Linear Algebra for Computer Science

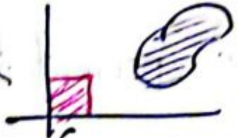
Lecture 20

Determinant

Determinant

$f(x) = Ax$

$A = [a_1\ a_2]$



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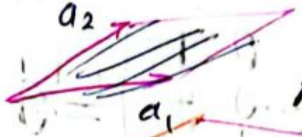
$\mathbb{R}^2$



$A = [a_1\ a_2] \in \mathbb{R}^{2\times 2}$ $\det(A) = V(a_1, a_2)$

$\mathbb{R}^3$

$A = [a_1\ a_3\ a_3] \in \mathbb{R}^{3\times 3}$

$\det(aA) = V(a_1, a_2, a_3)$

Three basic properties

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I) $\det(I) = V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = 1$

II)

$$\det\begin{bmatrix} a_1 & a_2 & a_3+a_3' & a_4 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} + \det\begin{bmatrix} a_1 & a_2 & a_3' & a_4 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3' & a_4 \end{vmatrix}$$

$$\cancel{\det} \begin{vmatrix} a_1 & a_2 & \beta a_3 & a_4 \end{vmatrix} = \beta \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

III) $\begin{vmatrix} a_1 & a_2 & a_2 & a_4 \end{vmatrix} = 0$

1. Determinant of Identity Matrix

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$\det(aA) = V($

$$\det(I) = V\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) = 1$$

2. Multilinear (Linear in each column)

$$\det\begin{bmatrix} a_1 & a_2 & a_3+a_3' & a_4 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} + \det\begin{bmatrix} a_1 & a_2 & a_3' & a_4 \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3' & a_4 \end{vmatrix}$$

~~de~~

$$\cancel{\det}\begin{vmatrix} a_1 & a_2 & \beta a_3 & a_4 \end{vmatrix} = \beta \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

3. Identical Columns

$$\begin{vmatrix} a_1 & a_2 & a_2 & a_4 \end{vmatrix} = 0$$

Swapping Columns

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$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} = -\begin{vmatrix} a_1 & a_3 & a_2 & a_4 \end{vmatrix}$$

Permutation matrix

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P is a permutation matrix $\det(P) \in \{-1, 1\}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

even perm. $|P| = 1$

odd perm. $|P| = -1$

one zero column

$$\det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & -\vec{0} & a_3 \end{bmatrix}$$

$$= -\det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a_1 & a_2 & \vec{0} & a_3 \end{bmatrix} = 0$$

Determinant of a 2x2 matrix

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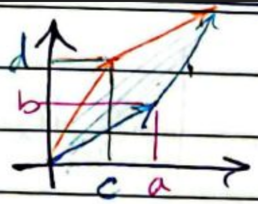
$$\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}$$

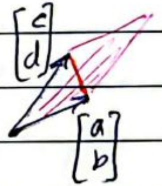


$$= \det\left[\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \quad \begin{matrix} c \\ d \end{matrix}\right] = \begin{vmatrix} a & c \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & d \end{vmatrix}$$

$$= \begin{vmatrix} a & c \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & c \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ b & d \end{vmatrix}$$

$$\overset{a_1 = a_2}{} \quad ac\underbrace{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}_{0} + ad\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1} + bc\underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{-1} + bd\underbrace{\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}}_{0}$$

$$\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$



Diagonal Matrix

Diagonal Matrix $\begin{vmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = d_1 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{vmatrix}$

$= d_1 d_2 d_3 d_4 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = d_1 d_2 d_3 d_4$

Scaling a matrix

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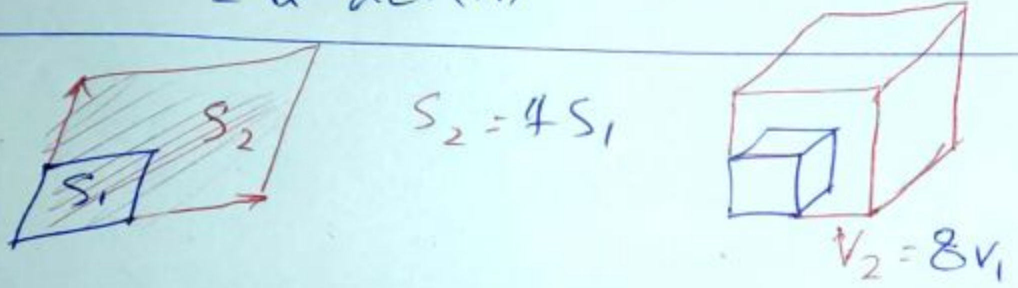
$$A \in \mathbb{R}^{n \times n} \qquad \alpha \in \mathbb{R} \qquad A = [a_1 \ a_2 \cdots a_n]$$

$$\det(\alpha A) = \det([\alpha a_1 \ \ \alpha a_2 \cdots \alpha a_n])$$

$$= \alpha^n \det(A)$$



Singular matrices

$$\begin{vmatrix} a_1 & a_2 & \alpha a_1 + \beta a_2 \end{vmatrix} = \alpha \underbrace{\begin{vmatrix} a_1 & a_2 & a_1 \end{vmatrix}}_{0} + \beta \underbrace{\begin{vmatrix} a_1 & a_2 & a_2 \end{vmatrix}}_{0} = 0$$

Singular Matrices

One column is a linear combination of $\underbrace{\text{others}}_{\text{(some of)}}$ IV

$$\begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & a_n \end{vmatrix}$$

$$a_n = \sum_{i=1}^{n-1} \beta_i \cdot a_i$$

$$\begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & \sum_{i=1}^{n-1} \beta_i \cdot a_i \end{vmatrix} \longleftrightarrow \sum_{i=1}^{n-1} \beta_i \begin{vmatrix} a_1 & a_2 \cdots & a_{n-1} & a_i \end{vmatrix} = 0$$

Singular Matrices

Determinant of a singular matrix is zero.

(is the converse true?)

Adding a multiple of another column

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2+\beta a_1 & a_3 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \beta \underbrace{\begin{vmatrix} a_1 & a_1 & a_3 & a_4 \end{vmatrix}}_{0}$$

$$= |A|$$

Adding a multiple of another column

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2+\beta a_1 & a_3 & a_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \end{vmatrix} + \beta \underbrace{\begin{vmatrix} a_1 & a_1 & a_3 & a_4 \end{vmatrix}}_{0}$$

$$= |A|$$

Elimination Algorithm on columns do not change the determinant!

Triangular Matrices

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University of Technology

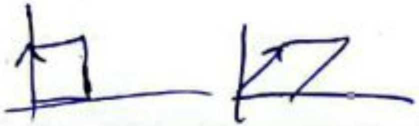
Triangular Matrices

$$\begin{array}{c}\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 6 & 3 \end{vmatrix} \\ \begin{array}{ccc} a_1 & a_2 & a_3 \end{array}\end{array} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ \cancel{15}\,0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times 2 \times 3$$

$a_2 \mathrel{-}= 2a_3$

$a_1 \mathrel{-}= 5a_3$

$a_1 \mathrel{-}= 2a_2$



Triangular Matrices

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Triangular Matrices

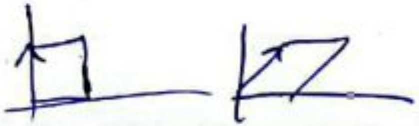
$$\begin{array}{c} \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 6 & 3 \end{vmatrix} \\ a_1 \quad a_2 \quad a_3 \end{array} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 15 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ \cancel{15}\,0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \times 2 \times 3$$

$a_2 -= 2\vec{a}_3$

$a_1 -= 5a_3$

$a_1 -= 2a_2$

What if a diagonal element is zero.



Triangular Matrices

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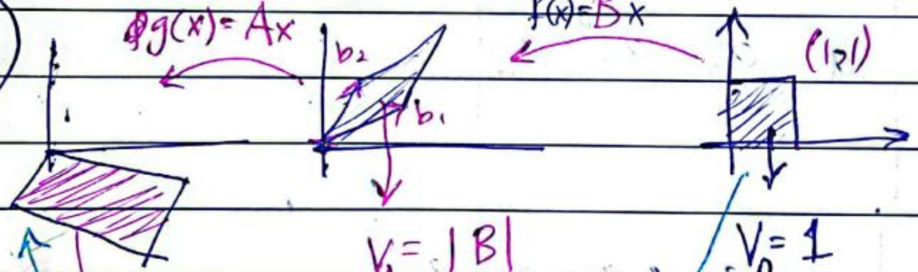
A is (Upper or Lower) Triangular Matrix

$$|A| = \text{product of } \cancel{\text{dig}}\text{diagonal elements}$$

$$= \prod_{i=1}^{n} a_{ii}$$

Product of matrices

$\det(AB)$



$V_1 = |B|$ $\qquad V_0 = 1$

$V_2 = |A| V_1 = |A||B|$

$$h(x) = (g \circ f)(x) = A(Bx) = (AB)x$$

$V_2 = |AB|$

$$\Rightarrow |AB| = |A|\,|B|$$

Product of matrices

$\det(A\,B) = \det(A)\,\det(B)$

try proving using the previous properties.

Inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\Rightarrow |AA^{-1}| = |I|$$

$$\Rightarrow |A||A^{-1}| = 1$$

Determinant of a non-singular matrix

$\det(A)\ \det(A^{-1}) = 1 \Rightarrow$ determinant of a non-singular matrix is non-zero.

Determinant of a non-singular matrix

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$\det(A)\ \det(A^{-1}) = 1 \Rightarrow$ determinant of a non-singular matrix is non-zero.

$\Rightarrow$ $\det(A) = 0$ if and only if A is singular.

Transpose of a matrix

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$\det(A^T) = ?$

LU decomposition

every $A \in R^{n\times n}$ ~~has~~ can be written as

$$\underbrace{P}_{\text{permutation}} A = \underbrace{L}_{\text{Lower triangular}} \underbrace{U}_{\text{upper triangular}}$$

$$PA = LU \Rightarrow |P|\,|A| = |L|\,|U| \Rightarrow \boxed{|A| = \frac{\prod_i l_{ii} \prod_i u_{ii}}{|P|}}$$

$$\Downarrow$$

$$A^T P^T = U^T L^T \Rightarrow |P^T|\,|A^T| = \underbrace{|L^T|}_{\text{upper triangular}}\,\underbrace{|U^T|}_{\text{Lower tria}} \Rightarrow \boxed{|A^T| = \frac{\prod l_{ii} \prod u_{ii}}{|P^T|}}$$

$|P| = |P^T| \Rightarrow$ $\left.\begin{array}{l} P^T P = I \Rightarrow |P|\,|P^T| = 1 \\ |P| \in \{1, -1\} \end{array}\right\} \Rightarrow \boxed{|P^T| = |P|} \Longrightarrow |A^T| = |A|$

$P^T = P^{-1}$

P is odd $\Rightarrow P^T$ is odd

Transpose of a matrix

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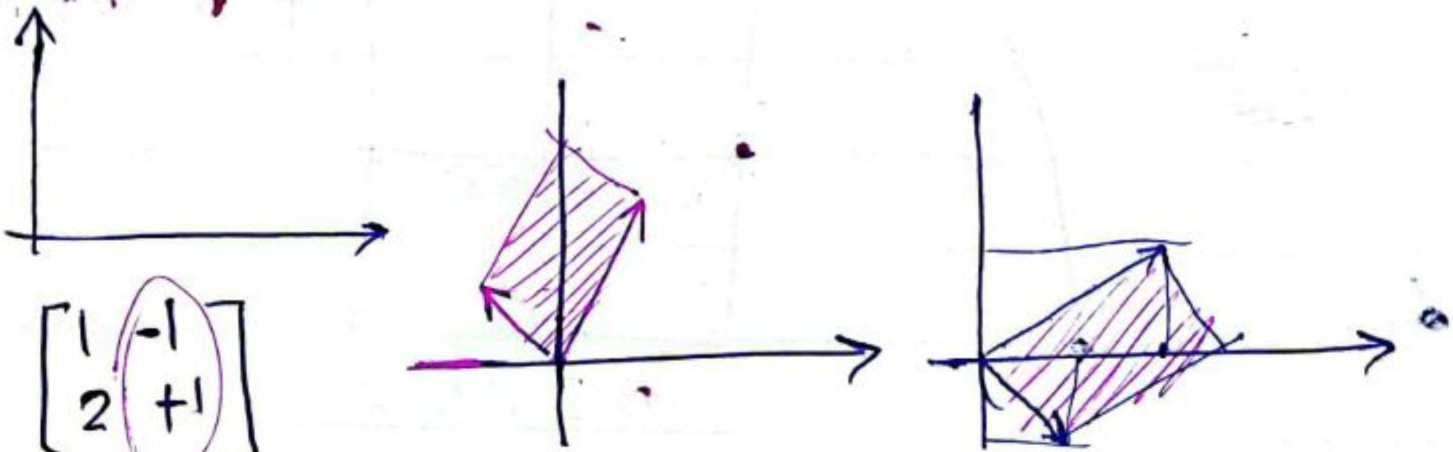
$$\det(A^T) = \det(A)$$

All the determinant properties about the columns of a matrix applies to the rows of a matrix.

Transpose of a matrix

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\mathrm{Vol}\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = \mathrm{Vol}\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)$$



$$\begin{bmatrix} 1 & -1 \\ 2 & +1 \end{bmatrix}$$