Linear Algebra for Computer Science

Lecture 22

Diagonalization, Eigenvalues and Eigenvectors

Diagonal Linear Transformations

eig1 ①

$x, y \in \mathbb{R}^n$

$A \in \mathbb{R}^{n \times n}$

$y = \underline{A} x \longrightarrow A @ x \longrightarrow O(n^2)$

تبدیل خطی

$$A = \begin{bmatrix} a_{11} & & & \emptyset \\ & a_{22} & & \\ & & \ddots & \\ \emptyset & & & a_{nn} \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11} x_1 \\ a_{22} x_2 \\ \vdots \\ a_{nn} x_n \end{bmatrix}$$

Diagonal Linear Transformations

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$y = Ax \qquad A \in \mathbb{R}^{n\times n} \qquad x \in \mathbb{R}^n$

$\Rightarrow O(n^2)$ operation in general.

$$A = \begin{bmatrix} a_1 & & & \emptyset \\ & a_2 & & \\ & & a_3 & \\ \emptyset & & & \ddots & a_n \end{bmatrix} \qquad Ax = \begin{bmatrix} a_1 & & \emptyset \\ & a_2 & \\ & & \ddots \\ \emptyset & & & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{bmatrix}$$

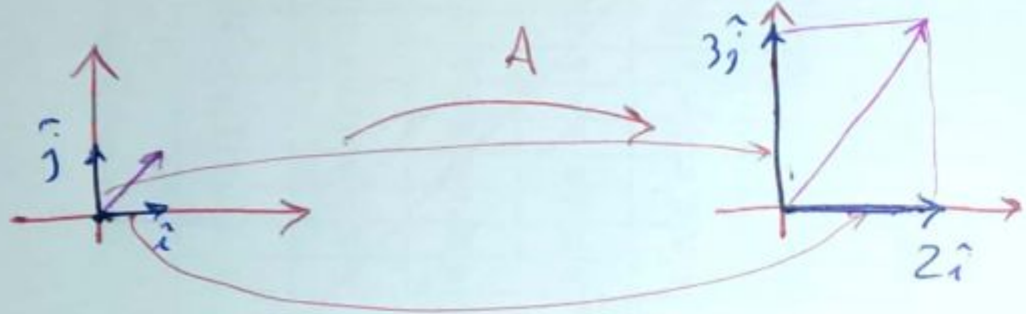
A diagonal $\quad y = Ax$ needs $O(n)$ operation

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Diagonal Linear Transformations

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 3x_2 \end{bmatrix}$$

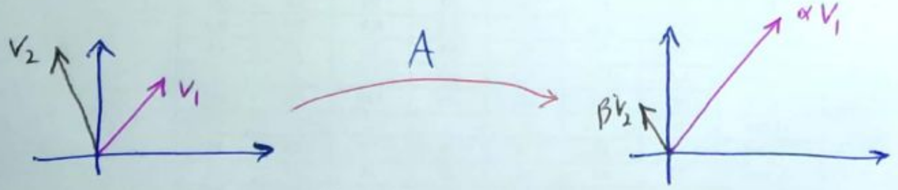


$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ do not change orientation.

Diagonal Transformation with Change of Basis

Consider a general linear ~~tran~~ transform $A \in \mathbb{R}^{n \times n}$.

Can we choose a basis in which the transformation acts ~~diagonally~~ like a diagonal matrix



Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. What are the coordinates of x in the basis (v_1, v_2)? $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $\cancel{x = \alpha \vec{v}_1 + \beta \vec{v}_2}$

$$x = z_1 \vec{v}_1 + z_2 \vec{v}_2 = \underbrace{\begin{bmatrix} v_1 & v_2 \end{bmatrix}}_{V} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Rightarrow z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} x$$

$$Ax = z_1 A \vec{v}_1 + z_2 A \vec{v}_2 = z_1 \alpha v_1 + z_2 \beta v_2 \qquad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha z_1 \\ \beta z_2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} x \Rightarrow \boxed{A = V \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} V^{-1}}$$

Diagonal Transformation with Change of Basis

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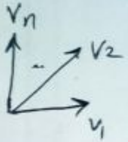

$x \in \mathbb{R}^n$

$\vec{x} = z_1\vec{v}_1 + z_2\vec{v}_2 + \cdots + z_n\vec{v}_n$

$v_1, v_2 \ldots v_n$ a basis for $\mathbb{R}^n$

$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \longrightarrow$ coordinates of x in the basis formed by $v_1, v_2 \ldots v_n$

$V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n \times n}$

$\vec{x} = [v_1\ v_2 \ldots v_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = V\vec{z}$

$\vec{z} = V^{-1}\vec{x}$

choose $v_1, v_2, \ldots v_n$ such that the transformation A is diagonal in this new basis.

Let D be the diagonal matrix performing the same transformation as A in the new basis.

$x' = Ax$

$z' = Dz,\ z = V^{-1}x,\ z' = V^{-1}x' \Rightarrow V^{-1}x' = DV^{-1}\cancel{z}\,x \Rightarrow V^{-1}Ax = DV^{-1}\cancel{z}\,x$

$Ax = VDV^{-1}x \Rightarrow \boxed{A = VDV^{-1}}$

Diagonalizable Matrices

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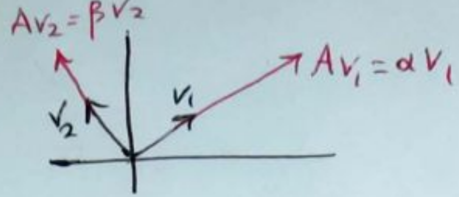
Can we find a basis in which the transformation $A \in R^{n\times n}$ acts like a diagonal matrix?



$\Updownarrow$

If there is a matrix $V \in R^{n\times n}$ such that

$$A = V D V^{-1}$$

where $D = diag([d_1, d_2, \dots, d_n])$ is a diagonal matrix.

In such case, we say that A is diagonalizable.

Diagonalizable Matrices in terms of Similarity

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In such case, we say that A is diagonalizable.

A and $B \in \mathbb{R}^{n\times n}$ are similar if $\exists\, V \in \mathbb{R}^{n\times n}$ such that

$$A = V B V^{-1}$$

A is diagonalizable $\Leftrightarrow$ A is similar to a diagonal matrix

not all matrices are diagonalizable!

Powers of a matrix using Diagonalization

$$Ax = \cancel{A}\; V D V^{-1} x$$

$$A^2 x = AAx = V D V^{-1} V D V^{-1} x = V D^2 V^{-1} x \qquad D^2 = \begin{bmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{bmatrix}$$

$$A^n x = V D^n V^{-1} x$$

Joint Diagonalization

$A, B \in \mathbb{R}^{n\times n}$ are jointly diagonalizable if there exist a nonsingular matrix $V \in \mathbb{R}^{n\times n}$ such that

$A = V D_1 V^{-1}$ $\quad B = V D_2 V^{-1}$ where D_1, D_2 are diagonal.

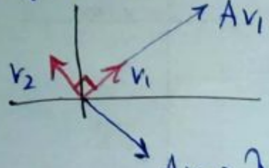
$BAx = V D_2 D_1 V^{-1} \Longrightarrow$ exactly what happens in the fourier basis.

circular Convolution $Ax \Rightarrow A$ is a circulant matrix

Diagonalization with a Orthogonal Basis?

Can we find a orthonormal basis in which the linear transformation $A \in \mathbb{R}^{n\times n}$ acts like a diagonal matirix

Example: $n=2$



$Av_1 = \lambda_1 v_1$

$Av_2 = \lambda_2 v_2$

$v_1 \perp v_2$

$v_1^T v_2 = 0$

$v_1^T v_1 = v_2^T v_2 = 1$

$V = [v_1 \; v_2]$ orthogonal matrix

$V^{-1} = V^T$

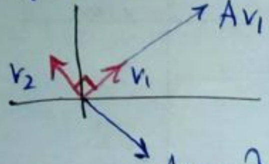
$A = VDV^T \Rightarrow \begin{cases} V \text{ orthogonal} \\ D \text{ diagonal} \end{cases}$

Diagonalization with an Orthonormal Basis?

Can we find a orthonormal basis in which the linear transformation $A \in \mathbb{R}^{n \times n}$ acts like a diagonal matirix

Example: $n=2$

$Av_1 = \lambda_1 v_1$



$Av_2 = \lambda_2 v_2$

$v_1 \perp v_2$

$v_1^T v_2 = 0$

$v_1^T v_1 = v_2^T v_2 = 1$

$V = [v_1 \; v_2]$ orthogonal matrix

$V^{-1} = V^T$

$A = VDV^T \Rightarrow \begin{cases} V \text{ orthogonal} \\ D \text{ diagonal} \end{cases}$

Eigenvectors

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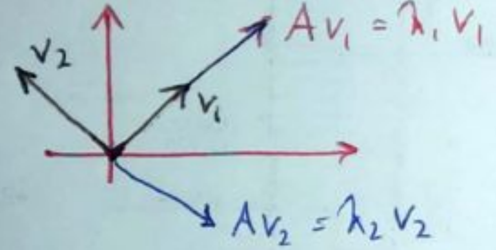

~~For A~~

For a matrix $A \in \mathbb{R}^{n\times n}$ is there a vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$

$v \neq 0$

Eigenvectors and Eigenvalues

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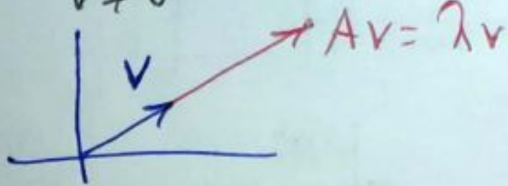
For a matrix $A \in \mathbb{R}^{n\times n}$ is there a vector $v \in \mathbb{R}^n$ such that $Av = \lambda v$

$v \neq 0$



if there exists such a vector $v \in \mathbb{R}^n$

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v is called an eigenvector of A.

λ is called an eigenvalue of A.

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Eigenvectors and Eigenvalues

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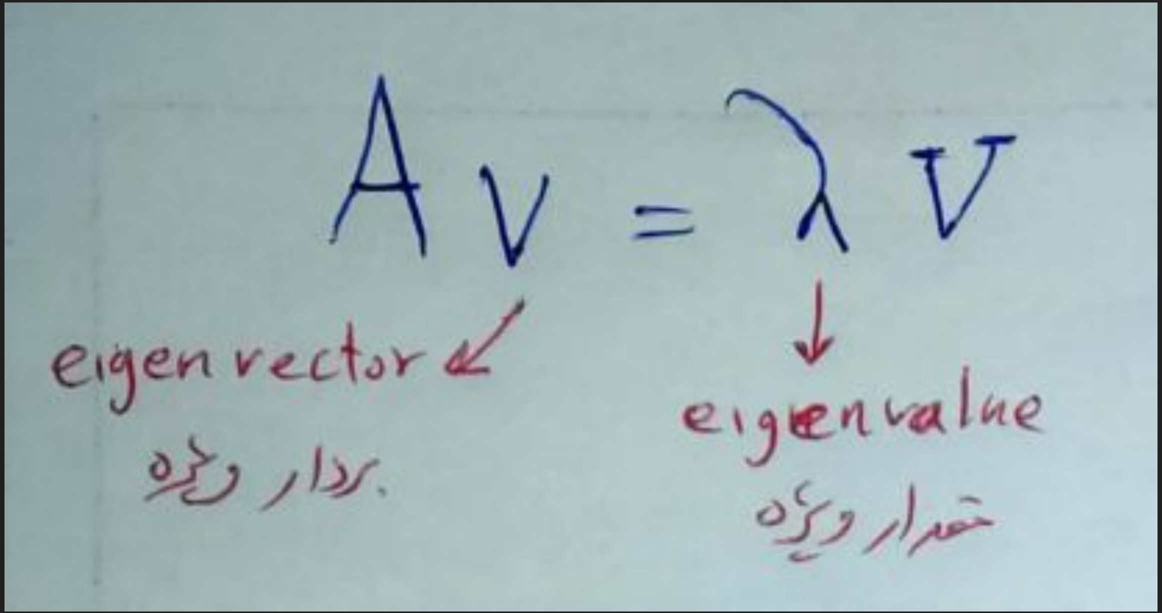
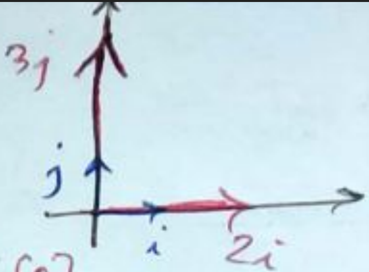


Example: Eigenparis of a Diagonal Matrices

Example A diagonal

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$



eigenvectors are $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

eigenvalue

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 2 \right), \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 3 \right)$$

eigenvector

corresponding eigenvalue

Length of Eigenvector does not matter

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$$Av = \lambda v \Rightarrow A(2v) = \lambda(2v)$$
$$A(\alpha v) = \lambda(\alpha v)$$

for eigenvectors the orientation matters (not lenght)

Example: All Eigenpairs of a diagonal matrix

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$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2v_1 \\ 3v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq 0$$

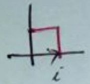
$$v_1 \neq 0 \text{ or } v_2 \neq 0$$

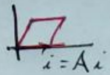
$$\Rightarrow \begin{bmatrix} (2-\lambda) v_1 \\ (3-\lambda) v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \lambda = 2, \; v_2 = 0 \\ \lambda = 3, \; v_1 = 0 \end{cases}$$

$$\Rightarrow \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 2 \right) \text{ and } \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 3 \right) \text{ are the only eigen pairs}$$

Example: Shear

$$A = \begin{bmatrix}1 & \alpha\\ 0 & 1\end{bmatrix}$$





$$A\hat{i} = A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1 & \alpha\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \left(\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{eigenvector}}, \underbrace{1}_{\text{eigenvalue}}\right)$$

$$Av = \begin{bmatrix}1 & \alpha\\ 0 & 1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1 + \alpha v_2\\ v_2\end{bmatrix} = \lambda\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}\lambda v_1\\ \lambda v_2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}(1-\lambda)v_1 + \alpha v_2\\ (1-\lambda)v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \begin{cases} v_2 = 0 \Rightarrow (1-\lambda)v_1 = 0 \overset{v_1 \neq 0}{\Rightarrow} \lambda = 1 \\ (1-\lambda) = 0,\ v_2 \neq 0 \\ \quad\Downarrow \\ \alpha v_2 = 0 \\ \quad\Downarrow \text{ only possible if } \alpha = 0 \end{cases}$$

Example: Shear

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$\Rightarrow A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ with $\alpha \neq 0$ only has a single eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\Rightarrow$ A is not diagonalizable.

Example: Identity Matrix

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Choose any $v \neq 0 \in \mathbb{R}^n$

$A = I \in \mathbb{R}^{n \times n}$

$Av = Iv = v \Rightarrow Av = 1\,v$

eigen value

Any vector $v \neq \emptyset$ is an eigen vector of I.

was in any case 1 is an eigenvalue.

Example: Identity matrix

$A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Choose any $v \neq 0 \in \mathbb{R}^n$

$A = I \in \mathbb{R}^{n\times n}$

$Av = Iv = v \Rightarrow Av = 1v$

eigen value

Any vector $v \neq \emptyset$ is an eigen vector of I.

in any case 1 is an eigenvalue.

Example: 2D rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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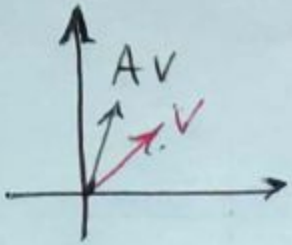
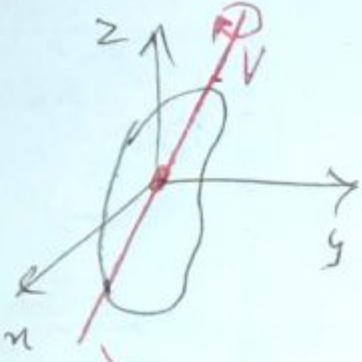
$\theta \neq 0$



No $\underbrace{}_{\text{(real)}}$ Eigenvectors

Example: 3D Rotation

$R \in \mathbb{R}^{3\times 3}$ is $\underbrace{\text{a}}_{3D}$ rotation matrix



$Rv = 1 \cdot v$

$(v, 1)$ only (real) eigenpair

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axis of rotation

Singular matrices

Let $A \in R^{n \times n}$ be singular. $\Rightarrow Dim(N(A)) > 0$

$\exists\, v \in N(A) \quad v \neq 0 \qquad A\vec{v} = \vec{0} = 0 \cdot \vec{v}$

Any $v \neq 0$ in $N(A)$ is an eigenvector of A with the corresponding eigenvalue $\lambda = 0$.

Projection matrix

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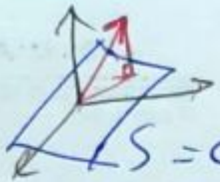
Projection matrix P



$$P = V \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} V^{-1}$$

$S = C(P)$

$v \in S \Rightarrow Pv = v \qquad \lambda = 1$

$v \in S^{\perp} \Rightarrow Pv = 0 = 0.v \Rightarrow \lambda = 0$

1. $P^2 = PP = P$

$$\left.\begin{aligned} Pv = \lambda v \Rightarrow P^2 v = PPv &= Pv = \lambda v \\ &= P(\lambda v) = \lambda^2 v \end{aligned}\right\} \begin{aligned} &\lambda = \lambda^2 \Rightarrow (\lambda^2 - \lambda) = 0 \\ &\lambda = 0,\ \lambda = 1 \end{aligned}$$