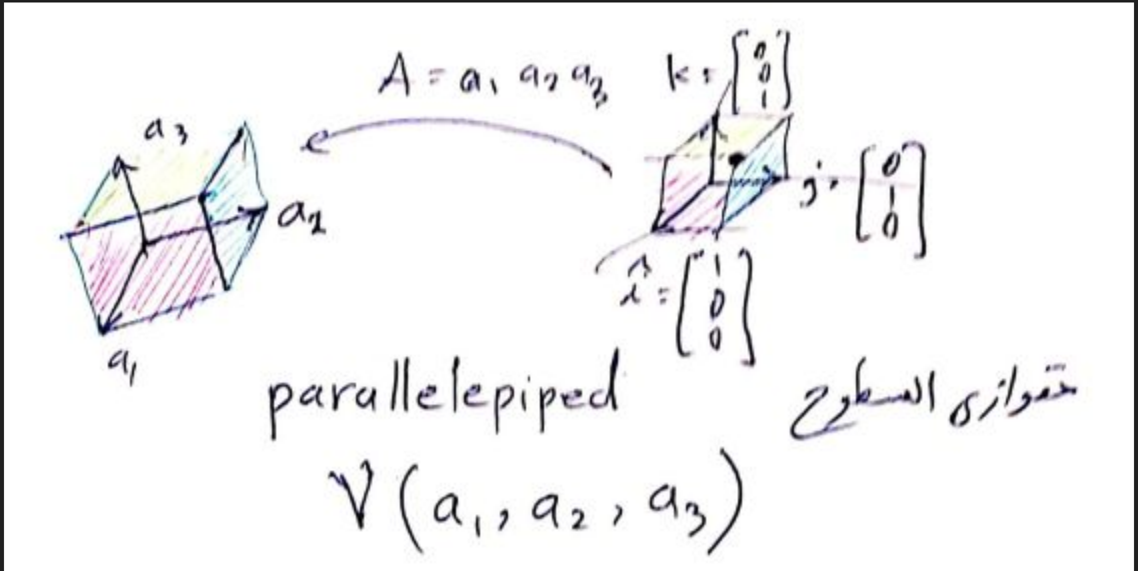
$$A = [a_1\ a_2]$$

$$V(a_1, a_2) = V(A)$$

3D Volume



3D volume



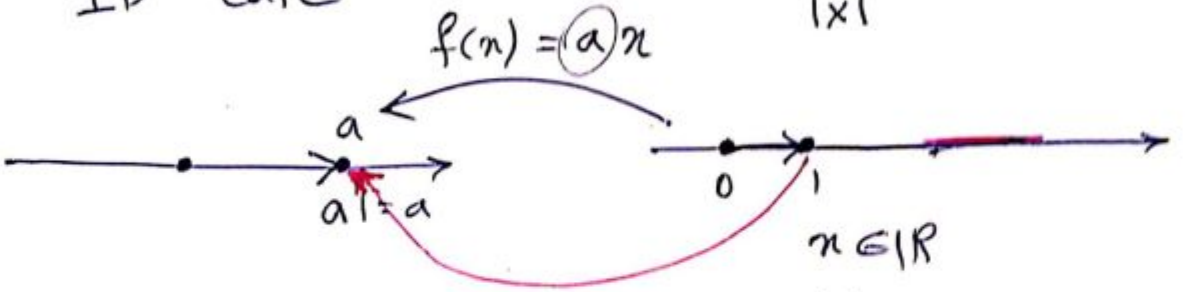
$A = a_1\ a_2\ a_3$
a_3
a_2
a_1
$k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
$\hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
parallelepiped
متوازی السطوح
$V(a_1, a_2, a_3)$

1D case

1D case

$A = a$

1×1



$$V(a) = a$$

$$f(x) \overset{\cancel{a=2}}{=} 2x \qquad \cancel{V(f)}\, V(\cdot) = 2$$

$$f(x) = 0.5x \qquad \cancel{V(f) =} \; V(\cdot) = 0.5$$

$$f(x) = 0 \cdot x = 0 \qquad \cancel{V(0) = 0} = 1$$

1D case

$f(x) = -1\ x$

$V(-1) = -1 \qquad V(.) = -1$

signed lenght

$f(x) = -4\ x$

1D-case $\quad A = [a] \quad V(A) = V(a) = \underline{a}$

Signed Volume

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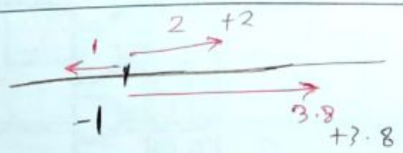
1D $x \in \mathbb{R}$ طول



signed lenght طول علامت دار

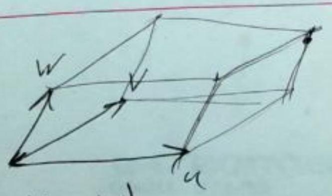
$SL(x) = x$

2D Area مساحت

SA: signed Area

$SA(x,y) = -SA(y,x)$

3D Volume حجم



SV: signed volume

$SV(u,v,w) = -SV(u,v,-w)$

$SV(u,v,w) = -SV(u,w,v)$

Signed Volume

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N-D
N-Dimensional

اَبَرحجم

Hyper-Volume

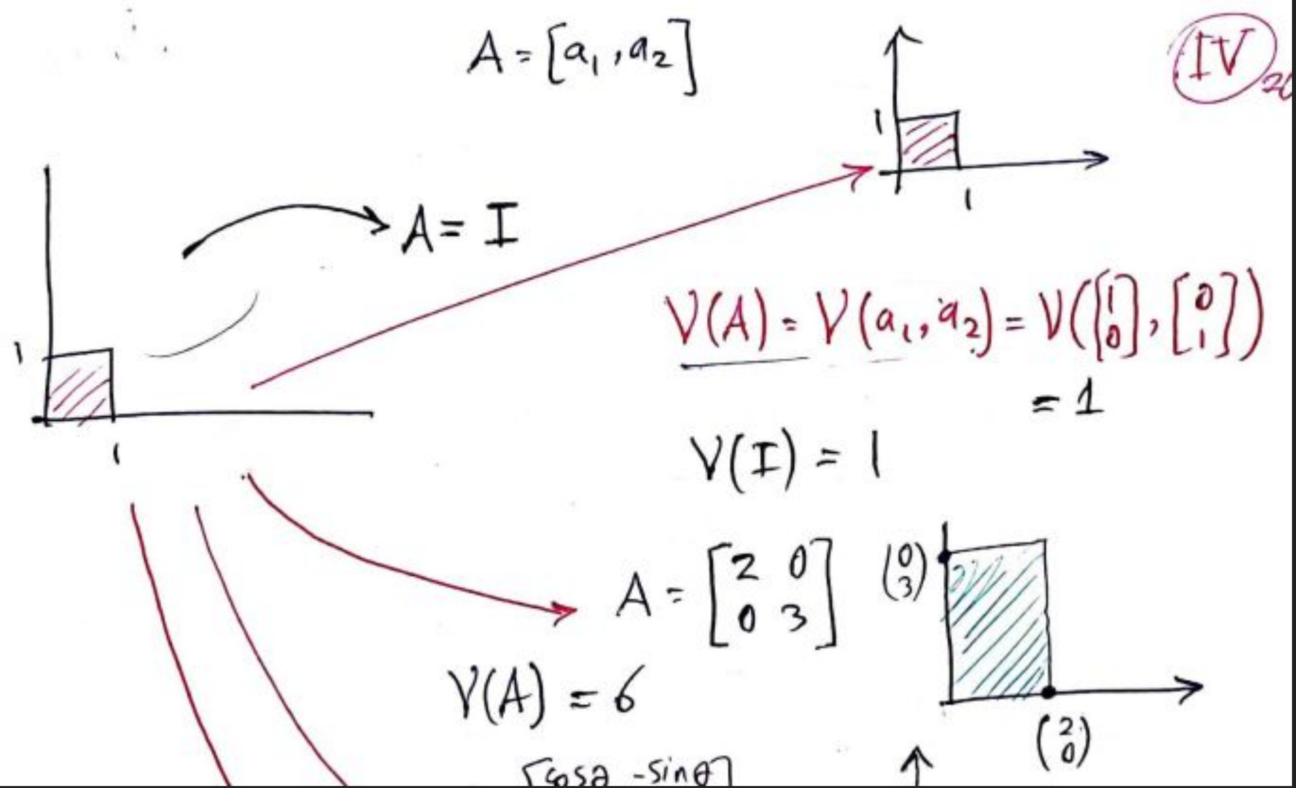
$V(a_1, a_2, \dots, a_N)$ $\quad a_i \in \mathbb{R}^N$ $\quad [a_1, a_2, \dots, a_N] \in \mathbb{R}^{N \times N}$

Signed Volume

$$A = [\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n] \in \mathbb{R}^{n\times n}$$

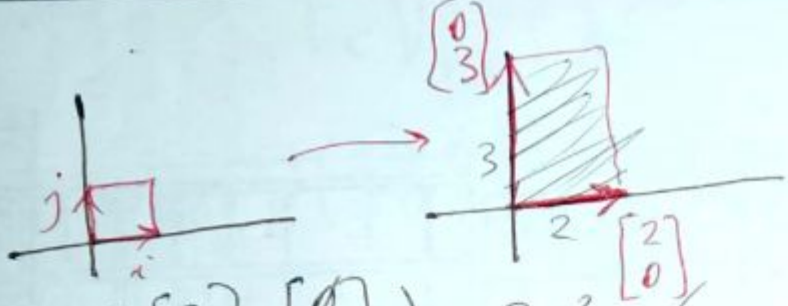
$$V(a_1, a_2, \ldots, a_n) = ?$$

Example: Non-uniform scaling



Example: diagonal matrics (Non-uniform scaling)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



$$V(A) = V(a_1, a_2) = V\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = 2 \times 3 = 6$$

$$V\left(\begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix}\right) = d_1 \cdot d_2 \cdots d_n = \prod_{i=1}^{n} d_i$$

Example: Rotation

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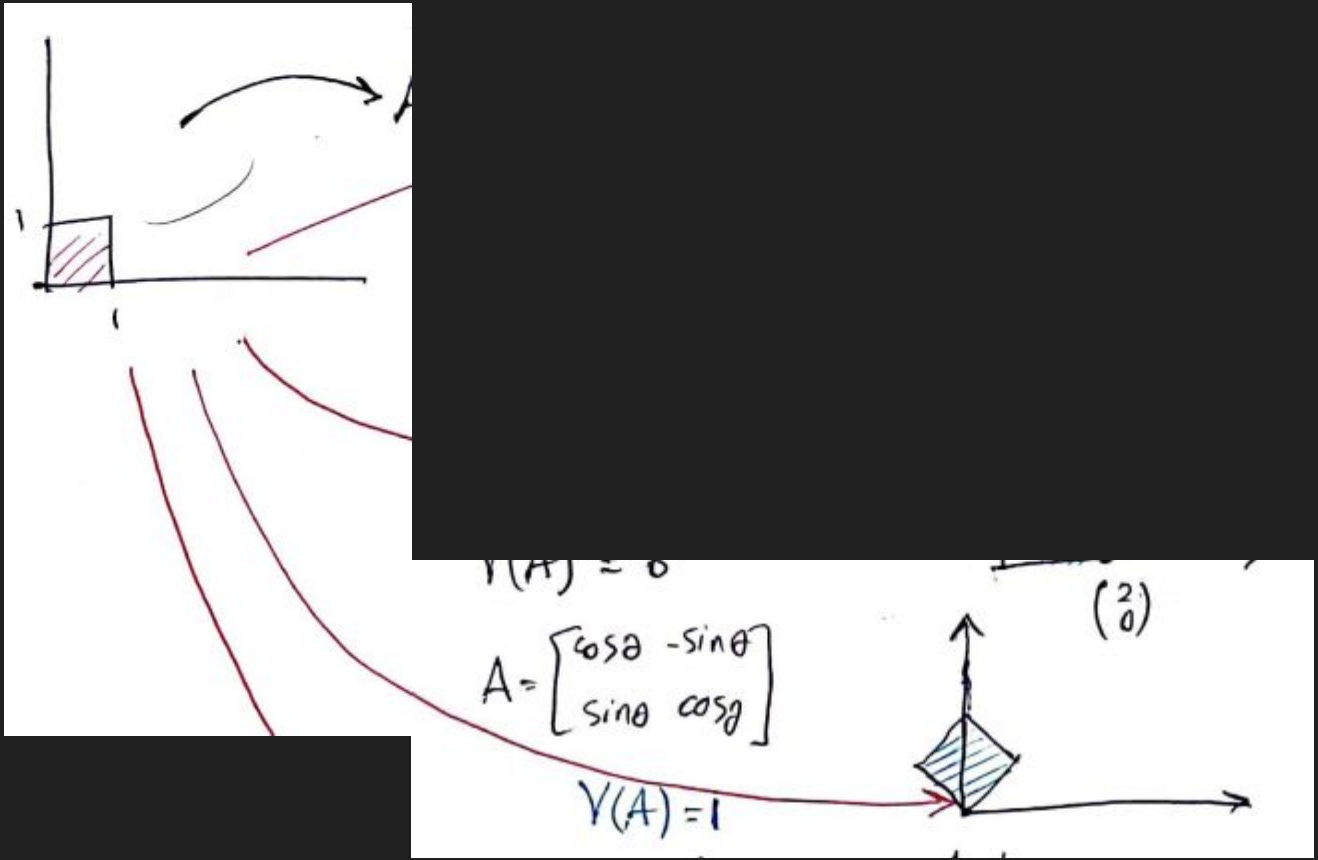

Example: Shear

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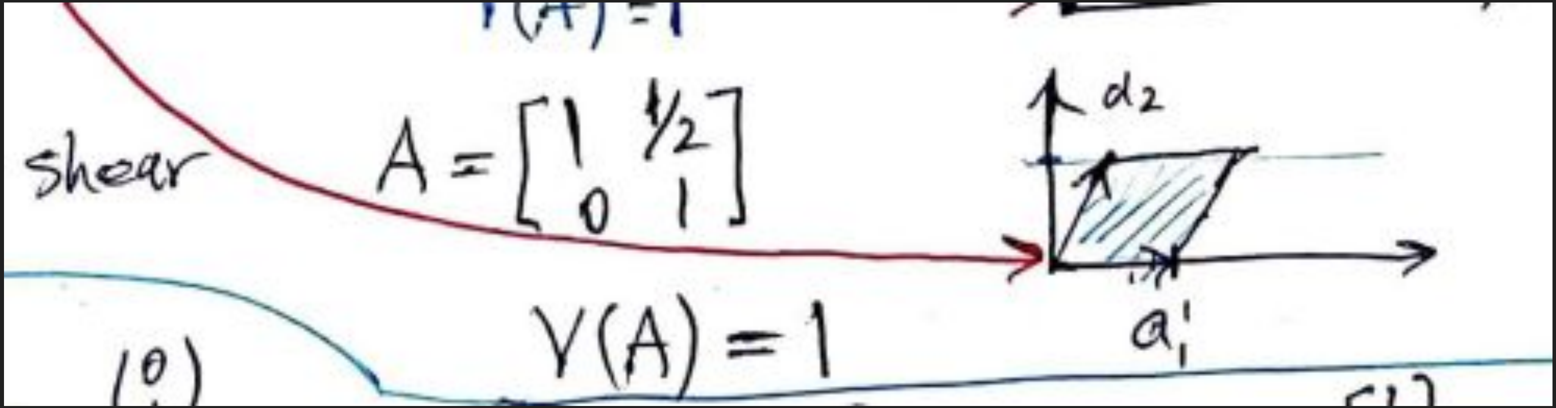
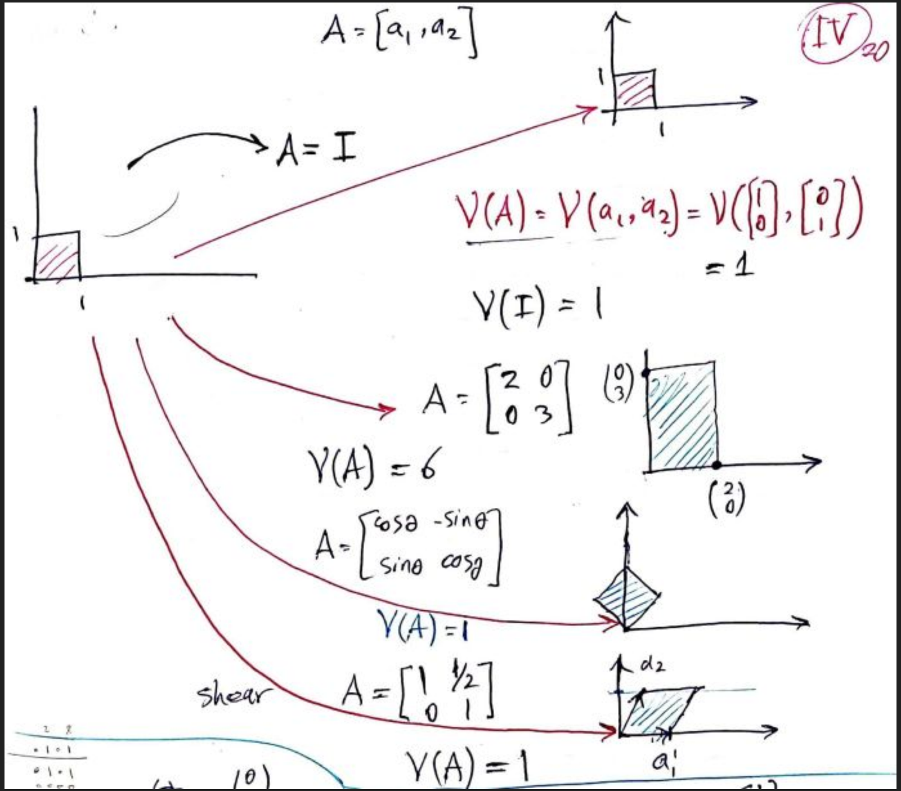
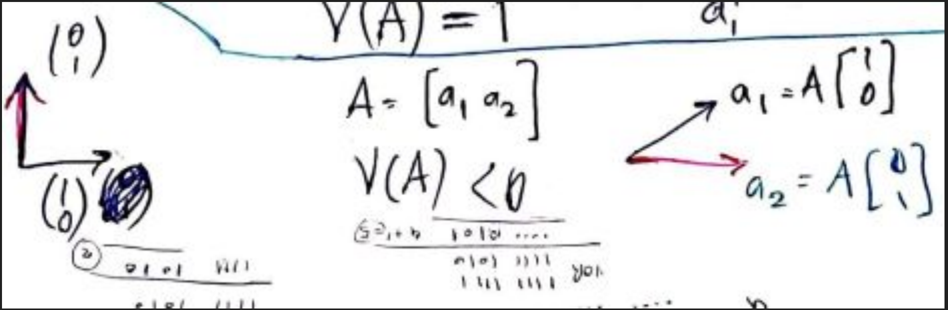


Example:

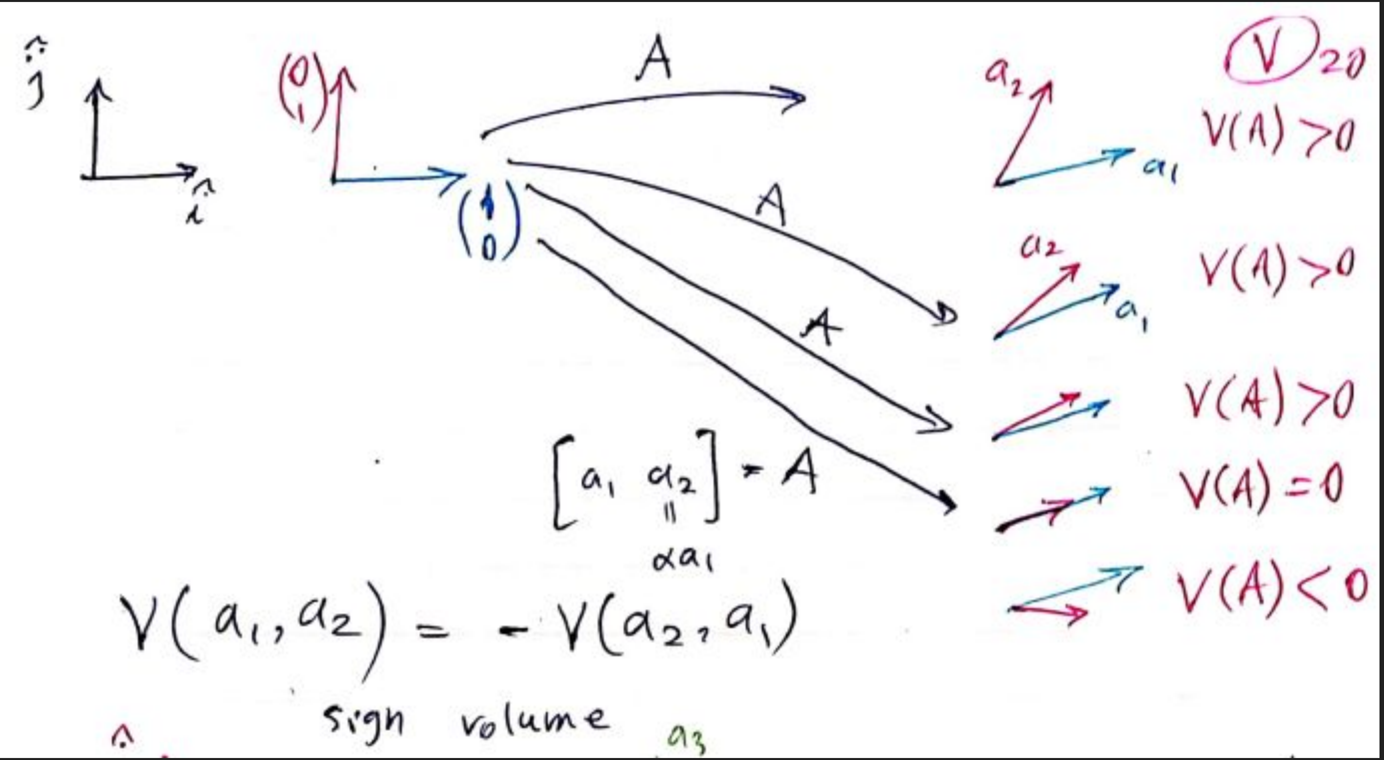


A = [a1 , a2]
IV
20
A = I
1
1
1
1
V(A) = V(a1 , a2) = V([1 0], [0 1])
= 1
V(I) = 1
A = [2 0 / 0 3]
(0 3)
(2 0)
V(A) = 6
A = [cosθ -sinθ / sinθ cosθ]
V(A) = 1
shear
A = [1 1/2 / 0 1]
a2
a1
V(A) = 1

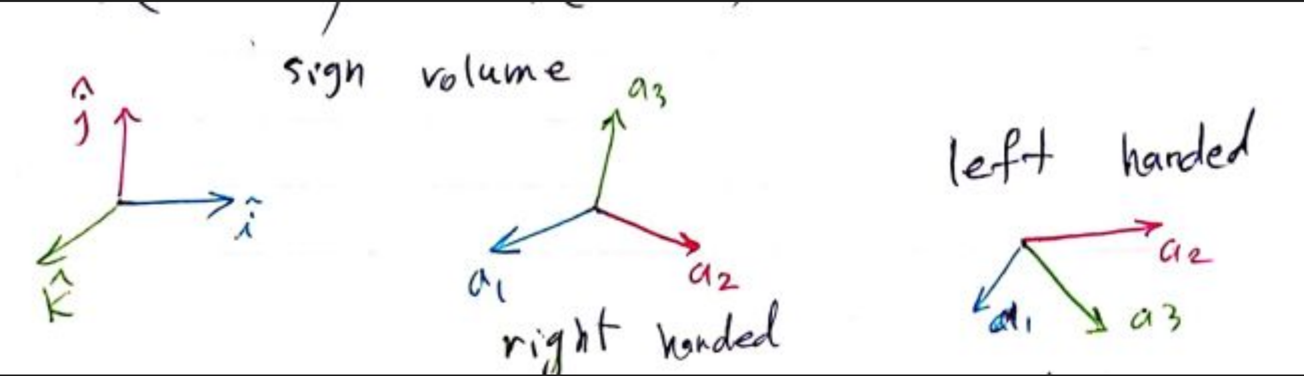
Signed Volume



Signed Volume



Signed Volume



Singular Case

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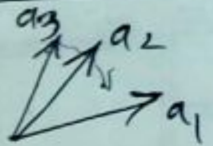


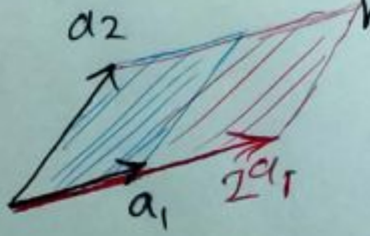
$A = [a_1 \ a_2 \ a_3] \in \mathbb{R}^{\cancel{n\times n}\, 3\times 3}$ $\quad a_i \in \mathbb{R}^3$

a_1, a_2, a_3 dependent $\Longrightarrow V(A) = V(a_1, a_2, a_3) = 0$

A singular

Scaling one vector

2D case



$V(a_1, a_2)$

$$V(2\vec{a}_1, \vec{a}_2) = 2V(a_1, a_2)$$

$$V(-a_1, a_2) = -V(a_1, a_2)$$

$$V(\beta a_1, a_2) = \beta V(a_1, a_2)$$

Multilinearity

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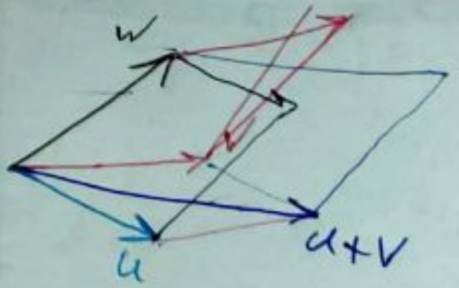
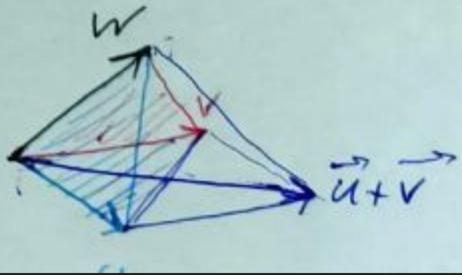
2D



$$V(u+v, w) = V(u, w) + V(v, w)$$

$$V(\alpha u + \beta v, w) = \alpha V(u, w) + \beta V(v, w)$$



Multilinearity

$a_i \in \mathbb{R}$

$V(a_1, a_2, \ldots, a_n)$ is multilinear.

linear in each argument

$$V(\underline{\alpha a_1 + \beta a_1'}, a_2, \ldots, a_n) = \alpha V(a_1, a_2, \ldots, a_n) + \beta V(a_1', a_2, \ldots, a_n)$$

Signed Volume of the Identity matrix

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$$V(I) = 1 = V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix} \cdots, \begin{bmatrix}0\\0\\0\\ \vdots \\1\end{bmatrix}\right) = 1$$

Repeated argument (column)

$$V(a_1, a_1, a_3, a_4, \dots a_n) = 0$$

Exchanging two arguments (two columns)

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$$V(x,y,z) \quad x,y,z \in \mathbb{R}$$

$$V(y,x,z) = ?$$

$$0 = V(x+y, x+y, z) = V(x, x+y, z) + V(y, x+y, z)$$

$$= \underbrace{\cancel{V(x,x,z)}}_{0} + V(x,y,z) + \underbrace{\cancel{V(y,y,z)}}_{0} + V(y,x,z) = 0$$

$$\Rightarrow V(x,y,z) = -V(y,x,z)$$

Signed Volume of a 2x2 matrix

$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = V\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$$V\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$aV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$ac\,V\Bigl(\underbrace{\cancel{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}}}_{0}\Bigr) + ad\,V\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bc\,V\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + bd\,\underbrace{\cancel{V\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}}_{0}$$

$$ad\,V\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + bc\,V\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

$$ad - bc\,V\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = ad - bc$$

Signed Volume = Determinant

LA 19 V

$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = V\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right)$$

$$V\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$aV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bV\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$ac\underbrace{\cancel{V\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}}_{0} + adV\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) + bcV\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + bd\underbrace{\cancel{V\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}}_{0}$$

$$ad\,V\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) + bcV\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$$

$$ad - bcV\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = ad - bc$$

$$V\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = ad - bc$$

determinant