Linear Algebra for Computer Science

Lecture 21

Cofactors, Minors, Cramer's Rule

Determinant of a 3x3 matrix

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$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & c \\ d & e & f \\ 0 & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{vmatrix}$$

$$\underbrace{\begin{vmatrix} a & b & c \\ 0 & 0 & f \\ 0 & 0 & i \end{vmatrix}}_{=0} + \begin{vmatrix} a & 0 & c \\ 0 & e & f \\ 0 & 0 & i \end{vmatrix} + \begin{vmatrix} a & 0 & c \\ 0 & 0 & f \\ 0 & h & i \end{vmatrix}$$

Determinant of a 3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ 0 & e & f \\ 0 & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & c \\ d & e & f \\ 0 & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & c \\ 0 & e & f \\ g & h & i \end{vmatrix} \quad \text{(IV)}$$

$$= \overset{0}{\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}} + \overset{\emptyset}{\begin{vmatrix} a & b & 0 \\ 0 & 0 & f \\ 0 & 0 & 0 \end{vmatrix}} + \overset{0}{\begin{vmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{vmatrix}} + \overset{\emptyset}{\begin{vmatrix} a & 0 & c \\ 0 & e & 0 \\ 0 & 0 & 0 \end{vmatrix}} + \overset{\emptyset}{\begin{vmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & 0 & 0 \end{vmatrix}} + \underset{\overset{\shortparallel}{aei}}{\begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & i \end{vmatrix}}$$

$$+ \cdots + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & h & i \end{vmatrix} \emptyset \qquad 27 \text{ terms}$$

Determinant of a 3x3 matrix

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$$\Rightarrow \begin{vmatrix} a & b & c\\ d & e & f\\ g & h & i\end{vmatrix} = \overset{aei}{\begin{vmatrix} a&0&0\\0&e&0\\0&0&i\end{vmatrix}} + \overset{-ahf}{\begin{vmatrix} a&0&0\\0&0&f\\0&h&0\end{vmatrix}} + \overset{-dbi}{\begin{vmatrix}0&b&0\\d&0&0\\0&0&i\end{vmatrix}}$$

$$+ \underset{dhc}{\begin{vmatrix}0&0&c\\d&0&0\\0&h&0\end{vmatrix}} + \underset{gbf}{\begin{vmatrix}0&b&0\\0&0&f\\g&0&0\end{vmatrix}} + \underset{-gec}{\begin{vmatrix}0&0&c\\0&e&0\\g&0&0\end{vmatrix}}$$

$$= a\begin{vmatrix} e & f\\ h & i\end{vmatrix} - d\begin{vmatrix} b & c\\ h & i\end{vmatrix} + g\begin{vmatrix} b & c\\ e & f\end{vmatrix}$$

Cofactors

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$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + b\begin{vmatrix} 0 & 1 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + c\begin{vmatrix} 0 & 0 & 1 \\ d & e & f \\ g & h & i \end{vmatrix}$$

elimination -> $$= a\begin{vmatrix} 1 & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{vmatrix} + b\begin{vmatrix} 0 & 1 & 0 \\ d & 0 & f \\ g & 0 & i \end{vmatrix} + c\begin{vmatrix} 0 & 0 & 1 \\ d & e & 0 \\ g & h & 0 \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Cofactors

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$$A_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \boxed{B} & \\ 0 & & & \end{bmatrix}$$

$(n-1)\times(n-1)$

$$\det(A) = ?$$

$$\det(A) = \det(B)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Cofactors

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{11} & b_{12} & b_{13} \\ 0 & b_{21} & b_{22} & b_{23} \\ 0 & b_{31} & b_{32} & b_{33} \end{bmatrix} \qquad \det(A) = \det(B)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Cofactors

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ b_{11} & 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{22} & b_{23} \\ b_{31} & 0 & b_{32} & b_{33} \end{bmatrix} \qquad \det(A) = -\det(B)$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Cofactors

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$$\begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$A = \begin{array}{c c} & \begin{array}{cccc} \phantom{b_{11}} & 2 & \phantom{b_{12}} & \phantom{b_{13}} \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \\ \end{array} & \begin{bmatrix} b_{11} & 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{22} & b_{23} \\ 0 & 1 & 0 & 0 \\ b_{31} & 0 & b_{32} & b_{33} \end{bmatrix} \end{array}$$

A_{32}

$$\det(A) = (-1)^3 \det(B)$$
$$= -\det(B)$$

$$i,j \rightarrow \det(A) = (-1)^{i+j} \det(B)$$

Cofactors

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 0 & 0 \end{vmatrix} + h\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 1 & 0 \end{vmatrix} + i\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{vmatrix}$$

$$= g\begin{vmatrix} 0 & b & c \\ 0 & e & f \\ 1 & 0 & 0 \end{vmatrix} + h\begin{vmatrix} a & 0 & c \\ d & 0 & f \\ 0 & 1 & 0 \end{vmatrix} + i\begin{vmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Minor

Minors

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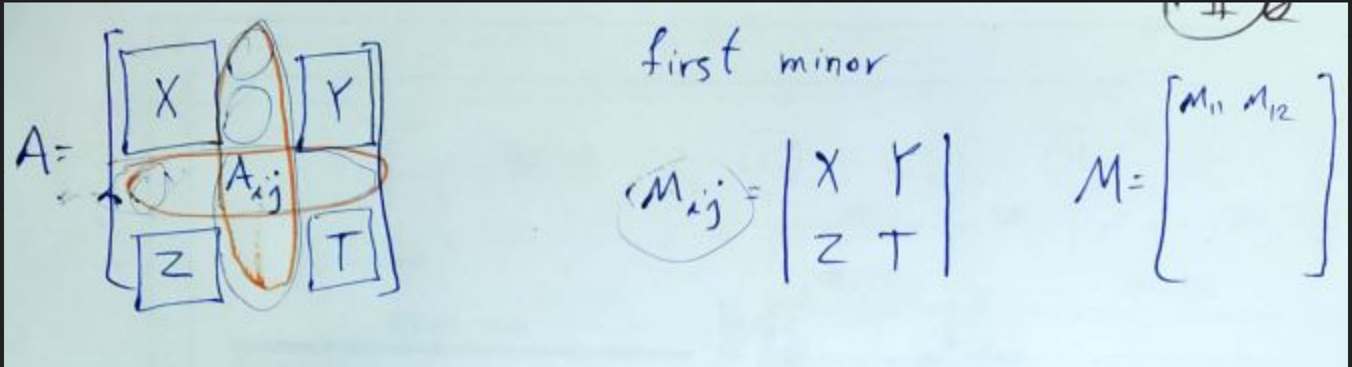

Cofactors

$|A|$ take the i-th row of A

$$|A| = (-1)^{i+1} a_{i1} M_{i1} + (-1)^{i+2} a_{i2} M_{i2} + \dots + (-1)^{i+n} a_{in} M_{in}$$

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

Cofactor

$$C_{ij} = (-1)^{i+j} M_{ij}$$

take the j-th column

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Cofactors

$$M_{ij} = \det\left(A_{-i,-j}\right) = \det\left(A \text{ with } i\text{-th row and } j\text{-th column removed}\right)$$

$$\begin{array}{c} j \\ \begin{bmatrix} & & \\ & M_{ij} & \\ & & \end{bmatrix} \end{array} = M \in \mathbb{R}^{n\times n}$$

Minor matrix

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C = \begin{array}{cc} & j \\ i & \begin{bmatrix} & & \\ & C_{ij} & \\ & & \end{bmatrix} \end{array}$$

Cofactor Matrix

Cofactors

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$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{vmatrix} = \boxed{a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \cdots + (-1)^{1+n} M_{1n}$$

$$i \begin{vmatrix} \vdots & & \\ a_{i1} & a_{i2} \cdots & a_{in} \\ \vdots & & \end{vmatrix} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

~~cofactor matirx~~

Cofactors

$i=1$ $\qquad$ $i=1$

cofactor matrix $\uparrow$

$$A = \begin{bmatrix} A_{11} & A_{1} & A_{1} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \qquad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}$$

$$\langle A[i,:],\ C[i,:] \rangle = |A|$$

$$\langle A[:,j],\ C[:,j] \rangle = |A|$$

$$\langle A[i,:],\ C[k,:] \rangle = ? \qquad i \neq k$$

Cofactor Matrix

$$A_{21}C_{31} + A_{22}C_{32} + A_{23}C_{33} + A_{24}C_{34}$$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix} = 0$$

Cofactor Matrix

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$$A\,C^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

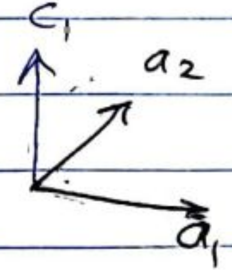
$$\Rightarrow \quad A\,C^T = |A|\, I \qquad \boxed{A^{-1} = \frac{1}{|A|} C^T}$$

Cofactor Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ & & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} a_{22}a_{33}-a_{22}a_{32} \\ \\ \end{bmatrix}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \qquad C = \begin{bmatrix} c_1^T \\ c_2^T \\ c_3^T \end{bmatrix}$$



$$c_1 = a_1 \times a_2$$

Cofactor Matrix

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \\ a_4^T \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_4^T \end{bmatrix}$$

$$c_1 = a_2 \wedge a_3 \wedge a_4$$

Cramer's Rule

$$Ax = b \qquad A \in \mathbb{R}^{n\times n}$$

$$x = A^{-1}b = \frac{1}{|A|} C^T b \qquad A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x = \frac{1}{|A|} \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_n^T \end{bmatrix} b$$

$$x_i = \frac{1}{|A|} c_i^T b = \frac{1}{|A|} \langle c_i, b \rangle$$

$$\langle c_i, b \rangle = \begin{vmatrix} a_1, & a_2, & \cdots & a_{i-1}, & b, & a_{i+1}, & \cdots, & a_n \end{vmatrix}$$

$A_i = A$ with i-th column replaced by b

$$Ax = b \qquad x = ? \qquad x_i = \frac{|A_i|}{|A|}$$