Linear Algebra for Computer Science

Lecture 23

Eigenvalues and Eigenvectors, Algebraic Multiplicity, Eigenspaces, Geometric Multiplicity

Calculate the Eigenvalues

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Eig2 (I)

$A \in \mathbb{R}^{n\times n}$

$Av = \underline{\lambda}\,\underline{v} \qquad v \in \mathbb{R}^n,\ \lambda \in \mathbb{R}$

$$Av = \lambda v \Rightarrow Av - \lambda v = 0 \Rightarrow \underbrace{A}_{n\times n} v - \underbrace{(\lambda I)}_{n\times n} v = 0$$

$\Rightarrow \overbrace{(A - \lambda I)}^{n\times n} v = 0 \Rightarrow (A - \lambda I)$ is singular

$v \neq 0$

$\det(A - \lambda I) = 0 \Rightarrow$ a polynomial on λ of degree n

چندجمله‌ای درجه n

Calculate the Eigenvalues

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Converse:

$\det(A-\lambda I)=0 \Rightarrow (A-\lambda I)$ singular

$\Rightarrow \exists v \in \mathbb{R}^n$ $(A-\lambda I)v=0 \Rightarrow Av=\lambda v$
$v \neq 0$

$\Rightarrow \lambda$ is an eigenvalue of A.

Calculate the Eigenvalues

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow A - \lambda I = A - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) \; \cancel{\det(A)} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda) - (a_{11}-\lambda)\, a_{32}\, a_{23}$$

$$a_{21}\, a_{12}\, (a_{33}-\lambda) \pm \cdots$$

$$\Rightarrow -\lambda^3 + \cdots + a\lambda + b$$

$\Rightarrow$ a polynomial of degree 3 $\Rightarrow$ at most 3 roots

$\Rightarrow$ at most 3 eigenvalue.

Example

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \Rightarrow A - \lambda I = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda & 3 \\ 3 & \lambda \end{bmatrix}$$

charcteristic c

چندجمله‌ای مشخصه

$$\det \begin{bmatrix} \lambda & 3 \\ 3 & \lambda \end{bmatrix} = 0 \Rightarrow \underline{\lambda^2 - 3^2} = 0 \Rightarrow (\lambda - 3)(\lambda + 3) = 0$$

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$$\lambda = 3 \Rightarrow \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3v_2 \\ 3v_1 \end{bmatrix} = \begin{bmatrix} 3v_1 \\ 3v_2 \end{bmatrix} \Rightarrow v_1 = v_2$$

$$v = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = -3 \Rightarrow \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3v_2 \\ 3v_1 \end{bmatrix} = \begin{bmatrix} -3v_1 \\ -3v_2 \end{bmatrix} \Rightarrow v_1 = -v_2$$

$$v = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 3\right), \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, -3\right)$$

choose eigenvector to be unit vectors

$$\left(\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, 3\right), \left(\begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, 3\right)$$

Example

eig2

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}$$

$$\underbrace{(2-\lambda)^2 - 3^2}_{\text{charcteristic polynomial of } A} = 0 \Rightarrow 2-\lambda = \pm 3 \Rightarrow \begin{cases} \lambda = -1 \\ \lambda = 5 \end{cases}$$

$$\lambda = -1 \Rightarrow A - \lambda I \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = 5 \Rightarrow A - \lambda I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example

Shear

$$A = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & \alpha \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1$$

ریشه مضاعف

repeated

$$(A-\lambda I)v = 0 \Rightarrow \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix} v = 0 \Rightarrow \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \alpha v_2 = 0 \Rightarrow \begin{cases} \alpha = 0 \end{cases}$$

$$\alpha v_2 = 0 \Rightarrow \begin{cases} \alpha = 0 & \text{(A is identity) every } v \neq 0 \in \mathbb{R}^2 \text{ is an eigenvector} \\ \alpha \neq 0 & \Rightarrow v_2 = 0 \Rightarrow v = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} \end{cases}$$

$$\Rightarrow \text{the only eigenvector is } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example: 2D Rotation

Rotation (2D) $\quad R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = \pi/2 \Rightarrow R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|R - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$ no real roots

what about complex roots? $\lambda = \pm i$

$\lambda = i \Rightarrow (R - \lambda I)v = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = -i \Rightarrow (R - \lambda I)v = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} v = 0 \Rightarrow v = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Example: 2D Rotation

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix}$$

eig2 III

$\Rightarrow$ no real eigenpairs $\Rightarrow$ there are complex eigenpairs

$$\left(\begin{bmatrix} 1 \\ i \end{bmatrix}, -i \right), \quad \left(\begin{bmatrix} i \\ 1 \end{bmatrix}, +i \right)$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -i \\ 0 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Number of Eigenvalues

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$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & b & c & d \\ e & f-\lambda & g & h \\ i & j & k-\lambda & l \\ m & n & o & p-\lambda \end{bmatrix}$$

$\Rightarrow \det(A - \lambda I)$ is a polynomial of degree 4 on λ

$A \in \mathbb{R}^{n \times n} \Rightarrow \det(A - \lambda I)$ is a polynomial of ~~deg~~ degree $\underline{\underline{n}}$ on λ

$\Rightarrow$ has at most $\underline{\underline{n}}$ roots

$\Rightarrow A \in \mathbb{R}^{n \times n} \Rightarrow A$ has at most n eigenvalues

$A \in \cancel{\mathbb{R}}\,\mathbb{C}^{n \times n} \Rightarrow$ " " " " n "

Eigenspace

Let λ be an eigenvalue of $A \in \mathbb{R}^{n \times n}$. eig2 IV

What are the set of ~~eig~~ eigenvectors corresponding to λ? $\Rightarrow N(A - \lambda I)$ $\Rightarrow$ is a linear subspace

null space

v_1, v_2 are eigenvectors corresponding to eigenvalue λ.

v_1, v_2 share a common eigenvalue λ

$$\left.\begin{array}{l} A v_1 = \lambda v_1 \\ A v_2 = \lambda v_2 \end{array}\right\} \quad A(\alpha v_1 + \beta v_2) = \cancel{A}\, \alpha A v_1 + \beta A v_2$$

$$= \alpha \lambda v_1 + \beta \lambda v_2$$

$$= \lambda (\alpha v_1 + \beta v_2)$$

$\Rightarrow \overbrace{\alpha v_1 + \beta v_2}^{\text{(every linear combination of } v_1, v_2)}$ is also an eigenvector of A.

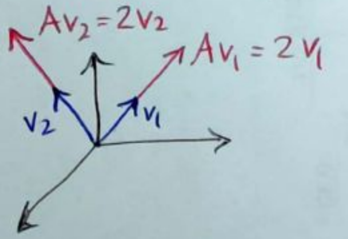
$\Rightarrow$ the set of eigenvectors corresponding to an eigenvalue λ is a <u>linear subspace</u>.

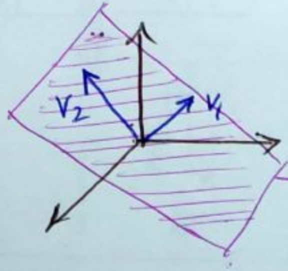
Eigenspace

In general for every eigenvalue λ there is a linear subspace V_λ such that for all $v \in V_\lambda$ we have $Av = \lambda v$

V_λ is called the eigenspace correspondig to λ.

$$V_\lambda = N(A - \lambda I)$$





eigenspace correspondig to $\lambda = 2$

Example: Identity matrix

eig2 V

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ the only eigenvalue is $\lambda = 1$

eigenspace $(\lambda = 1) = V_\lambda = \mathbb{R}^2 \Rightarrow$ 2D eigenspace

$Av = v = 1 \cdot v$

Identity matrix, geometric and algebraic multiplicity

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$$A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix}1-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} = (1-\lambda)^2 = 0$$

$$\lambda = 1$$

Algebraic mult $= 2$

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_x\\ v_y\end{bmatrix} = 0$$

eigenspace $(\lambda = 1) = \mathbb{R}^2$

geo mult. $= 2$

Example:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda = 2 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \quad 1D \\ \lambda = 3 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\} \quad 1D \end{array}$$

Example:

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 2 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\} \quad \text{1D}$$

$$\lambda = 3 \Rightarrow V_\lambda = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R} \right\} \quad \text{2D}$$

for every eigen space

Example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow |A - \lambda I| = \begin{vmatrix} 2-\lambda & & \emptyset \\ \emptyset & 3-\lambda & \\ & & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 = 0$$

$$\lambda_1 = 2 \longrightarrow \text{alg mult} = 1 \qquad \cancel{\text{age}}\ e \longrightarrow \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2, \lambda_3 = \underline{3} \longrightarrow \text{alg mult} = 2 \longrightarrow \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\lambda = 2 \implies V(\lambda) = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

geometric mult = algeb. mult. = 1

$$\lambda = 3 \implies V(\lambda) = \left\{ \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}$$

2 = geo. mult = alg mult